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Fault-Tolerant Sensor Reconciliation Schemes based on Unknown Input Observers

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This paper proposes two fault-tolerant sensor reconciliation design methods for over-sensed plants (see Fig. 1). The aim of the reconciliator is to detect, at each time instant, the presence of faults on the existing physical sensors y and hide the corrupted measurements in the generation of the virtual output z (with $\dim y \geq \dim z$), which one would like to be generated in a reliable way in spite of fault occurrences and hence trustfully usable for control purposes. If the reconciliation scheme were effective, it would provide a reliable output z for feedback control and would exclude the need to reconfigure the nominal control law in case of faults. The sensor faults here considered are limited to variations of both sensor gain and offset values. The proposed approach envisages the use of an Unknown Input Observers (UIO) coupled with an "ad-hoc" parameters estimator used to estimate on-line the sensor effectiveness matrix at each time instant. In the paper, two design methodologies are described, based one on the Linear Parameter Varying (LPV) polytopic formulation and the other on the Linear Fractional Transformation (LFT) paradigm. All main properties of the sensor reconciliation schemes are investigated and rigorously proved. A final simulation example is included where both the LPV and LFT schemes are compared.

Keywords: Sensor Reconciliation, Virtual Sensors, Fault Detection, Unknown Input Observer

1. Introduction

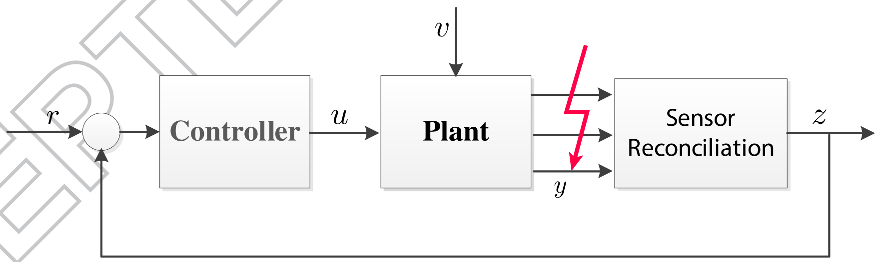

Figure 1.: Fault-tolerant sensor reconciliator basic scheme

In control systems applications the capability to detect faulty sensors and recover uncorrupted data has gained importance in the last two decades. Specifically, in traditional control schemes, faulty sensors give wrong information about the system status, which could cause instability when used in feedback loops. Even in the fortuitous cases where stability is preserved, inaccurate sensor values may lead to poor regulation or tracking performance, which may be highly undesirable for

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many high precision control applications (Djath et al. (2000); Mirabadi et al. (2003); Romero et al. (2010)).

To cope with these situations a Sensor Reconciliation (SR) scheme (Vachhani et al. (2001)) is often exploited in order to recover useful data from the pool of redundant sensors whenever unpredicatable fault events may eventually occur. In Figure 1, a quite general SR scheme is depicted. There, the SR block can be seen as a *virtual sensor* (Steffen (2005)), in charge of translating measurements from the possibly faulty sensors y into the reliable virtual sensors vector z that can be trustfully used for control purposes.

Many SR approaches based on Figure 1 are often proposed in the literature as part of a Fault Tolerant Control (FTC) scheme and coupled with traditional controllers. Such a choice avoids the usage of complex control reconfiguration strategies to accommodate sensor faults. In this respect, relevant contributions include De Doná et al. (2009); Sun and Deng (2004); Yetendje et al. (2011), where the sensor information are fused in a decentralized way by exploiting local estimators. Another class of SR FTC based strategies is considered in (Berbra, Lesecq, and Martinez (2008); Romero et al. (2010)), where a switching mechanism involving sensors and related observers is exploited to implicitly detect the healthy components of the system. The estimates provided by the observers are compared at each sampling time by a switching logic that allows one to select the sensors-observer pair with the smallest estimation error.

All the above mentioned approaches are mainly focused on the accomplishment of two fundamental tasks: (i) identification of faults in the sensors, (ii) correction of sensor measurements. In this respect many effective methods have been developed for the estimation of either actuator or sensor faults. See e.g. (Alwi et al. (2011); Cristofaro and Zaccarian (2016); Han et al. (2016); He et al. (2013)). See also (Crowe (1996); Mah et al. (1976); Romagnoli and Stephanopoulos (1981)) for relevant works in sensors rectification.

This paper aims at presenting a general SR method for linear discrete-time systems with redundant physical sensors possibly subject to loss of effectiveness (gain) and offset (bias) faults. Differently from the above mentioned SR methods, the scheme here discussed is based on the Unknown Input Observer (UIO) approach (Guan and Saif (1991)). In the present context, the UIO methodology has been widely investigated for the design of fault detection and isolation schemes for LTI continuous-time systems but limited to the detection of sensor bias faults (J. Chen et al. (1996); W. Chen and Saif (2006); Duan et al. (2002)).

Here we move to the discrete-time system domain and extend the ideas of Rodrigues et al. (2005); Zhou et al. (2013), where Linear Matrix Inequality (LMI) based procedures have been proposed to synthesize UIOs with constant observer gains, to address the more challenging case of jointly detecting both bias and gain sensor faults. To this end, the key idea here is that of considering admissible ranges of current sensor gain estimates as structural uncertainty in the plant matrices. Such a choice leads to a non-convex uncertainty representation in the UIO equations. To deal with this issue, two different approaches are proposed. The first one relies upon a polytopic embedding of the uncertain matrices that allows the design, via a specific LMI procedure, of a polytopic Linear Parameter Varying (LPV) UIO observer. Preliminary results about this approach have been given in Behzad et al. (2016) and are presented here in a more formal and complete way. The second method to deal with the non-convexity of UIO equations involves the well-known Linear Fractional Transformation (LFT) formulation (Cockburn and Morton (1997)). In this way, it is possible to build up a time-varying observer by solving LMI feasibility problems that are characterized by a lower numerical complexity with respect to the general LPV case. The resulting computational burden required to numerically synthesize the observer gain is comparable to that of the standard linear time-invariant case.

The proposed schemes consist each of three interconnected modules: (i) a Parameter Estimator unit implemented via a constrained weighted least-squares batch method used, within a windowing data processing approach, to estimate the current gain sensor faults, (ii) a UIO unit in charge of combining the corrupted information gathered by multiple sensors to reconstruct, on the basis of the output of the Parameter Estimator, the state of the system and estimating the current bias; (iii) a Sensor Reconciliation unit used to reconcile the sensor measures.

Properties of the proposed LPV-UIO and LFT-UIO schemes are formally proved and discussed and complete computational procedures are provided for their design. A final numerical example is reported where comparisons involving both the approaches are provided.

Notation

Let $\mathbb{R}$ denote the set of real numbers whereas $\mathbb{N}$ that of natural numbers. Let $v' \in \mathbb{R}^{1\times n}$ denote the transpose of a vector $v \in \mathbb{R}^n$, $\|\cdot\|_2$ the 2-norm of a vector (i.e. $\|x\|_2 := \sqrt{x'x}$) and $\|\cdot\|_{l_2}$ the l_2-norm of a signal $w(t) \in l_2$ (i.e. $\|w(\cdot)\|_{l_2} := \sqrt{\sum_{i=0}^{\infty} w(i)'w(i)}$). Given a matrix $M \in \mathbb{R}^{n\times m}$, the i-th row of M is denoted as $M^{(i)}$. For a matrix $M \in \mathbb{R}^{n\times m}$ having linearly independent rows, the *Moore-Penrose Pseudoinverse* is defined as $M^\dagger \in \mathbb{R}^{n\times m}$ and is computed as $M^\dagger := A'(AA')^{-1}$. Linear Fractional Transformations (LFTs) are extensively used in the paper. For properly sized matrices N and

$$M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

the lower LFT is defined as

$$LFT(M,N) := M_{11} + M_{12}N(I - M_{22}N)^{-1}M_{21}$$

For $\mathcal{P} \subset \mathbb{R}^p$ and $\mathcal{Q} \subset \mathbb{R}^q$ being two polytopes, their *Cartesian Product* is defined as

$$\mathcal{P} \times \mathcal{Q} = \{(x,y) : x \in \mathcal{P}, y \in \mathcal{Q}\}$$

The Polytope $\mathcal{S}_l := \{\xi \in \mathbb{R}^l | 0 \le \xi_i \le 1,\ i = 1,...,l, \sum_{i=1}^{l} \xi_i = 1\}$ is a l-dimensional *Unit Simplex*. For l matrices $M_i \in \mathbb{R}^{n\times m}$, $i = 1,...,l$, their *Convex Hull*, denoted by $\mathrm{Co}\{M_i\}, i = 1,...,l$, is the polytope arising by all convex combinations of matrices M_i i.e $\{\sum_{i=1}^{l} \rho_i M_i, [\rho_1,...,\rho_l]' \in \mathcal{S}_l\}$ with $\mathcal{S}_l$ being a l-dimensional unit simplex.

