Research Paper

OBJECTIVE BAYESIAN ANALYSIS FOR A TWO-PARAMETERS EXPONENTIAL DISTRIBUTION

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ABSTRACT. In any Bayesian inference problem, the posterior distribution is a product of the likelihood and the prior: thus, it is affected by both in cases where one possesses little or no information about the target parameters in advance. In the case of an objective Bayesian analysis, the resulting posterior should be expected to be universally agreed upon by everyone, whereas . subjective Bayesianism would argue that probability corresponds to the degree of personal belief. In this paper, we consider Bayesian estimation of two-parameter exponential distribution using the Bayes approach needs a prior distribution for parameters. However, it is difficult to use the joint prior distributions. Sometimes, by using linear transformation of reliability function of two-parameter exponential distribution in order to get simple linear regression model to estimation of parameters. Here, we propose to make Bayesian inferences for the parameters using non-informative priors, namely the (dependent and independent) Jeffreys' prior and the reference prior. The Bayesian estimation was assessed using the Monte Carlo method. The criteria mean square error was determined evaluate the possible impact of prior specification on estimation. Finally, an application on a real dataset illustrated the developed procedures.

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1. Introduction and Background

The two-parameter exponential distribution plays vital role in survival analysis and has extensive applications in reliability, engineering, queueing theory, and medical sciences. The researchers studied different estimation methods for a two-parameter exponential distribution. The most important method is the Bayesian method. Bayes estimation depends on its applications in assuming that the parameter is a random variable with an ability density function. The main challenge was determining the prior probability distribution of the parameters in the two cases with information and non-information. [1], [11], [12], [8] and [3] have used approximate methods.

In this paper, we will prove that Jeffrey's prior leads to a proper posterior while the reference prior leads to an improper posterior. We also show how to represent the two-parameter exponential distribution in a hierarchical form by augmenting the model with a latent variable, making the Bayesian computations easier to implement. This representation would also allow

the user to implement inferences using all-purpose Bayesian statistical packages, like Win BUGS [9] or [10]. [2] estimate the parameters of two-parameter exponential distribution using the linear transformation (LT) of the reliability function. They mainly explore prior distributions for estimation parameters of a two-parameter exponential distribution. One of these prior distributions is Jeffreys' joint prior distribution. Nevertheless, we show that the prior distribution given by [12] is proper and applicable.

This paper proposes a method for estimating the two-parameter exponential distribution by modeling a simple linear regression model based on the cumulative distribution function. The proposed method estimates the distribution parameters based on the Bays method and then compares the proposed method with the approximation method [11] using the mean squares error.

The main goal of this article is to present Bayesian estimators with appropriate posterior distribution related to the prior distribution of the same, which is shown by the point that the posterior distribution is finite in the parameter space. and rendering estimators compared to estimators provided by [2], [5] and [4]. So, according to the methods of estimating the parameters of the two-parameter exponential distribution presented by many researchers. We have theoretically shown that the posterior distribution is finite with respect to objective Bayes and the estimators will perform better as a result. Recently a large number of researchers, including [13], [5], and [14] used objective Bayes estimation to estimate parameters. In this article, the objective Bayes estimators of the parameters of the two-parameter exponential distribution are compared to the estimators provided by [7], [12], [11], and [2] and show that the MSE of these estimators are less.

The remainder of this paper is organized as follows. In Section 2, we present the two-parameters exponential distribution and list some of its properties, and we deal with the method of maximum likelihood in estimating μ and θ. In section 3.1, we formulated the Bayesian estimation using linear transformation of reliability function. In section 3.2, we provided the Bayesian estimation by using non-informative priors. In section 4.1, a simulation study is presented. The methodology is illustrated on the real dataset in section 4.2.

2. Model Definitions

Here we use the model definition from [11] and [12].

2.1. **Definition.** A continuous random variable X has a two-exponential distribution with parameters μ and θ if its probability density function is given by

$$(2.1) \qquad f(x|\mu,\theta) = \frac{1}{\theta}e^{-\frac{(x-\mu)}{\theta}}, \quad x > \mu,$$

where the parameters $\mu > 0$ and $\theta > 0$ are interpreted as a measure of guarantee and failure rate, respectively. We refer to this distribution as $\text{Exp}(\mu,\theta)$. The median is $(\mu - \theta \ln(2))$. To estimate the parameters, suppose that $X_1, X_2, ..., X_n$ is a random sample of size n from (2.1). The likelihood function can be written as

$$(2.2) \qquad L(x_1, x_2, ..., x_n|\mu,\theta) = L(\underline{x}|\mu,\theta) = \frac{1}{\theta^n}e^{-\frac{1}{\theta}\sum_{i=1}^{n}(x_i-\mu)} = \frac{1}{\theta^n}e^{-\frac{1}{\theta}(n\bar{x}-n\mu)}.$$

Let $l(\mu,\theta) = \log(L(\underline{x}|\mu,\theta))$, then the maximum likelihood estimation of μ and θ are $\hat{\mu} = \min(X_1, X_2, ..., X_n)$ and $\hat{\theta} = \bar{X} - \hat{\mu}$, respectively.

3. **Bayesian Estimation**

3.1. **Bayesian Estimation using a linear transformation of the reliability function.**
[2] transformed the reliability function of the two-parameter exponential distribution to esti-
mate the distribution parameters by taking the natural logarithm to the facility of (2.1). The
solution is as follows.

$$(3.1) \qquad R(x_i) = \exp\left(-\frac{x_i - \mu}{\theta}\right).$$

Taking the natural logarithm for both sides of (2.1), then

$$(3.2) \qquad -\ln R(x_i) = \theta^{-1} x_i - \mu\theta^{-1}.$$

This equation is similar to the following simple linear regression model.

$$(3.3) \qquad y_i = \beta_0 + \beta_1 x_i + e_i, \quad i = 1, 2, ..., n.$$

So, the estimators of the two-parameter exponential distribution are as follows.

$$(3.4) \qquad \hat{\theta} = \hat{\beta}_{1L}^{-1}, \quad \hat{\mu} = -\beta_{0L}\hat{\theta}.$$

In general, model (3.3) will be

$$(3.5) \qquad \underline{Y} = X\underline{\beta} + \underline{U}, \quad \underline{U} \sim N_2(\underline{0}, \sigma^2 I_2),$$

where $\underline{Y}$ and $\underline{U}$ are vector of size $(n \times 1)$, X is a matrix of size $(n \times 2)$, and $\underline{\beta}$ is a vector of
regression parameters of size (2×1). Then, the likelihood function is

$$(3.6) \qquad L(\underline{\beta}, \sigma^2 | \underline{x}) = \frac{e^{-\frac{1}{2\sigma^2}(\underline{Y} - X\underline{\beta})(\underline{Y} - X\underline{\beta})}}{(2\pi)^{\frac{n}{2}}(\sigma^2)^{\frac{n}{2}}}.$$

As $\underline{\beta}$ and σ^2 are unknown, the conditional conjugate prior distribution for $\underline{\beta}$ and σ^2 are given,
respectively of

$$(3.7) \qquad \underline{\beta} | \sigma^2 \sim N_2(\underline{\beta}_0, \sigma^2 V_0),$$

$$(3.8) \qquad \sigma^2 \sim IG\left(\frac{a_0}{2}, \frac{b_0}{2}\right),$$

where V_0 is a symmetric positive defined matrix of size (2×2). From (3.7) and (3.8), the
joint prior distribution for $\underline{\beta}$ and σ^2 is defined as

$$(3.9) \qquad p(\underline{\beta}, \sigma^2) = p(\underline{\beta} | \sigma^2)p(\sigma^2).$$

The joint posterior distribution for $(\underline{\beta}, \sigma^2)$ can be found by equations (3.6) and (3.9) as follows.

$$(3.10) \qquad p(\underline{\beta}, \sigma^2 | data) \propto (\sigma^2)^{-\frac{a_0 + n}{2} + 1} e^{-\frac{s_T^2}{2\sigma^2}} \frac{1}{\sigma^2} e^{-\frac{1}{2\sigma^2}(\hat{\underline{\beta}}_{Ls} - c)' D(\hat{\underline{\beta}}_{Ls} - c)}.$$

It represents the kernel of $IG\left(\frac{a_0 + n}{2}, \frac{s_T^2}{2}\right) N(\underline{c}, \sigma^2 D^{-1})$, where $\underline{c} = \binom{c_1}{c_2} = (I_2 + \nu_0^{-1})^{-1}(\nu_0^{-1}\beta_0 + \hat{\beta})$ and $D^{-1} = I_2 + \nu_0^{-1}$. This can be written as follows.

$$D^{-1} = \begin{bmatrix} d_{11}^* & d_{12}^* \\ d_{21}^* & d_{22}^* \end{bmatrix}, \quad s_T^2 = b_0 + (n-1)s_e^2 + d, \quad s_e^2 = \frac{(\underline{Y} - X\hat{\underline{\beta}}_{Ls})'(\underline{Y} - X\hat{\underline{\beta}}_{Ls})}{n-1},$$

$d = (\hat{\underline{\beta}}_0 - \hat{\underline{\beta}}_{Ls})'((I_2 + \nu_0)^{-1})(\hat{\underline{\beta}}_0 - \hat{\underline{\beta}}_{Ls})$, this is constant.