2. Problem Formulation

Let us consider a plant whose dynamics is described by the following discrete-time state-space representation

$$x_p(t+1) = Ax_p(t) + Bu(t) + Ev(t) \tag{1}$$

$$y(t) = \Delta\big(\gamma(t)\big)C_y x_p(t) + Fb(t) \tag{2}$$

$$z(t) = H_z C_y x_p(t) \tag{3}$$

where $x_p(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is a known input while $v(t) \in \mathbb{R}^{n_v}$ is an unknown input. Moreover, $y(t) \in \mathbb{R}^m$ represents the *plant output* provided by physical redundant sensors possibly effected by both bias $b(t) \in \mathbb{R}^q$ and loss of effectiveness faults, the latter being modeled by the gain matrix $\Delta(\gamma) \in \mathbb{R}^{m\times m}$ that, for simplicity, we assume hereafter to have the following elementary structure:

$$\Delta(\gamma) = \begin{bmatrix} \gamma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \gamma_m \end{bmatrix} \tag{4}$$

Finally, $z(t) \in \mathbb{R}^r$, with $r \leq m$, is defined as the *virtual output* of the system and represents the healthy information we need to get from the plant for control purposes, which we would like to be free from the effects of faults possibly occurring on the physical sensors y.

It is clear that in the absence of faults one would have $\Delta(\gamma) = I_m$ and $b(t) = 0_q$. However, in the more general case $b(t) \neq 0_q$ and $\Delta(\gamma) \neq I_m$, with γ confined in the generic polytope

$$\Gamma \subseteq \mathcal{S} := \{\gamma : 0_m \leq \gamma \leq 1_m\} \tag{5}$$

Notice that Γ, implicitly defined by fulfilling next Assumptions 1 and 2, is always a proper subset of $\mathcal{S}$. Because of faults, it is not convenient to evaluate the signal $z(t)$ as $z(t) = H_z y(t)$ because it would be affected by possibly corrupted information brought by $y(t)$. However, because the state $x_p(t)$ is assumed not directly measurable, $z(t)$ cannot be evaluated as simply as in (3), but a more sophisticated machinery is required. This aspect motivates the design of the *Sensor Reconciliator* (virtual sensor) unit depicted in Figure 1, which basically aims at addressing the following problem:

Sensor Reconciliaton Design Problem (SRDP-Problem) :
Given the system (1)-(3)*, compute, at each time* $t \geq 0$ *on the basis of the real output* $y(t)$ *measurements, a suitable estimate* $\hat{z}(t)$ *of the virtual output* $z(t) := H_z C_y x_p(t)$*, despite the presence of both fault occurrences, corrupting the vector* $y(t)$*, and disturbances* $v(t)$*.*

The problem just stated can be solved in principle by evaluating an estimate $\hat{x}_p(t)$ of the state $x_p(t)$ that is exploited to compute the corresponding estimate $\hat{z}(t)$ of $z(t)$ through the following equation

$$\hat{z}(t) = H_z C_y \hat{x}_p(t) \tag{6}$$

Anyway such an approach require to face two crucial issues: 1) How to estimate the fault occurrences corrupting $y(t)$? 2) How to get a good estimation $\hat{x}_p(t)$ in presence of an unknown input $v(t)$ and time-varying sensor gains and bias? Next section is devoted to answer to these questions.

3. Virtual Sensor Architecture

A solution for the problem presented in the previous section is here described by introducing the *virtual sensor* architecture depicted in Fig. 2 that consists of three modules: an *Unknown Input Observer* (UIO) unit, which is the core of this scheme and it is designed to compute estimates of $x_p(t)$ and $b(t)$; a *Parameter Estimator* whose output is an estimate of effectiveness matrix (4) and a *Reconciliator Unit* that simply performs the computation indicated in (6).

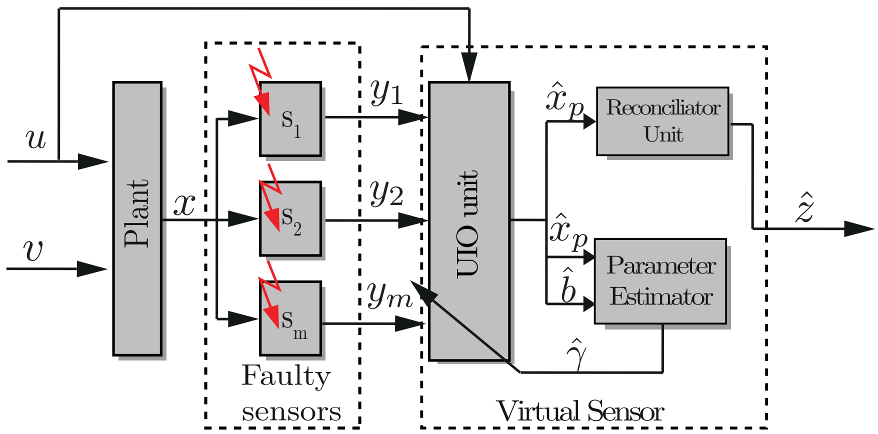

Figure 2.: Virtual Sensor Architecture

3.1 *Sensor Fault Augmented Model*

In order to design the UIO, the following augmented state is considered including the bias fault $b(t)$ among its components

$$x(t) = \begin{bmatrix} x_p(t) \\ b(t) \end{bmatrix} \tag{7}$$

In this way, the related augmented model can be described as

$$\begin{aligned} x(t+1) &= \bar{A}x(t) + \bar{B}u(t) + \bar{E}v(t) + \bar{F}\Delta b(t) \\ y(t) &= \bar{C}_\gamma x(t) \end{aligned} \tag{8}$$

where

$$\bar{A} := \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, \bar{B} := \begin{bmatrix} B \\ 0 \end{bmatrix}, \bar{E} := \begin{bmatrix} E \\ 0 \end{bmatrix}, \bar{F} := \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{C}_\gamma \quad := \begin{bmatrix} \triangle(\gamma)C_y & F \end{bmatrix}, \; \Delta b(t) := b(t+1) - b(t)$$

Moreover, the following technical assumptions are required

Assumption 1: *The pair* $(\bar{C}_\gamma, \bar{A})$ *is observable* $\forall \gamma \in \Gamma$

Assumption 2:

$$rank\{\bar{C}_\gamma \bar{E}\} \geq rank(\bar{E}), \forall \gamma \in \Gamma \tag{9}$$

The above assumptions are mandatory for the existence of the Unknown Input Observer presented in the next section. In particular, concerning the latter assumption please notice that the structure of the set Γ in (5) guarantees that for every full-rank matrix $\bar{E}$, rank$\{\bar{C}_\gamma \bar{E}\} \geq 1, \forall \gamma \in \Gamma$.

3.2 *Unknown Input Observer*

In this section we describe the basic ingredients of the proposed UIO. Let us assume to be provided with an estimation $\hat{\gamma}(t)$ of $\gamma(t)$ at each time t. Then, a possible structure for an unknown input observer for the model (8) is given by

$$\hat{x}(t+1) \;=\; T_{\hat{\gamma}(t)}\bar{A}\hat{x}(t) + T_{\hat{\gamma}(t)}\bar{B}u(t) + L_{\hat{\gamma}(t)}\big(y(t) - \hat{y}(t)\big) + Q_{\hat{\gamma}(t)}y(t+1) \tag{10}$$

where $T_{\hat{\gamma}} \in \mathbb{R}^{(n+q)\times(n+q)}$, $L_{\hat{\gamma}} \in \mathbb{R}^{(n+q)\times m}$ and $Q_{\hat{\gamma}} \in \mathbb{R}^{(n+q)\times m}$ represent design parameters all depending on the effectiveness matrix (4). In particular, if $T_{\hat{\gamma}}$ were chosen to satisfy

$$T_{\hat{\gamma}} + Q_{\hat{\gamma}}\bar{C}_{\hat{\gamma}} = I_{n+q} \tag{11}$$

under the condition

$$T_{\hat{\gamma}}\bar{E} = 0, \forall\hat{\gamma} \in \Gamma \tag{12}$$

the system (8) could be represented as

$$x(t+1) = T_{\hat{\gamma}(t)}\bar{A}x(t) + T_{\hat{\gamma}(t)}\bar{B}u(t) + T_{\hat{\gamma}(t)}\bar{F}\Delta b(t) + Q_{\hat{\gamma}(t)}y(t+1) \tag{13}$$

Please notice that (12) is satisfied if $Q_{\hat{\gamma}}$ is chosen as

$$Q_{\hat{\gamma}} := \bar{E}(\bar{C}_{\hat{\gamma}}\bar{E})^{\dagger}, \forall\hat{\gamma} \in \Gamma \tag{14}$$

where the existence of the matrix $(\bar{C}_{\hat{\gamma}}\bar{E})^{\dagger}$ is guaranteed $\forall\hat{\gamma} \in \Gamma$ by **Assumption 2**.