So, the Bayesian estimation for β_0 and β_1 are defined as the followed value respectively,

$$(3.11) \qquad \hat{\beta}_{0L} = c_1, \quad \hat{\beta}_{1L} = c_2.$$

By using substitute values of $\hat{\beta}_{0L}, \hat{\beta}_{1L}$ in the equation (3.4), then the estimated values of the exponential distribution parameters in the proposed method will be as follows.

$$(3.12) \qquad \hat{\theta}_{LT} = c_2^{-1}, \quad \hat{\mu}_{LT} = -c_1 c_2^{-1}.$$

Note that in any Bayesian inference problem, the posterior is a product of the likelihood and the prior and thus is affected by both. In cases where one possesses little or no information about the target parameters in advance. In objective Bayesian, the resulting posterior should be expected to be universally agreed upon by everyone, whereas a subjective Bayesian would argue that probability corresponds to the degree of personal belief.

3.2. **Prior specifications.** The Bayesian estimation approach has received significant attention failure time data analysis. It uses one's prior knowledge about the parameters and also considers the available data. If one's prior knowledge about the parameter is available, it is suitable to use an informative prior. However, one does not have any prior knowledge about the parameter and cannot obtain vital information from experts in this regard, then a non-informative prior will be a suitable alternative to use ([6]). [2] used the joint Jeffreys' prior as

$$(3.13) \qquad p(\mu, \theta) \propto \frac{1}{\theta} I_{(0,\infty)}(\mu), \quad \theta > 0.$$

According to (2.2), the posterior density for μ and θ is

$$(3.14) \qquad p(\mu, \theta | \underline{x}) = \frac{n s^{n-1}}{\Gamma(n-1)\theta^{n+1}} e^{-\frac{1}{\theta}\{s + n(x_{(1)} - \mu)\}},$$

where $s = \sum_{i=1}^{n}(x_i - x_{(1)})$. The marginal posterior density of μ is given by

$$p(\mu | \underline{x}) = \int_0^\infty p(\mu, \theta | \underline{x}) d\theta$$

$$= n(n-1) \frac{s^{n-1}}{\{s + n(x_{(1)} - \mu)\}^{n-1}}.$$

So

$$(3.15) \qquad \hat{\mu}_{Bayes} = E(\mu | \underline{x}) = x_{(1)} - \frac{s}{n(n-2)}.$$

By using (3.14), the marginal posterior density of θ is given by

$$p(\theta | \underline{x}) = \int_0^{x_{(1)}} p(\mu, \theta | \underline{x}) d\theta = \frac{s^{n-1}}{\Gamma(n-1)} \frac{\theta^{-\frac{s}{\theta}}}{\theta^n}.$$

Then

$$(3.16) \qquad \hat{\theta}_{Bayes} = E(\theta | \underline{x}) = \frac{s}{(n-2)}.$$

We conclude that the posterior distribution by using this prior is proper.

Let us recall the hierarchical form for two-parameter exponential distribution which is presented in Section 2. The conditional density of μ and θ for the prior distribution given by [11] and [12] can be obtained as follows.

$$(3.17) \qquad \pi_{J_2}(\mu, \theta) = \pi(\mu)\pi(\theta) = \frac{1}{\mu\theta}, \quad \mu > 0, \ \ \theta > 0,$$

$$(3.18) \qquad \pi(\mu, \theta) \propto \frac{1}{\theta^{\nu_0+1}} \exp\left(-\frac{\delta_0 - \lambda_0\mu}{\theta}\right), \quad 0 < \mu \leq \eta_0, \ \ \theta > 0,$$

such that $\nu_0 \leq \lambda_0 \leq \frac{\delta_0}{\eta_0}$.

Proposition 3.1. *For any sample size, the posterior distribution under the independent Jeffrey's prior (3.17) is improper.*

Proof. Using an independent Jeffrey's prior, the joint posterior density of μ and θ is given by

$$\pi(\mu, \theta | \underline{x}) \propto \frac{n}{\theta^{n+1}} \frac{1}{\mu} e^{-\frac{1}{\theta}\sum_{i=1}^{n}(x_i - \mu)}, \quad \mu > 0, \ \ \theta > 0.$$

We next show that the integral of this expression is infinite.

$$
\begin{aligned}
\int_0^\infty \int_0^\infty \frac{n}{\theta^{n+1}} \frac{1}{\mu} e^{-\frac{n\bar{x}}{\theta} - \frac{n\mu}{\theta}} d\,\mu d\theta &= \int_0^\infty \frac{n}{\theta^{n+1}} e^{-\frac{n\bar{x}}{\theta}} \left(\int_0^\infty \frac{1}{\mu} e^{\frac{n\mu}{\theta}}\right) d\theta \\
&= \int_0^\infty \frac{n^2}{\theta^{n+2}} e^{-\frac{n\bar{x}}{\theta}} \left(\int_0^\infty \sum_{t=0}^\infty \left(\frac{\left(\frac{n\mu}{\theta}\right)^{t-1}}{t!}\right) d\mu\right) d\theta \\
&= \int_0^\infty \frac{n^2}{\theta^{n+2}} e^{-\frac{n\bar{x}}{\theta}} \left(\sum_{t=0}^\infty \frac{1}{t!} \int_0^\infty \left(\frac{n\mu}{\theta}\right)^{t-1} d\mu\right) d\theta \\
&= \int_0^\infty \frac{n^2}{\theta^{n+2}} e^{-\frac{n\bar{x}}{\theta}} \left(\sum_{t=0}^\infty \frac{1}{t!} \left[\frac{1}{t} \left(\frac{n}{\theta}\right)^{t-1} \mu^{t-1} \Big|_0^\infty\right]\right) d\theta \\
&= \int_0^\infty \frac{n^2}{\theta^{n+2}} e^{-\frac{n\bar{x}}{\theta}} (\infty) d\theta = \infty.
\end{aligned}
$$

$\square$

Proposition 3.2. *For any sample size, the posterior distribution under the prior given by [11] and [12] or (3.18) is proper.*

Proof. Under this prior distribution, the joint posterior density of μ and θ is given by

$$(3.19) \qquad \pi(\mu, \theta | \underline{x}) \propto \frac{1}{\theta^{n+\nu_0+1}} \exp\left(-\frac{\delta_0 - \lambda_0\mu}{\theta} + \frac{n\mu - n\bar{x}}{\theta}\right), \quad 0 < \mu \leq \eta_0, \ \ \theta > 0.$$

We next show that the integral of this expression is finite.

$$\int_0^\infty \int_0^{\eta_0} \pi(\mu, \theta|\underline{x}) d\mu \ d\theta \propto \int_0^\infty \int_0^{\eta_0} \frac{1}{\theta^{n+\nu_0+1}} \exp\left(-\frac{\delta_0 - \lambda_0\mu}{\theta} + \frac{n\mu - n\bar{x}}{\theta}\right) d\mu \ d\theta$$

$$= \int_0^\infty \frac{1}{\theta^{n+\nu_0+1}} \int_0^{\eta_0} \exp\left(-\frac{\lambda_0\mu + n\mu}{\theta}\right) d\mu \ d\theta$$

$$= \int_0^\infty \frac{1}{\theta^{n+\nu_0+1}} \exp\left(-\frac{\delta_0 + n\bar{x}}{\theta}\right) \left[\frac{\theta}{n + \lambda_0} \exp\left(\frac{(n + \lambda_0)\eta_0}{\theta}\right) \Big|_0^{\eta_0}\right] d\theta$$

$$= \frac{k}{n + \lambda_0} \int_0^\infty \frac{1}{\theta^{n+\nu_0}} \exp\left(-\frac{\delta_0 + n\bar{x}}{\theta}\right) \left[\exp\left(\frac{(n + \lambda_0)\eta_0}{\theta}\right) - 1\right] d\theta$$

$$= \frac{1}{n + \lambda_0} \int_0^\infty \frac{\exp\left(-\frac{\delta_0 + n\bar{x} - n\eta_0 - n\lambda_0}{\theta}\right)}{\theta^{n+\nu_0}} d\theta - \frac{1}{n + \lambda_0} \int_0^\infty \frac{\exp\left(-\frac{\delta_0 + n\bar{x}}{\theta}\right)}{\theta^{n+\nu_0}} d\theta.$$

So

$$\int_0^\infty \int_0^{\eta_0} \pi(\mu, \theta|\underline{x}) d\mu \ d\theta \propto \frac{1}{n + \lambda_0} \int_0^\infty \frac{\exp\left(-\frac{\delta_0 + n\bar{x} - n\eta_0 - n\lambda_0}{\theta}\right)}{\theta^{n+\nu_0}} d\theta - \frac{1}{n + \lambda_0} \int_0^\infty \frac{\exp\left(-\frac{\delta_0 + n\bar{x}}{\theta}\right)}{\theta^{n+\nu_0}} d\theta$$

$$= \frac{1}{n + \lambda_0} \frac{\Gamma(n + \nu_0 - 1)}{(\delta_0 + n\bar{x} - n\eta_0 - n\lambda_0)^{n+\nu_0-1}} - \frac{1}{n + \lambda_0} \frac{\Gamma(n + \nu_0 - 1)}{(\delta_0 + n\bar{x})^{n+\nu_0-1}} < \infty.$$

$\square$

Therefore, we conclude that the posterior distribution by using this prior is proper. Let us recall the hierarchical form for two-parameter exponential distribution which is presented in Section 2, we obtain conditional density of μ and θ for the prior distribution given by [11] and [12] as follows.

$$\pi(\mu, \theta|\underline{x}) \propto \frac{1}{\theta^{N+1}} \exp\left(-\frac{\delta_0 + n\bar{x} - (n + \lambda_0)\mu}{\theta}\right)$$