In this respect, it is worth pointing out that unfortunately the matrix T_γ does not depend linearly on the parameter γ. As a consequence, the related uncertainty representation results non-convex. For this reason, in order to take advantages of existing LMI optimization techniques, we present two different approaches, respectively based on LPV polytopic embeddings and LFT formulations, for the design of the observer gain (10).

3.2.1 LPV formulation

In this section we assume to be provided by a polytopic embedding approximation for matrices T_γ and $\bar{C}_\gamma$ given by (see Figure 3 for a graphic idea)

$$\begin{cases} \bar{C}_\rho = \sum_{i=1}^{l} \rho_i(\gamma)\tilde{C}_i, \\ T_\rho = \sum_{i=1}^{l} \rho_i(\gamma)\tilde{T}_i \end{cases} \tag{15}$$

for a certain continuous functions $\rho_i : \Gamma \to \mathbb{R}$ of γ and pair of matrices $(\tilde{T}_i, \tilde{C}_i)$, $i = 1, ..., l$. In addition, we assume that the map $\rho : \Gamma \to \mathcal{R}^l$ given by $\rho := (\rho_1, ..., \rho_l)^T$ always returns values into the unit simplex $\mathcal{S}_l$. Hence, for each $\gamma \in \Gamma$, the pair $(T_\rho, \tilde{C}_\rho)$ lies in the convex hull $\mathrm{Co}\{(\tilde{T}_i, \tilde{C}_i)\}, i = 1, ..., l$.

Moreover, the above representations have to guarantee that the following Assumptions 3 and 4 hold true:

Assumption 3: $\left(T_\rho\bar{A}, \bar{C}_\rho\right)$ *is detectable* $\forall\rho \in \mathcal{S}_l$

Assumption 4: $T_\rho\bar{E} = 0$, $\forall\rho \in \mathcal{S}_l$

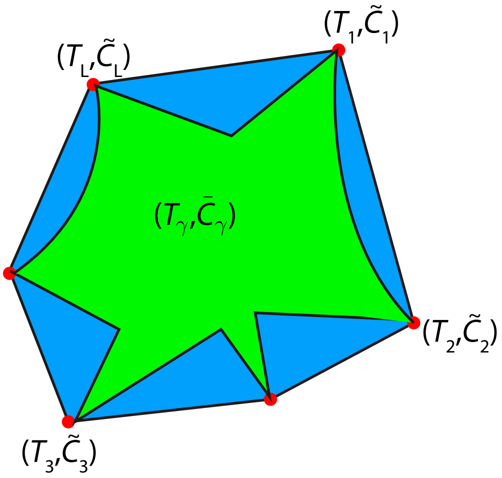

Figure 3.: Non convex representation (green region) and related polytopic embedding (blue region)

Now, we have all the ingredients to design a LPV gain $L_{\hat{\rho}}$ defined as follows

$$L_{\hat{\rho}} = \sum_{i=1}^{l} \hat{\rho}_i(\gamma) L_i \tag{16}$$

where the gains L_i, $i = 1, ..., l$ are properly chosen to stabilize the state estimation error, provided that an estimation $\hat{\rho}(t) := \hat{\rho}(\hat{\gamma}(t))$ is available. In fact, notice that the state estimation error sequence $e(t)$ satisfies the following recurrent equation

$$e(t+1) = N_{\hat{\rho}(t)} e(t) + T_{\hat{\rho}(t)} \bar{E} v(t) + F_{\hat{\rho}(t)} w(t) \tag{17}$$

with

$$e(t) := x(t) - \hat{x}(t), N_\rho := (T_\rho \bar{A} - L_\rho \bar{C}_\rho), F_\rho := \begin{bmatrix} T_\rho \bar{F} & I \end{bmatrix}, \quad w(t) := \begin{bmatrix} \Delta b(t) \\ L_\rho (C_\gamma - C_{\hat{\gamma}}) x(t) \end{bmatrix}$$

More formally we are interested to find a parameter-dependent gain $L_{\hat{\rho}(t)}$ such that the recurrent equation (17) is stable for any arbitrary time variation of the parameters $\hat{\rho}(t) \in \mathcal{S}_l$ and such that, for any input $w(t) \in \ell_2$, the error $e(t)$ is bounded

$$||e(\cdot)||_{l_2} < \sigma ||w(\cdot)||_{l_2} \tag{18}$$

A convex optimization methodology to solve the above stated design problem is provided in the next Theorem 1.

Theorem 1: *Assume symmetric positive definite matrices $P_i = P_i' > 0$ and matrices G_i and Y_i, $i = 1, ..., l$ exist such that the optimization problem*

$$\min_{P_i, G_i, Y_i, \mu} \mu$$

subject to

$$\begin{aligned} &\Xi_{ij} := \begin{bmatrix} G_i + G_i' - P_j & Q_{12} & G_i F_i \\ \star & P_i - I & 0 \\ \star & \star & \mu I \end{bmatrix} > 0, \\ &Q_{12} := G_i \tilde{T}_i \bar{A} - Y_i \tilde{C}_i,\ i = 1, ..., l,\ j = 1, ..., l \end{aligned} \tag{19}$$

$$\begin{gathered} \Xi_{ijk} := \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ * & P_i + P_k - I & 0 \\ * & * & \mu I \end{bmatrix} > 0 \\ i = 1, ..., l-1, j = 1, ..., l, k = i+1, ..., l \\ R_{11} := G_i + G_i' + G_k + G_k' + P_j \\ R_{12} := G_i \tilde{T}_k \bar{A} + G_k \tilde{T}_i \bar{A} - Y_i \tilde{C}_k - Y_k \tilde{C}_i, R_{13} := G_i F_k + G_k F_i \end{gathered} \tag{20}$$

has a solution. Then, the convergence of the observer estimation error dynamically characterized by equation (17) is ensured and a guaranteed $\mathcal{H}_\infty$ performance gain (18) *is achieved with*

$$\sigma = \sqrt{\mu^\star},\ \mu^\star = \min \mu \tag{21}$$

Moreover, the observer gain vertices defined in (16) *are given by*

$$L_i = G_i^{-1} Y_i \tag{22}$$

and stabilize the observer for any arbitrary time variation of the parameter $\hat{\rho}(t)$ in the polytope $\mathcal{S}_l$.

Proof: Consider the parameter-dependent Lyapunov function

$$V\big(e(t)\big) = e'(t) P_{\hat{\rho}(t)} e(t) \tag{23}$$

with

$$P_{\hat{\rho}(t)} = \sum_{i=1}^{l} \hat{\rho}_i(t) P_i, P_i = P_i',\ i = 1, ..., l \tag{24}$$

The related one-step ahead evolution of the Lyapunov function on the observer error trajectory is given by

$$V\big(e(t+1)\big) = e'(t+1) P_{\hat{\rho}(t+1)} e(t+1) \tag{25}$$

where $P_{\hat{\rho}(t+1)}$ can be written as

$$P_{\varrho(t)} = \sum_{j=1}^{l} \varrho_j(t) P_j\,, P_j = P_j',\ j = 1, ..., l \tag{26}$$

Using (26), one can recast (25) into

$$V(e(t+1)) = \Big(N_{\hat{\rho}(t)} e(t) + F_{\hat{\rho}(t)} w(t)\Big)' P_{\varrho(t)} \Big(N_{\hat{\rho}(t)} e(t) + F_{\hat{\rho}(t)} w(t)\Big) \tag{27}$$

Then, the Lyapunov function increments derived by (23) and (27) result to be given by

$$\begin{aligned}\Delta V(e(t)) &= V(e(t+1)) - V(e(t)) \qquad (28)\\ &= e'(t)\Big(N'_{\hat{\rho}(t)}P_{\varrho(t)}N_{\hat{\rho}(t)} - P_{\hat{\rho}(t)}\Big)e(t) + 2e'(t)N'_{\hat{\rho}(t)}P_{\varrho(t)}F_{\hat{\rho}(t)}w(t) + w'(t)F'_{\hat{\rho}(t)}P_{\varrho(t)}F_{\hat{\rho}(t)}w(t)\end{aligned}$$

It is well-known that the stability of system with $\mathcal{H}_\infty$ guaranteed performance (18) is ensured if

$$\Delta V(e(t)) < -e'(t)e(t) + \mu w'(t)w(t), \ \ \forall t \in \mathbb{N} \qquad (29)$$