(3.20)
$$= \frac{1}{\theta^{N+1}} \exp\left(-\frac{n\bar{x} + \delta_0}{\theta} + \frac{(n + \lambda_0)\mu}{\theta}\right),$$

such that $N = n + \nu_0$. Then

$$\pi(\mu|\underline{x}) \propto \int_0^\infty \frac{1}{\theta^{N+1}} \exp\left(-\frac{n\bar{x} + \delta_0}{\theta} + \frac{(n + \lambda_0)\mu}{\theta}\right) d\theta = \frac{\Gamma(N)}{((n\bar{x} + \delta_0) - (n + \lambda_0)\mu)^N}.$$

So

(3.21)
$$\pi(\mu|\underline{x}) = \frac{k_1\Gamma(N)}{((n\bar{x} + \delta_0) - (n + \lambda_0)\mu)^N}, \quad 0 < \mu < \eta_0,$$

where $\frac{1}{k_1} = \int_0^{\eta_0} \frac{\Gamma(N)}{((n\bar{x}+\delta_0)-(n+\lambda_0)\mu)^N} d\mu$. It easy to show that

$$k_1 = \frac{(N-1)(n + \lambda_0)}{\Gamma(N)\left[((n\bar{x} + \delta_0) - (n + \lambda_0)\eta_0)^{-N+1} - (n\bar{x} + \delta_0)^{-N+1}\right]}$$

$$= \frac{(n + \lambda_0)}{\Gamma(N-1)\left[((n\bar{x} + \delta_0) - (n + \lambda_0)\eta_0)^{-N+1} - (n\bar{x} + \delta_0)^{-N+1}\right]},$$

and

$$\pi(\mu|\underline{x}) \propto \int_0^{\eta_0} \frac{1}{\theta^{N+1}} \exp\left(-\frac{n\bar{x}+\delta_0}{\theta} + \frac{(n+\lambda_0)\mu}{\theta}\right) d\mu$$

$$= \frac{1}{\theta^{N+1}} \exp\left(-\frac{n\bar{x}+\delta_0}{\theta}\right) \int_0^{\eta_0} \exp\left(\frac{(n+\lambda_0)\mu}{\theta}\right) d\mu$$

$$(3.22) \qquad = \frac{1}{(n+\lambda_0)} \frac{1}{\theta^N} \exp\left[\left(-\frac{(n\bar{x}+\delta_0)}{\theta} + \frac{(n+\lambda_0)\eta_0}{\theta}\right) - \exp\left(-\frac{(n\bar{x}+\delta_0)}{\theta}\right)\right].$$

Then
$$(3.23)$$

$$\pi(\theta|\underline{x}) = \frac{1}{(n+\lambda_0)} \frac{k_2}{\theta^N} \exp\left[\left(-\frac{(n\bar{x}+\delta_0)}{\theta} + \frac{(n+\lambda_0)\eta_0}{\theta}\right) - \exp\left(-\frac{(n\bar{x}+\delta_0)}{\theta}\right)\right], \quad \theta > 0,$$

where

$$k_2 = \frac{(n+\lambda_0)}{\Gamma(N-1)\left[((n\bar{x}+\delta_0)-(n+\lambda_0)\eta_0)^{-N+1} - (n\bar{x}+\delta_0)^{-N+1}\right]}.$$

Here, to obtain Bayes estimation of μ and θ by using the equation, we have

$$\hat{\mu}_{Obj-B} = E(\mu|\underline{x}) = \int_0^{\eta_0} \frac{k_1 \Gamma(N)\mu}{((n\bar{x}+\delta_0)-(n+\lambda_0)\mu)^N} d\mu.$$

Let $u = \mu$, $du = d\mu$, and $\frac{1}{((n\bar{x}+\delta_0)-(n+\lambda_0)\mu)^N} d\mu = d\nu$. So

$$\nu = \frac{1}{(N-1)(n+\lambda_0)} \left((n\bar{x}+\delta_0)-(n+\lambda_0)\mu\right)^{-N+1}.$$

Then

$$\hat{\mu}_{Obj-B} = E(\mu|\underline{x}) = k_1 \Gamma(N) \left\{ \frac{\eta_0 \left((n\bar{x}+\delta_0)-(n+\lambda_0)\eta_0\right)^{-N+1}}{(n+\lambda_0)(N-1)} - \int_0^{\eta_0} \frac{\left((n\bar{x}+\delta_0)-(n+\lambda_0)\mu\right)^{-N+1}}{(n+\lambda_0)(N-1)} d\mu \right\}$$

$$= k_1 \Gamma(N) \left\{ \frac{\eta_0 \left((n\bar{x}+\delta_0)-(n+\lambda_0)\eta_0\right)^{-N+1}}{(n+\lambda_0)(N-1)} - \frac{1}{(n+\lambda_0)(N-1)} \left[\frac{\left((n\bar{x}+\delta_0)-(n+\lambda_0)\mu\right)^{-N+2}}{(n+\lambda_0)(N-2)} \Big|_0^{\eta_0} \right] \right\}$$