By replacing $\Delta V(e(t))$ with the expression (28), one is able to rewrite inequality (29) as $\Gamma'(t)U\Gamma(t) < 0$ with $\Gamma(t) := \begin{bmatrix} e'(t) & w'(t)\end{bmatrix}'$ and

$$U := \begin{bmatrix} U_{11} & U_{12} \\ \star & U_{22}\end{bmatrix}, \qquad U_{11} := N'_{\hat{\rho}(t)}P_{\varrho(t)}N_{\hat{\rho}(t)} - P(\rho(t)) + I$$
$$U_{12} := N'_{\hat{\rho}(t)}P_{\varrho(t)}F_{\hat{\rho}(t)}, U_{22} := F'_{\hat{\rho}(t)}P_{\varrho(t)}F_{\hat{\rho}(t)} - \mu I$$

Clearly, by imposing $U < 0$ it is possible to guarantee (29) for all $e(t) \neq 0$ and $w(t) \neq 0$. The latter inequality, thanks to the use of a Schur's complement argument, is equivalent to

$$\begin{bmatrix} P_{\varrho(t)} & P_{\varrho(t)}N_{\hat{\rho}(t)} & P_{\varrho(t)}F_{\hat{\rho}(t)} \\ \star & P_{\hat{\rho}(t)} - I & 0 \\ \star & \star & \mu I \end{bmatrix} > 0 \qquad (30)$$

that can be recast into

$$MUM' > 0 \text{ with } M = \begin{bmatrix} G_{\hat{\rho}(t)}P^{-1}_{\varrho(t)} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \qquad (31)$$

and, in turn, into

$$\begin{bmatrix} G_{\hat{\rho}(t)}P^{-1}_{\varrho(t)}G'_{\hat{\rho}(t)} & G_{\hat{\rho}(t)}N_{\hat{\rho}(t)} & G_{\hat{\rho}(t)}F_{\hat{\rho}(t)} \\ \star & P_{\hat{\rho}(t)} - I & 0 \\ \star & \star & \mu I \end{bmatrix} > 0 \qquad (32)$$

Using previously defined matrices and considering that $\varrho \in \mathcal{S}_l$ and $\hat{\rho} \in \mathcal{S}_l$, inequality (32) can be written as

$$\sum_{i=1}^{l}\hat{\rho}_i^2(t)\sum_{j=1}^{l}\varrho_j(t)\Xi_{ij} + \sum_{i=1}^{l-1}\sum_{k=i+1}^{l}\hat{\rho}_i(t)\hat{\rho}_k(t)\sum_{j=1}^{l}\varrho_j(t)\Xi_{ijk} > 0 \qquad (33)$$

with Ξ_{ij} defined in (19) and Ξ_{ijk} defined in (20). □

Remark 1: It is worth remarking that it is not hard to get polytopic representations in the form (15) and a number of procedures exist in the literature dealing with the above task (see for instance Tanaka and Wang (2004); Tóth (2010))

Remark 2: Please notice that if $\tilde{C}_i$ and $\tilde{T}_i$ are chosen as $\tilde{C}_i = \bar{C}_{\gamma_i}$, $\tilde{T}_i = I_{n+q} - (\bar{E}(\tilde{C}_i\bar{E})^{\dagger})\tilde{C}_i$ for fixed $\gamma_i \in \Gamma, i = 1, ..., N$, being by assumption $\tilde{T}_{\gamma}\bar{E} = 0, \forall\gamma \in \Gamma$, then Assumption 4 is directly fulfilled by the polytopic representation (15) thanks to convexity arguments.

Remark 3: It is also worth pointing out that in the nominal case where $\hat{\rho}(t) = \rho(t), \forall t \in \mathbb{N}$, the LPV gain (46) guarantees asymptotic convergence to zero of the estimation error. In the more general case $\hat{\rho}(t) \neq \rho(t)$, only a bounded steady-state estimation errors can be achieved.

3.2.2 LFT formulation

The approach presented in the previous section can lead to design procedures characterized by a huge number of LMIs. In order to get a less computation demanding observer design, in this section we propose an LFT based UIO formulation. To this end we assume to be provided with a LFT representations of $T_{\hat{\gamma}} = LFT(T, \theta_T)$ and $\bar{C}_{\hat{\gamma}} = LFT(C, \theta_C)$ for certain matrices T and C respectively and

$$\begin{aligned}\theta_T(\gamma) &:= \mathrm{diag}\{\theta_{T,1}(\gamma), ..., \theta_{T,n}(\gamma)\}, |\theta_{T,i}| \leq 1, i = 1, ..., n\\ \theta_C(\gamma) &:= \mathrm{diag}\{\theta_{C,1}(\gamma), ..., \theta_{C,m}(\gamma)\}, |\theta_{C,i}| \leq 1, i = 1, ..., m\end{aligned}$$

Such representations can be exploited to get (13) in LFT form

$$\begin{aligned}x(t+1) &= T_{11}\bar{A}x(t) + T_{11}\bar{B}u(t) + T_{11}\bar{F}\Delta b(t) + T_{12}p(t) + Q_{\hat{\gamma}}y(t+1)\\ y(t) &= C_{11}x(t) + C_{12}p(t)\\ q(t) &= C_{qx}x(t) + D_{qu}u(t) + D_{qb}\Delta b(t) + D_{qp}p(t)\\ p(t) &= \Theta(\hat{\gamma})q(t)\end{aligned} \tag{34}$$

where

$$C_{qx} := \begin{bmatrix} T_{21}\bar{A} \\ C_{21} \end{bmatrix}, D_{qu} := \begin{bmatrix} T_{21}\bar{B} \\ 0 \end{bmatrix}, D_{qb} := \begin{bmatrix} T_{21}\bar{F} \\ 0 \end{bmatrix}, \quad D_{qp} := \begin{bmatrix} C_{22} & 0 \\ 0 & T_{22} \end{bmatrix}$$

with $\Theta(\gamma)$ being an uncertain parameter obeying to the following structure

$$\Theta(\gamma) := \begin{bmatrix} \theta_T(\gamma) & 0 \\ 0 & \theta_C(\gamma) \end{bmatrix} \tag{35}$$

Then, a possible structure for an unknown input observer for the model (34) is given by

$$\begin{aligned}\hat{x}(t+1) &= T_{11}\bar{A}\hat{x}(t) + T_{11}\bar{B}u(t) + L\big(y(t) - \hat{y}(t)\big) + T_{12}\hat{p}(t) + Q_{\hat{\gamma}(t)}y(t+1)\\ \hat{y}(t) &= C_{11}\hat{x}(t) + C_{12}\hat{p}(t)\\ \hat{q}(t) &= C_{qx}\hat{x}(t) + D_{qu}u(t) + D_{qp}\hat{p}(t)\\ \hat{p}(t) &= \Theta(\hat{\gamma})\hat{q}(t)\end{aligned} \tag{36}$$

As a consequence, the one-step ahead evolution of the state estimation error

$$e(t) := x(t) - \hat{x}(t), \quad \tilde{p}(t) := p(t) - \hat{p}(t), \quad \tilde{q}(t) := q(t) - \hat{q}(t)$$

would take the following form

$$
\begin{aligned}
e(t+1) &= Ne(t) + N_e\tilde{p}(t) + F_e w(t) && (37)\\
\tilde{q}(t) &= C_{qx}e(t) + F_w w(t) + D_{qp}\tilde{p}(t) && (38)\\
\tilde{p}(t) &= \Theta(\hat{\gamma})\tilde{q}(t) && (39)
\end{aligned}
$$

where

$$N := T_{11}\bar{A} - LC_{11}, N_e := \begin{bmatrix} T_{12} & -LC_{12}\end{bmatrix}, F_e := \begin{bmatrix} T_{11}\bar{F} & I\end{bmatrix},$$

$$F_w := \begin{bmatrix} T_{21}\bar{F} & 0 \\ 0 & 0 \end{bmatrix}, w(t) := \begin{bmatrix} \Delta b(t) \\ L(C_\gamma - C_{\hat{\gamma}})x(t) \end{bmatrix}$$

In this case we want to determine a gain L such that difference equation (37) is stable for any arbitrary time variation of the variables $\tilde{p}(t)$ and $\tilde{q}(t)$ and for any input $w(t) \in \ell_2$. As a consequence, the error $e(t)$ is bounded as

$$\left\|e(\cdot)\right\|_{l_2} < \sigma \left\|w(\cdot)\right\|_{l_2} \tag{40}$$

A convex optimization methodology to solve the above stated design problem is provided in the next Theorem 2.

Theorem 2: *Assume that a symmetric positive matrix Q, a matrix S and positive scalars μ and λ exist such that the following optimization problem has a solution*

$$\min_{Q,S,\mu,\lambda} \mu \tag{41}$$

subject to:

$$\begin{bmatrix} Q & QT_{11}\bar{A} - SC_{11} & [QT_{12} \quad -SC_{12}] & QF_e \\ \star & Q - I - \lambda C_{qx}' C_{qx} & -\lambda C_{qx}' D_{qp} & -\lambda C_{qx}' F_w \\ \star & \star & \lambda I - \lambda D_{qp}' D_{qp} & -\lambda D_{qp}' F_w \\ \star & \star & \star & \mu I - \lambda F_w' F_w \end{bmatrix} > 0 \tag{42}$$

Then, the boundedness of the observer estimation error as in (40) is ensured with guaranteed $\mathcal{H}_\infty$ performance gain

$$\sigma = \sqrt{\mu^\star},\ \mu^\star = \min \mu \tag{43}$$

by choosing $L = Q^{-1}S$.

Proof: Consider the Lyapunov function

$$V\big(e(t)\big) = e'(t)Qe(t) \tag{44}$$

The related one-step ahead evolution of the above function on the observer error trajectory is given by

$$V\big(e(t+1)\big) = e'(t+1)Qe(t+1) \tag{45}$$

Using (37), one can recast (45) into

$$V\big(e(t+1)\big) = \big(Ne(t) + N_e\tilde{p}(t) + F_e w(t)\big)'Q\big(Ne(t) + N_e\tilde{p}(t) + F_e w(t)\big)$$

Then, the Lyapunov function increments derived by (44) and (46) result to be given by

$$\begin{array}{rcl} \Delta V(t) & = & V\big(e(t+1)\big) - V\big(e(t)\big) \\ & = & e'(t)\big(N'QN - Q\big)e(t) + 2\tilde{e}'(t)N'N_e\tilde{p}(t) \\ & & +2\tilde{p}'(t)N_e'F_e w(t) + 2\tilde{e}'(t)N'F_e w(t) + \tilde{p}'(t)N_e'QN_e\tilde{p}(t) + \tilde{w}'(t)F_e'QF_e\tilde{w}(t) \end{array} \tag{46}$$

It is well-known that the stability of system (37)-(39) with the $\mathcal{H}_\infty$ guaranteed performance (40) is ensured if

$$\Delta V(t) \le -e'(t)e(t) + \mu w'(t)w(t) \tag{47}$$

for each $\tilde{q}(t)$, $\tilde{p}(t)$ satisfying

$$\big\|\tilde{p}(t)\big\|_2^2 \le \big\|\tilde{q}(t)\big\|_2^2 \tag{48}$$

By replacing $\Delta V(e(t))$ with the expression (46), one is able to rewrite inequality (47) as

$$\begin{bmatrix} e(t) \\ \tilde{p}(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} N'QN - Q + I & N'QN_e & N'QF_e \\ N_e'QN & N_e'QN_e & N_e'QF_e \\ F_e'QN & F_e'QN_e & F_e'QF_e - \mu I \end{bmatrix} \begin{bmatrix} e(t) \\ \tilde{p}(t) \\ w(t) \end{bmatrix} < 0$$

while inequality (48) can be recast as

$$\begin{bmatrix} e(t) \\ \tilde{p}(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} -C_{qx}'C_{qx} & -C_{qx}'D_{qp} & -C_{qx}'F_w \\ -D_{qp}'C_{qx} & I - D_{qp}'D_{qp} & -D_{qp}'F_w \\ -F_w'C_{qx} & -F_w'D_{qp} & -F_w'F_w \end{bmatrix} \begin{bmatrix} e(t) \\ \tilde{p}(t) \\ w(t) \end{bmatrix} < 0$$

by substituting $q(t)$ in (48) with its expression in (37). As a consequence, by means of the S-procedure, we can state that the above inequalities are true if and only if there exists a scalar λ such that

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ \star & U_{22} & U_{23} \\ \star & \star & U_{33} \end{bmatrix} < 0 \tag{49}$$

$$\begin{array}{rclrcl} U_{11} & := & N'QN - Q + I + \lambda C_{qx}'C_{qx}, & U_{12} & := & N'QN_e + \lambda C_{qx}'D_{qp} \\ U_{13} & := & N'QF_e + \lambda C_{qx}'F_w, & U_{22} & := & N_e'QN_e - \lambda I + D_{qp}'D_{qp} \\ U_{23} & := & N_e'QF_e + \lambda D_{qp}'F_w, & U_{33} & := & F_e'QF_e - \mu I + \lambda F_w'F_w \end{array}$$

Notice that, by using Schur's complement lemma, (49) is equivalent to

$$\begin{bmatrix} Q & QN & QN_e & QF_e \\ \star & Q-I-\lambda C_{qx}'C_{qx} & -\lambda C_{qx}'D_{qp} & -\lambda C_{qx}'F_w \\ \star & \star & \lambda I-\lambda D_{qp}'D_{qp} & -\lambda D_{qp}'F_w \\ \star & \star & \star & \mu I-\lambda F_w'F_w \end{bmatrix} > 0$$

Finally, by taking the change of variable $QL = S$ into account, the inequality (42) results. □

3.3 *Parameter Estimator*

In this section the Parameter Estimator unit of Figure 2 is described. Its task consists in estimating the gain faults on the matrix $\Delta(\hat{\gamma})$ via a constrained batch least-mean-squares approach (Liew (1976)) used within a windowing data processing strategy. Such an approach is based on an algorithm that is able to detect constant or slowly-varying gain faults in systems of the form (2). The basic idea relies on finding a matrix $\Delta(\hat{\gamma}(t))$ that matches as much as possible the plant measured signals and the estimated state in the last N time instants, being N an arbitrarily chosen positive integer. In this respect the last N samples of both the *physical outputs* $y(t)$ and state estimation $\hat{x}(t)$ of the augmented system (8) are assumed to be provided at the generic time instant t. In this way, by considering $\hat{x}(t) = x(t)$ (*certainty equivalence hypothesis*), the following consistency equations can be imposed to the matrix $\Delta(\hat{\gamma}(t))$

$$y(t-i) = \Delta(\hat{\gamma}(t))C_y\hat{x}_p(t-i) + F\hat{b}(t-i),\ i = 0, ..., N-1 \tag{50}$$

that are equivalent to

$$y(t-i) - F\hat{b}(t-i) = X(t-i)\hat{\gamma}(t),\ i = 0, ..., N-1 \tag{51}$$

where

$$X(t-i) := \mathrm{diag}\left\{C_y^{(1)}\hat{x}_p(t-i), ..., C_y^{(m)}\hat{x}_p(t-i)\right\}$$

This allows one to recast the problem in the classical regressor form:

$$Y(t) = \varphi(t)\gamma(t) \tag{52}$$

where

$$Y(t) := \begin{bmatrix} y(t) - F\hat{b}(t) \\ \vdots \\ y(t-N+1) - F\hat{b}(t-N+1) \end{bmatrix}$$

are the measures and

$$\varphi(t) := \left[\ X(t), \ldots, X(t-N+1)\ \right]'$$

collects the linear regressors. Then, the variable $\hat{\gamma}(t)$ can be estimated through the resolution of the following quadratic program with linear constraints

$$\begin{array}{rc} \hat{\gamma}(t) := \arg\min\limits_{\gamma} & \frac{1}{2}\|(Y(t) - \varphi(t)\gamma)\|_2^2 \\ \text{subject to} & \gamma \in \Gamma \end{array} \tag{53}$$

Under a constant $\gamma(t) = \gamma^\star$, it is possible to prove (Casavola and Garone (2010)) that a sufficient condition to guarantee convergence of $\hat{\gamma}(t)$ to $\gamma^\star$ for some $t^\star \gg N$ is

$$\text{rank}\{\varphi(t^\star)\} = n \tag{54}$$

In particular, if $\bar{C}_y$ has not zero columns, a sufficient condition to ensure (54) is

$$\text{rank}\{\hat{X}_p(t)\} = n \tag{55}$$

where matrix $\hat{X}_p(t)$ is defined as

$$\hat{X}_p(t) := \left[\hat{x}_p(t), \ldots, \hat{x}_p(t-N)\right]' \tag{56}$$

Such a property can be guaranteed if the state estimation problem for $\hat{x}_p(t)$ is solved under a persistent excitation condition on the measurements provided by the physical sensors or by a suitable artificial dither injected in the state estimation $\hat{x}_p(t)$ sent to the Parameter Estimator, so as to force that signal to be persistently excited and make condition (55) hold true.