$$(3.24)$$

$$= \frac{k_1 \Gamma(N)}{N-1} \left\{ \frac{\eta_0 \left((n\bar{x}+\delta_0)-(n+\lambda_0)\eta_0\right)^{-N+1}}{(n+\lambda_0)} - \frac{\left((n\bar{x}+\delta_0)-(n+\lambda_0)\eta_0\right)^{-N+2}}{(n+\lambda_0)^2(N-2)} + \frac{(n\bar{x}+\delta_0)^{-N+2}}{(n+\lambda_0)^2(N-2)} \right\},$$

and

$$\hat{\theta}_{Obj-B} = E(\theta|\underline{x}) = \frac{k_2}{(n+\lambda_0)} \int_0^\infty \frac{\theta}{\theta^N} \left[\exp\left(-\frac{((n\bar{x}+\delta_0)-(n+\lambda_0)\eta_0)}{\theta}\right) - \exp\left(\frac{(n\bar{x}+\delta_0)}{\theta}\right) \right] d\theta$$

$$= \frac{k_2}{(n+\lambda_0)} \left[\int_0^\infty \frac{1}{\theta^{N-1}} \exp\left(-\frac{((n\bar{x}+\delta_0)-(n+\lambda_0)\eta_0)}{\theta}\right) d\theta - \int_0^\infty \frac{1}{\theta^{N-1}} \exp\left(\frac{(n\bar{x}+\delta_0)}{\theta}\right) d\theta \right]$$

$$= \frac{k_2}{(n+\lambda_0)} \Gamma(N-2) \left[((n\bar{x}+\delta_0)-(n+\lambda_0)\eta_0)^{-N+2} - (n\bar{x}+\delta_0)^{-N+2} \right]$$

$$(3.25)$$

$$= \frac{1}{N-2} \frac{\left[((n\bar{x}+\delta_0)-(n+\lambda_0)\eta_0)^{-N+2} - (n\bar{x}+\delta_0)^{-N+2} \right]}{\left[((n\bar{x}+\delta_0)-(n+\lambda_0)\eta_0)^{-N+1} - (n\bar{x}+\delta_0)^{-N+1} \right]}.$$

4. **Practical part**

4.1. **Simulation.** In order to apply what was mentioned in the theoretical part, the simulation approach was used. The Monte Carlo method was used the cumulative distribution function and to generate data followed two-parameter exponential distribution. Samples were generated with sizes $n = 10(10)90$ for the two-parameter exponential distribution. The model was then estimated using the posterior obtained from Jeffreys' priors. Let $\hat{\theta}^{(j)}$ be the estimate of parameter θ for the j-th replication, $j = 1, 2, ..., N$. These are the parameter posteri modes calculated from the $N = 10000$ simulated values for each replication. In order to evaluate the estimation method, the criteria was considered the mean square error (MSE), which is defined as

$$MSE = \frac{1}{N} \sum_{j=1}^{N} (\hat{\theta}^{(j)} - \theta)^2.$$

4.2. **Empirical Results.** The results are presented in the following tables, including a comparison between the estimations of scale and location. The μ and θ parameters were Bayes estimator from the formulas as in (3.15) and (3.16) and based on objected Bayes formulas as (3.24) and (3.25) with different initial values for $\eta_0 = 2.05$, $\delta_0 = 12$, $\nu_0 = 0.85$, and $\lambda_0 = 2$. The results of MSE criteria values for the estimator are also presented. Tables 1-4 show that the MSE values of the estimators in the objective method are less than using the Bayes method. Also, the MSE value of parameters decreases as the sample size increases.

Note that in all tables based on MSE the estimators of μ and θ using of objective method is better. By considering results of Tables 3 and 4, we can say that for fixed value of μ with decreasing of θ, the MSE decreases when the sample size increased. Therefore, the use of prior distribution given in [11] and [12] recommended when we consider estimators based on objective Bayesian method. Here to compare the objective Bayesian estimator with the linear transformation of reliability function according to [2], the results are given in Table 5. One should noted that the MSE values of the estimators in the objective method are less than its values by using the Bayes method.