3.4 *Reconciliation Algorithms*

Finally, the proposed sensor reconciliation methods can be summarized in the following algorithms. The first one is related to the LPV-UIO formulation

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Algorithm 1: LPV-UIO based Sensor Reconciliator (LPV-UIO-SR)

INITIALIZATION:

1: **compute** L_i, $i = 1, ..., l$ according to Theorem 1;
2: **chose** horizon N for the Parameter Estimator;
3: **set** $\Delta(\hat{\gamma}(t)) = I_m$ and $\hat{b}(t) = 0$ for $t = 0, ..., N-1$;
4: **store** L_i, $i = 1, ..., l$, N, $\Delta(\hat{\gamma}(t))$ and $\hat{b}(t)$, $t = 0, ..., N-1$.

ON-LINE PHASE (generic time $t \geq N$):

1: **receive** $y(t)$ from the sensors;
2: **compute** $\hat{\rho}(\hat{\gamma}(t-1))$ on the basis of the polytopic representation (15)
3: **compute** $Q_{\hat{\gamma}(t-1)}$ as in (14);
4: **set** $T_{\hat{\gamma}(t-1)} := I_{n+q} - Q_{\hat{\gamma}(t-1)}\bar{C}_{\hat{\gamma}(t-1)}$;
5: **estimate** plant state and bias by evaluating

$$\hat{x}(t) = T_{\hat{\gamma}(t-1)}\bar{A}\hat{x}(t-1) + T_{\hat{\gamma}(t-1)}\bar{B}u(t-1) + L_{\hat{\rho}(\hat{\gamma}(t-1))}\Big(y(t-1) - \hat{y}(t-1)\Big) + Q_{\hat{\gamma}(t)}y(t)$$

6: **estimate** $\hat{\gamma}(t)$ by solving (53)
7: **compute** the estimated *real output* as $\hat{y}(t) = \bar{C}_{\hat{\gamma}(t)}\hat{x}(t)$
8: **return** the *virtual output* $\hat{z}(t) = H_z\hat{y}(t)$
9: **set** $t := t+1$
10: **go to** step 1

The second algorithm is related to the LFT-UIO based approach

Algorithm 2: LFT-UIO based Sensor Reconciliator (LFT-UIO-SR)

INITIALIZATION:
1: **compute** L, according to Theorem 2
2: **chose** horizon N for the Parameter Estimator;
3: **set** $\Delta(\hat{\gamma}(t)) = I_m$ and $\hat{b}(t) = 0$ for $t = 0, ..., N-1$;
4: **store** L, N, $\Delta(\hat{\gamma}(t))$ and $\hat{b}(t)$, $t = 0, ..., N-1$.

ON-LINE PHASE (generic time $t \geq N$):
1: **receive** $y(t)$ from the sensors;
2: **compute** $Q_{\hat{\gamma}(t-1)}$ as in (14);
3: **set** $T_{\hat{\gamma}(t-1)} := I_{n+q} - Q_{\hat{\gamma}(t-1)}\bar{C}_{\hat{\gamma}(t-1)}$;
4: **estimate** plant state and bias by evaluating

$$\hat{x}(t) = T_{\hat{\gamma}(t-1)}\bar{A}\hat{x}(t-1) + T_{\hat{\gamma}(t-1)}\bar{B}u(t-1) + L(y(t-1) - \hat{y}(t-1)) + Q_{\hat{\gamma}(t)}y(t)$$

5: **estimate** $\hat{\gamma}(t)$ by solving (53)
6: **compute** the estimated *real output* as $\hat{y}(t) = \bar{C}_{\hat{\gamma}(t)}\hat{x}(t)$
7: **return** the *virtual output* $\hat{z}(t) = H_z\hat{y}(t)$
8: **set** $t := t + 1$
9: **go to** step 1

4. Illustrative Example

In this section, the effectiveness of the proposed **LFT-UIO-SR** scheme is investigated by considering the three-tank benchmark model of Ding (2008) depicted in Figure 4 whose dynamics is modeled by the following differential equations

$$\begin{aligned} A\dot{h}_1 &= Q_1 - a_1 S_{13} Sgn(h_1 - h_3)\sqrt{2g\,|h_1 - h_3|} \\ A\dot{h}_2 &= Q_2 + a_3 S_{23} Sgn(h_3 - h_2)\sqrt{2g\,|h_3 - h_2|} + a_2 s_0\sqrt{2gh_2} \\ A\dot{h}_3 &= a_1 S_{13} Sgn(h_1 - h_3)\sqrt{2g\,|h_1 - h_3|} - a_3 S_{23} Sgn(h_3 - h_2)\sqrt{2g\,|h_3 - h_2|} \end{aligned} \tag{57}$$

where parameters $A = 0.0154m^2$ and $S_{ij} = 5 \times 10^{-5}m^2$ are the cross-section areas of the tanks and the cross-section areas of the pipes respectively.

Notice that in the above formulation Q_1 and Q_2 are the incoming mass flows, while $h_i(t), \quad i = 1, 2, 3$ are the measured water levels in each tank. After linearization at the operating point $h_1 = 45cm$, $h_2 = 15cm$ and $h_3 = 30cm$, $Q_1 = Q_2 = 0,35 \cdot 10^{-4}m^3/s$ and discretization of the continuous time model, the following discrete-time linear (nominal) model is achieved

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Ev(t) \\ y(t) &= \Delta_\gamma Cx + Fb(t) \end{aligned} \tag{58}$$

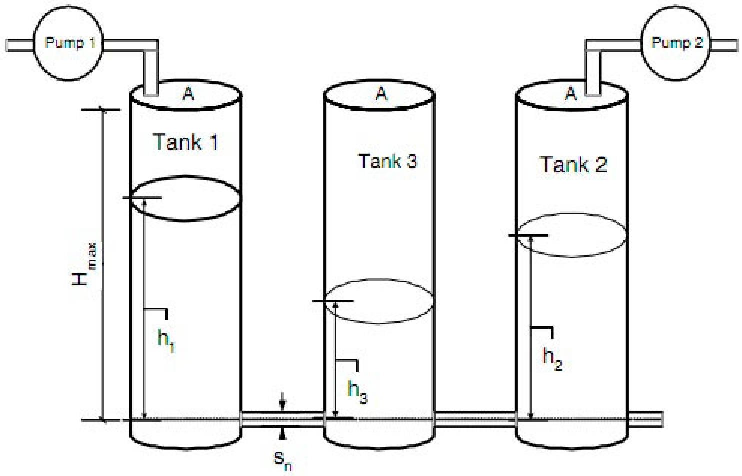

Figure 4.: Benchmark: three-tank system of Ding (2008)

with matrices

$$A = \begin{bmatrix} 0.9916 & 0 & 0.008393 \\ 0 & 0.9807 & 0.008249 \\ 0.008393 & 0.008249 & 0.9833 \end{bmatrix}, \quad B = \begin{bmatrix} 0.006473 & 7.649e-008 \\ 7.649e-008 & 0.006437 \\ 2.739e-005 & 2.697e-005 \end{bmatrix}$$

$$C = \begin{bmatrix} 10 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{59}$$

with γ supposed to be confined within the polytope $\Gamma := \left\{\gamma : [\underline{\gamma}_1, \underline{\gamma}_2, \underline{\gamma}_3, \underline{\gamma}_4]' \leq \gamma \leq [\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3, \overline{\gamma}_4]'\right\}, \underline{\gamma}_i = 0.01, \overline{\gamma}_i = 1,\ i = 1, 2, 3, 4.$

The goal of this simulation is to verify the capability of the proposed method of extracting the first component of the state $x_p(t)$ into the *virtual output* $z(t) = H_z C_y x_p(t)$ with the sensor reconciliation matrix given by $H_z = \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \end{bmatrix}$. Along the simulation, the known input $u(t)$ and the unknown input $v(t)$ are supposed to be those depicted in Figure 5.

Moreover, we assume that the bias profiles of the three available physical sensors change along the simulation according to the profile depicted in Figure 6 and that faults on the matrix effectiveness gain will affect the first two sensors of C_y as depicted in Figure 7. In this scenario, without any sensor reconciliator block, the *virtual output* would result to be falsified as depicted in Figure 9 (green dashed line).

In order to exploit the **LPV-UIO-SR** described in Section 3.2.1, the plant has to be recast in the augmented form (8) by following the procedure described in Behzad et al. (2016).

In the case of the **LFT-UIO-SR** scheme presented in Section 3.2.2, the plant has to be recast in the augmented LFT form (34)-(35). In this respect, please notice that

$$\bar{C}_\gamma \bar{E} = \Delta(\gamma) G, \ G := \begin{bmatrix} 10 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

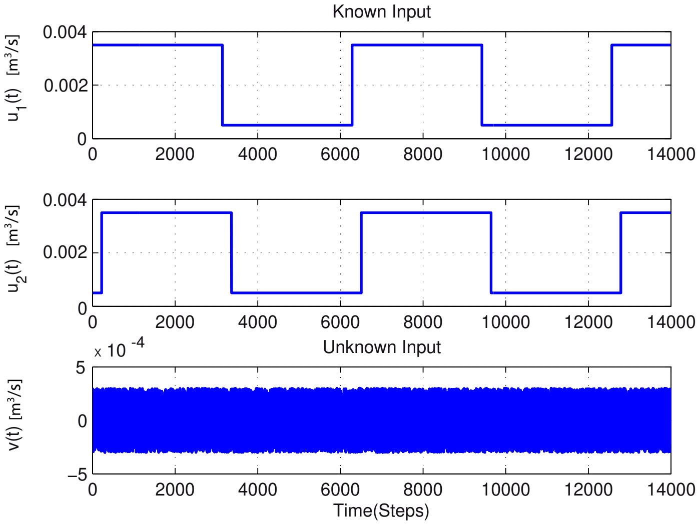

Figure 5.: Known Input(up) and Unknown Input(down)

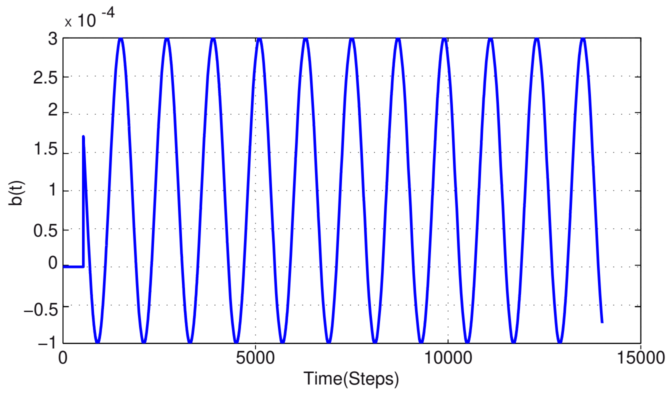

Figure 6.: Bias fault profile

and

$$Q_\gamma = \bar{E}(\bar{C}_\gamma \bar{E})' \left((\bar{C}_\gamma \bar{E})(\bar{C}_\gamma \bar{E})'\right)^{-1} = \bar{E}G'(GG')^{-1}\Delta^{-1}(\gamma)$$

where

$$\bar{C}_\gamma := \begin{bmatrix} 10\gamma_1 & 0 & 0 & 1 \\ \gamma_2 & 0 & 0 & 1 \\ 0 & \gamma_3 & 0 & 1 \\ \gamma_4 & \gamma_4 & \gamma_4 & 1 \end{bmatrix} \tag{60}$$

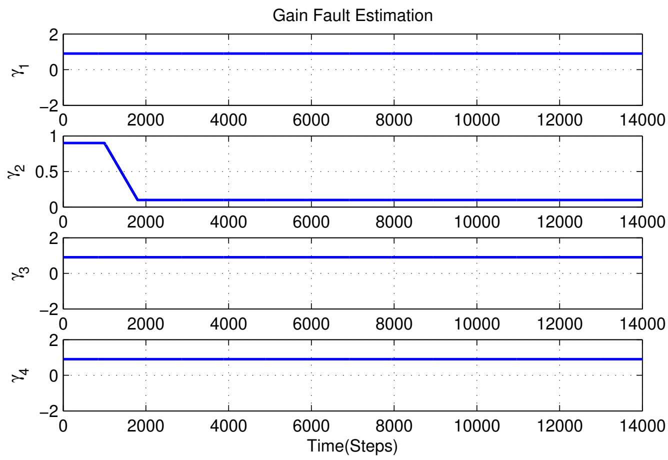

Figure 7.: Fault loss of effectiveness profiles

As a consequence, the matrix $T_{\hat{\gamma}}$ can be rewritten as

$$\begin{aligned}
T_{\hat{\gamma}} &= I_{n+q} - Q_{\hat{\gamma}}\bar{C}_{\hat{\gamma}} \\
&= I_{n+q} - \bar{E}G'(GG')^{-1}\Delta^{-1}(\hat{\gamma})\bar{C}_{\hat{\gamma}} \\
&= I_{n+q} - \bar{E}G'(GG')^{-1}\Delta^{-1}(\hat{\gamma})\begin{bmatrix}\Delta(\gamma)C_y & F\end{bmatrix} \\
&= I_{n+q} - \bar{E}G'(GG')^{-1}\Delta^{-1}(\hat{\gamma})\left(\Delta(\gamma)\begin{bmatrix}C_y & 0_4\end{bmatrix} + \begin{bmatrix}0_{4\times 3} & F\end{bmatrix}\right) \\
&= I_{n+q} - \bar{E}G'(GG')^{-1}\begin{bmatrix}C_y & 0_4\end{bmatrix} - \bar{E}G'(GG')^{-1}\Delta^{-1}(\hat{\gamma})\begin{bmatrix}0_{4\times 3} & F\end{bmatrix} \qquad (61)
\end{aligned}$$

A standard normalization for $\Delta^{-1}(\hat{\gamma})$ is required. It can be achieved e.g. by following the approach described in Cockburn (1998). As a result, one gets

$$\gamma_i^{-1} = a_i + b_i\delta_i, \quad |\delta_i| \leq 1$$

with

$$a_1 = \frac{1}{2}(\overline{\gamma} + \underline{\gamma}) = 101.1, \quad b_1 = \frac{1}{2}(\overline{\gamma} - \underline{\gamma}) = 98.99$$

Then, a LFT representation for $T_{\hat{\gamma}}$ can be obtained as

$$\begin{aligned}
T_{\hat{\gamma}} &= I_{n+q} - \bar{E}G'(GG')^{-1}\begin{bmatrix}C_y & 0_4\end{bmatrix} - \bar{E}G'(GG')^{-1} \\
&\qquad \left(\mathrm{diag}\{a_1, a_2, a_3, a_4\} + \mathrm{diag}\{b_1\delta_1, b_2\delta_2, b_3\delta_3, b_4\delta_4\}\begin{bmatrix}0_{4\times 3} & F\end{bmatrix}\right) \\
&= I_{n+q} - \bar{E}G'(GG')^{-1}\begin{bmatrix}C_y & 0_4\end{bmatrix} - \bar{E}G'(GG')^{-1}\mathrm{diag}\{a_1, a_2, a_3, a_4\}\begin{bmatrix}0_{4\times 3} & F\end{bmatrix} \\
&\qquad -\bar{E}G'(GG')^{-1}\mathrm{diag}\{b_1\delta_1, b_2\delta_2, b_3\delta_3, b_4\delta_4\}\begin{bmatrix}0_{4\times 3} & F\end{bmatrix} \\
&= LFT(T, \theta_T(\delta))
\end{aligned}$$

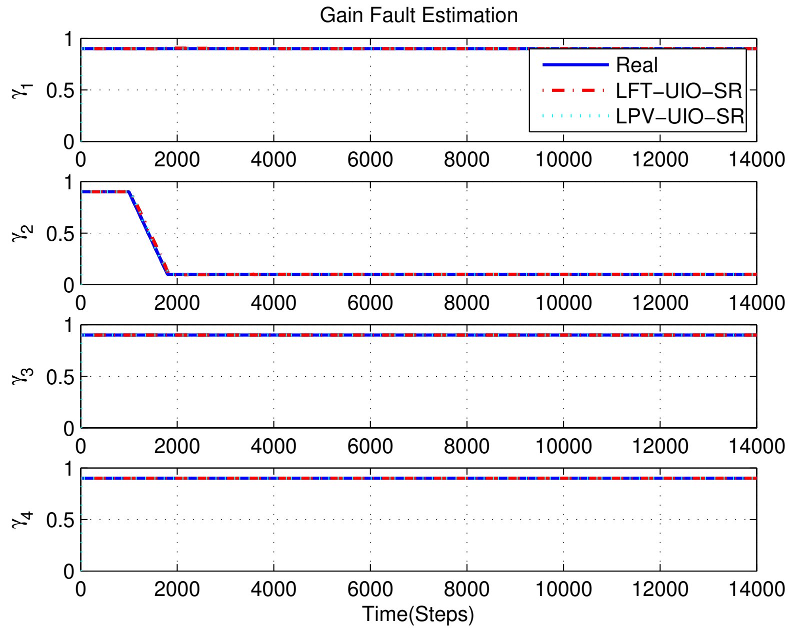

Figure 8.: Effectiveness Matrix Estimation

where

$$\begin{aligned} T_{11} &:= I_{n+q} - \bar{E}G'(GG')^{-1}\begin{bmatrix} C_y & 0_4 \end{bmatrix}, \\ T_{12} &:= -\bar{E}G'(GG')^{-1}, \\ T_{21} &:= \begin{bmatrix} 0_{4\times 3} & F \end{bmatrix},\ T_{22} := \begin{bmatrix} 0_{4\times 4} \end{bmatrix} \\ \theta_T(\delta) &:= (\mathrm{diag}\{a_1, a_2, a_3, a_4\} + \mathrm{diag}\{b_1\delta_1, b_2\delta_2, b_3\delta_3, b_4\delta_4\}) \end{aligned}$$