n	$\hat{\mu}_{Obj-B}$	$\hat{\theta}_{Obj-B}$	$\hat{\mu}_{Bayes}$	$\hat{\theta}_{Bayes}$	$MSE_{\hat{\mu}_{Obj-B}}$	$MSE_{\hat{\theta}_{Obj-B}}$	$MSE_{\hat{\mu}_{Bayes}}$	$MSE_{\hat{\theta}_{Bayes}}$
10	1.4772	1.6873	1.4747	1.7015	$3.7571e-04$	$5.4111e-02$	$2.4210e-02$	$3.0795e-01$
20	1.4086	1.6091	1.4968	1.6020	$1.0293e-04$	$4.9818e-02$	$5.9040e-03$	$1.3144e-01$
30	1.4217	1.6055	1.4986	1.5469	$4.4235e-05$	$4.5297e-02$	$2.2281e-03$	$8.9572e-02$
40	1.4280	1.6214	1.4979	1.5450	$2.0407e-05$	$3.5998e-02$	$1.2963e-03$	$6.1509e-02$
50	1.4323	1.6317	1.4985	1.5255	$1.0753e-05$	$2.9077e-02$	$8.8633e-04$	$4.4924e-02$
60	1.4351	1.6499	1.4994	1.5174	$7.1678e-06$	$2.7553e-02$	$6.3381e-04$	$3.8525e-02$
70	1.4370	1.6328	1.4996	1.5254	$4.4332e-06$	$2.2981e-02$	$4.6399e-04$	$3.1050e-02$
80	1.4385	1.6410	1.4986	1.5303	$3.2701e-06$	$2.1988e-02$	$2.9238e-04$	$2.8742e-02$
90	1.4398	1.6358	1.5008	1.5181	$2.4239e-06$	$2.0516e-02$	$3.1804e-04$	$2.6217e-02$

TABLE 1. MSEs and average values of estimates when $\mu = 1.5$, $\theta = 1.5$.

4.3. **Actual data.** The following data were selected from Tires Factory, where the working time (hours) between failures were deduced by the time recorded in the internal statements of the factory for six months. A test was carried out in **R** software. The results showed that the

n	$\hat{\mu}_{Obj-B}$	$\hat{\theta}_{Obj-B}$	$\hat{\mu}_{Bayes}$	$\hat{\theta}_{Bayes}$	$MSE_{\hat{\mu}_{Obj-B}}$	$MSE_{\hat{\theta}_{Obj-B}}$	$MSE_{\hat{\mu}_{Bayes}}$	$MSE_{\hat{\theta}_{Bayes}}$
10	1.8527	2.3776	1.9327	4.5373	$2.5000e-03$	$3.8380e-01$	$1.7210e-01$	$21.899e-01$
20	1.9168	2.9300	1.9915	4.2720	$7.2980e-04$	$3.5422e-01$	$4.1984e-02$	$9.3469e-01$
30	1.9516	3.1482	1.9965	4.1253	$3.1451e-04$	$3.2211e-01$	$1.5844e-02$	$6.3696e-01$
40	1.9707	3.3272	1.9944	4.1201	0.00014511	0.2559875	0.00921882	0.4374008
50	1.9844	3.4090	1.9961	4.0680	$7.6469e-05$	$2.0677e-01$	$6.3028e-03$	$3.1946e-01$
60	1.9938	3.4793	1.9985	4.0464	$5.0971e-05$	$1.9593e-01$	$4.5071e-03$	$2.7395e-01$
70	2.0005	3.5626	1.9991	4.06775	$3.1525e-05$	$1.6342e-01$	$3.2995e-03$	$2.2080e-01$
80	2.0058	3.6223	1.9964	4.0809	$2.3254e-05$	$1.5636e-01$	$2.0791e-03$	$2.0439e-01$
90	2.0104	3.6389	2.0022	4.0482	$1.7236e-05$	$1.4589e-01$	$2.2616e-03$	$1.8643e-01$

TABLE 2. MSEs and average values of estimates when $\mu = 2$, $\theta = 4$.

n	$\hat{\mu}_{Obj-B}$	$\hat{\theta}_{Obj-B}$	$\hat{\mu}_{Bayes}$	$\hat{\theta}_{Bayes}$	$MSE_{\hat{\mu}_{Obj-B}}$	$MSE_{\hat{\theta}_{Obj-B}}$	$MSE_{\hat{\mu}_{Bayes}}$	$MSE_{\hat{\theta}_{Bayes}}$
10	2.7769	3.3312	2.9564	5.5238	$3.5000e-03$	$6.0620e-01$	$3.0590e-1$	$33.710e-1$
20	2.8595	4.1966	2.9926	5.2185	$1.1000e-03$	$5.6200e-01$	$6.1400e-1$	$14.3960e-1$
30	2.9054	4.6270	2.9961	5.1309	$4.6460e-04$	$4.7689e-01$	$3.0179e-02$	$9.3315e-01$
40	2.9323	4.9399	2.9943	5.1620	$2.2025e-04$	$3.8862e-01$	$1.4963e-02$	$6.5352e-01$
50	2.9525	5.0696	2.9967	5.0804	$1.3368e-04$	$3.6149e-01$	$1.0526e-02$	$5.5228e-01$
60	2.9661	5.1973	2.9993	5.0669	$7.6342e-05$	$2.9346e-01$	$6.9892e-03$	$4.2124e-01$
70	2.9766	5.2783	2.9978	5.0436	$4.0829e-05$	$2.6349e-01$	$5.1691e-03$	$3.5948e-01$
80	2.9843	5.3847	3.0007	5.0716	$3.7625e-05$	$2.5299e-01$	$4.8152e-03$	$3.2984e-01$
90	2.9911	5.4163	2.9965	5.0398	$2.4663e-05$	$2.0875e-01$	$2.9360e-03$	$2.6730e-01$

TABLE 3. MSEs and average values of estimates when $\mu = 3$, $\theta = 5$.

n	$\hat{\mu}_{Obj-B}$	$\hat{\theta}_{Obj-B}$	$\hat{\mu}_{Bayes}$	$\hat{\theta}_{Bayes}$	$MSE_{\hat{\mu}_{Obj-B}}$	$MSE_{\hat{\theta}_{Obj-B}}$	$MSE_{\hat{\mu}_{Bayes}}$	$MSE_{\hat{\theta}_{Bayes}}$
10	2.8533	2.3673	2.9643	3.3678	$1.6000e-3$	$2.4580e-1$	$1.1180e-1$	$14.2490e-1$
20	2.9184	2.8952	2.9988	3.1568	$3.8994e-04$	$1.8907e-01$	$2.7496e-02$	$4.9172e-01$
30	2.9506	3.1791	2.9959	3.1340	$1.7494e-04$	$1.7915e-01$	$9.9781e-03$	$3.5355e-01$
40	2.9710	3.3141	2.9942	3.0779	$7.4747e-05$	$1.3185e-01$	$5.1218e-03$	$2.2342e-01$
50	2.9839	3.4336	2.9957	3.0782	$4.7603e-04$	$1.2871e-01$	$3.0341e-03$	$1.9621e-01$
60	2.9935	3.4998	2.9993	3.0530	$2.7009e-05$	$1.0382e-01$	$2.8620e-03$	$1.4813e-01$
70	3.0006	3.5504	2.9975	3.0406	$1.8286e-05$	$9.4799e-02$	$1.6164e-03$	$1.2946e-01$
80	3.0061	3.5923	3.0000	3.0304	$1.2744e-05$	$8.5692e-02$	$1.4115e-03$	$1.1357e-01$
90	3.0104	3.6397	2.9981	3.0420	$9.3748e-06$	$7.9348e-02$	$9.9925e-04$	$1.0110e-01$

TABLE 4. MSEs and average values of estimates when $\mu = 3$, $\theta = 3$.

data had two-parameter exponential distribution with scale parameter $\theta = 97.85$ and location parameter $\mu = 0.25$.

The following table depicts the estimated value by proposed and approximate Bayes methods when the values of the positive definite matrix are fixed at $\nu_{110} = 8$, $\nu_{220} = 8$, $\nu_{120} = \nu_{210} = 0.1$, and selecting initial values of parameters close to estimated values $\beta_{00} = -0.028$,

n	$\hat{\mu}_{Obj-B}$	$\hat{\theta}_{Obj-B}$	$\hat{\mu}_{Bayes}$	$\hat{\theta}_{Bayes}$	$MSE_{\hat{\mu}_{Obj-B}}$	$MSE_{\hat{\theta}_{Obj-B}}$	$MSE_{\hat{\mu}_{Bayes}}$	$MSE_{\hat{\theta}_{Bayes}}$
10	1.9298	2.0947	1.3693	1.9358	0.004925	0.008975	0.006523	0.000716
25	1.9298	2.0947	1.6239	1.9898	0.004925	0.008975	0.007969	0.000127
50	1.9298	2.0947	1.7449	2.0146	0.004925	0.008975	0.005851	0.000021
100	1.9298	2.0947	1.8539	2.0318	0.004925	0.008975	0.003269	0.000013

TABLE 5. MSEs and average values of estimates when $\theta = \mu = 2$ (Initial values for $\beta_{00} = -28$ & $\beta_{01} = 0.3$)

$\beta_{01} = 0.01$, $\theta_0 = 97$, and $\mu_0 = 0.25$ by using different sample sizes.
140.5 312 22.5 48.75 72.5 49.75 218.25 22.25 9.25 68.25 75.5 22.5 23.25 22 63 23 58.75 237.5 193.5 30.25 17.75 141 146.5 127.5 257.75 352 42.5 138.5 51 35.75 173 41.75

Method	Object Bayes	Pl-Bayes
$\hat{\mu}$	6.045425	0.4920
$\hat{\theta}$	74.209891	97.0620
$-\log L(\hat{\mu}, \hat{\theta})$	178.8498	179.611
$AIC = 2k - 2l(\hat{\mu}, \hat{\theta})$	361.6978	363.2219
$BIC = k\log(n) - 2l(\hat{\mu}, \hat{\theta})$	364.6293	366.1534

TABLE 6. Estimated value, Log-likelihood and its AIC and BIC of the real example.

The results in Table 6 show that the estimator of parameters based on objective Bayes estimation is proper. Therefore, we conclude that objective Bayes estimation should be used.

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