For the matrix $\bar{C}_\gamma$, the LFT representation is achieved in a simpler manner and it is given by

$$\bar{C}_\gamma = \begin{bmatrix} 0_{4\times 3} & F \end{bmatrix} + \Delta(\gamma)\begin{bmatrix} C_y & 0_4 \end{bmatrix} = LFT(C, \theta_C(\gamma))$$

where

$$C_{11} := \begin{bmatrix} 0_{4\times 3} & F \end{bmatrix}, \quad C_{12} := I_4, \quad C_{21} := \begin{bmatrix} C_y & 0_4 \end{bmatrix}$$

Finally, a windowing horizon $N = 50$ has been chosen for the Parameter Estimator. All simulations have been performed by using the Yalmip interpreter (Lofberg (2005)) and the Sedumi solver; all running under MATLAB 8.6 environment on an Intel Core i5-3330 machine with 3.3 GHz and 8GB RAM.

Simulative comparisons have been depicted in Figures (8)-(9). There the proposed schemes have been compared with the method described in Ding (2008), where the multiplicative fault has been considered as additive fault. A better state estimation (Figures 11 and 12) and a more accurate *virtual output* generation(Figure 9) arise from those Figures when comparing to the *Ding's* approach. Furthermore the **LPV-UIO-SR** scheme exhibits a slight better behavior with respect to **LFT-UIO-SR**, both in estimating the state and the bias. This is mostly due to the fact that the observer gain is time-varying scheduled in the **LPV-UIO-SR** scheme and it is able to "adapt" itself more quickly with respect to changes in the effectiveness matrix. Such an aspect translates in a better effectiveness parameter (gain matrix) estimation (Figure 8) and in a more accurate *virtual*

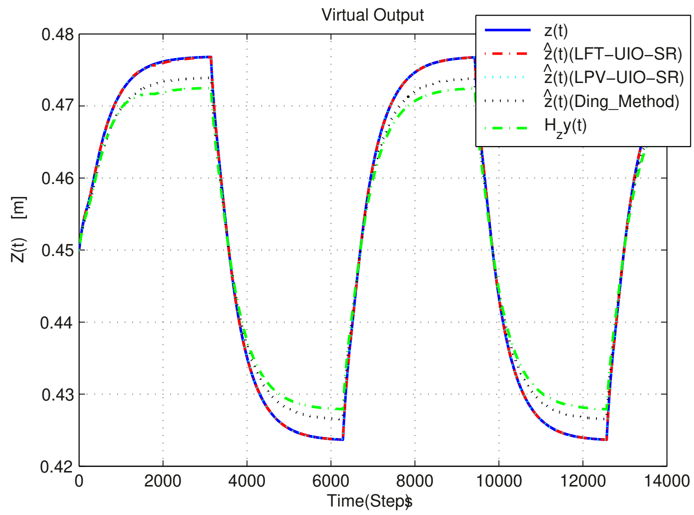

Figure 9.: Virtual Output

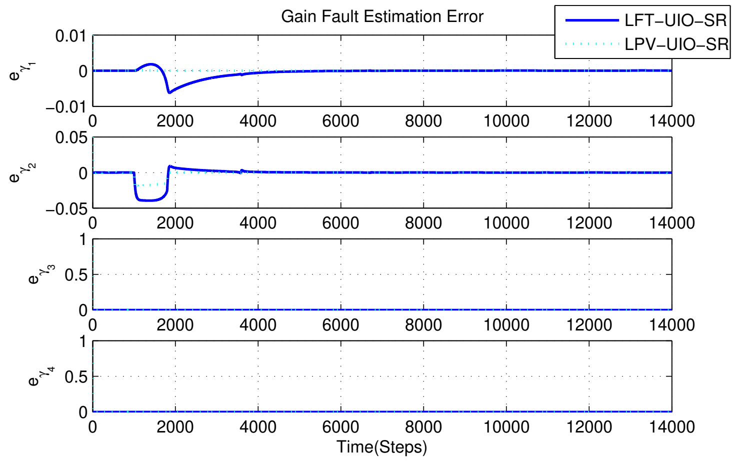

Figure 10.: Effectiveness Matrix Error

output generation (Figure 9). However, it is worth commenting that, although, **LPV-UIO-SR** achieve better performance, it involves a time-expensive ($\approx$ 9 hours) design procedure with respect to **LFT-UIO-SR** ($\approx$ 6 minutes) that can be impracticable in the case of systems with a large number physical sensors to be monitored.

5. Conclusions

In this paper a fault-tolerant sensor reconciliation scheme based on unknown input observers has been presented for linear discrete-time systems subject to possible faults on sensor gain and bias. The role of the observer relies on the estimation of both the state of the system and the current bias of the physical sensors whereas a least-squares batch algorithm provides estimates of the current effectiveness matrix of the physical sensors. For the design procedure of the observer two approaches have been proposed. The first one has been achieved by resorting to LPV paradigm while the second one have exploited the LFT formalism. Both approaches have been compared in the finally simulation example where good performance in recovering useful data from the pool of

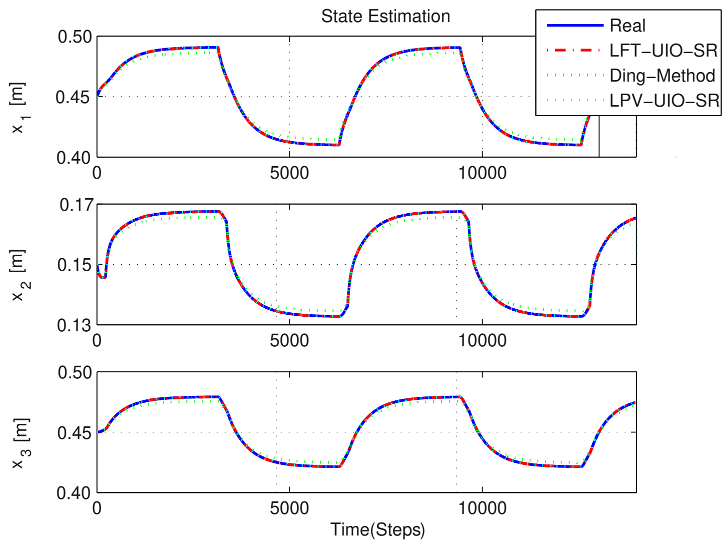

Figure 11.: State Estimation

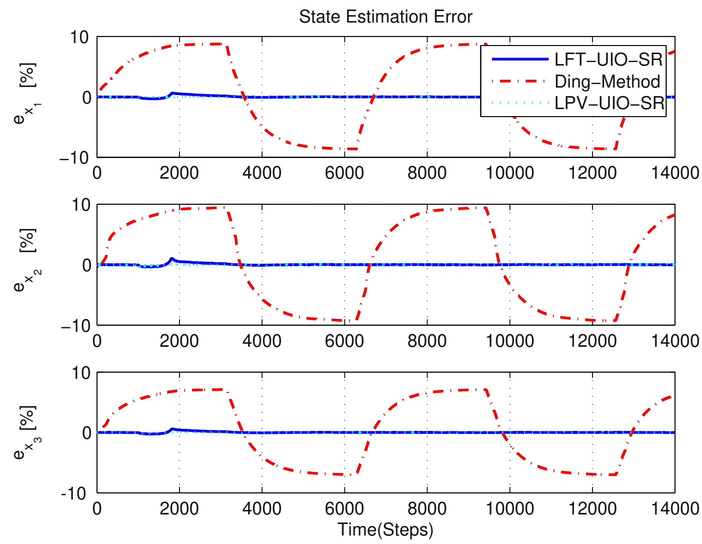

Figure 12.: State Estimation Error

redundant sensors have been observed.

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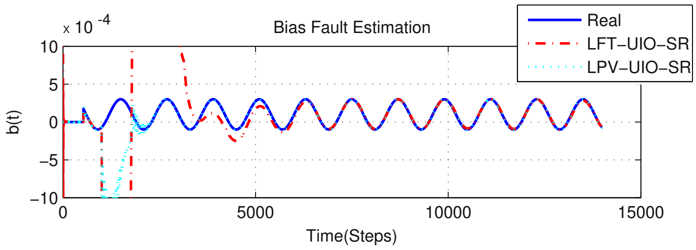

Figure 13.: Bias Estimation

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