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COMPUTATIONAL ASPECTS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS USING FINITE ELEMENT METHOD

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Abstract. This article describes the computational aspects of stochastic parabolic differential equations driven by additive noise. A fully discrete approximation of the stochastic problem is provided based on piecewise linear finite elements for the spacial discretization and the implicit Euler method for the temporal discretization. The computational aspects of the method are illustrated with a numerical test.

1. Introduction

Stochastic partial differential equations (SPDEs) are widely used models in applied sciences, engineering, and finance. Hence, the design of efficient computational methods for such problems is of great importance. In particular, the convergence analysis of numerical methods for approximating the solution of SPDEs is one of the most recently developed areas [1, 2, 3]. The aim of this work is to illustrate numerically the convergence properties of finite element method combined with implicit Euler method for a class parabolic semilinear SPDE, of the form

$$du(t) + Au(t)dt = F(u(t))dt + \sigma(t)dW_Q(t),\ u(0) = u_0 \tag{1.1}$$

in a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\| = (\cdot,\cdot)^{\frac{1}{2}}$. Here, A is assumed to be a linear, self adjoint, positive definite, not necessarily

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bounded operator with compact inverse. Moreover, $F : \mathcal{H} \to \mathcal{H}$ is a smooth nonlinearity and $\sigma : [0,T] \times \mathcal{H} \to \mathcal{H}$ is a deterministic mapping. $\{W_Q(t)\}_{t\geq 0}$ is considered to be a Q-Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$. The following assumptions are standard in the literature on the numerical approximation of stochastic PDEs [4, 5]. Let $\{e_i\}_{i\in\mathbb{N}}$ be a complete orthonormal basis of the Hilbert space $\mathcal{H}$ and the covariance operator Q be the linear, bounded, self adjoint operator on $\mathcal{H}$ such that $Qv = \sum_{i=1}^{\infty} q_i \langle v, e_i\rangle e_i$, where $\{q_i\}_{i\in\mathbb{N}}$ is a sequence of non-negative real numbers. We assume $\{W_Q(t)\}_{t\geq 0}$ is a Q- Wiener process defined as follows:

$$W_Q(t) = \sum_{i=1}^{\infty} \sqrt{q}_i \beta_i(t) e_i, \tag{1.2}$$

where $\{\beta_i\}_{i\in\mathbb{N}}$ is a family of independent standard real valued Wiener processes. We assume that the nonlinear operator F in (1.1) is globally Lipschitz continuous. We also assume that the deterministic function $\sigma : [0,T] \times \mathcal{H} \to \mathcal{H}$ satisfies

$$\|A^{\frac{\beta-1}{2}} \sigma(t)\|_{L_2^0} \leq C,\ \ \beta \in [0,1]. \tag{1.3}$$

In this work, we are concerned with full discrete approximation of stochastic problem (1.1) based on the finite element spatial discretization combined with linear implicit Euler method for the temporal discretization. Let $\Delta t = \frac{T}{N}$ denote the time step size and $t_i = i\Delta t,\ i = 1, 2, \ldots, N$. The full discrete method is defined by

$$u_h^{n+1} = E_{h,\Delta t}^n u_h^n + \Delta t E_{h,\Delta t}^n P_h F(u_h^n) + E_{h,\Delta t}^n P_h \sigma(t) \Delta W_Q^n, \tag{1.4}$$

for $n = 1, \ldots, N$, where $E_{h,\Delta t}^n := (1 + \Delta t A_h)^{-1}$, with the initial condition $u_h^0 = P_h u_0$. In (1.4), —the Wiener increments are denoted by $\Delta W_Q^n = W_Q((n+1)\Delta t) - W_Q(n\Delta t)$.

Theorem 1.1. [5] *Let $u(t)$ be the solution of* (1.1) *and let u_h^n be given by* (1.4). *Then, under the given assumptions, it holds that*

$$\|u(t_n) - u_h^n\|_{L_2(\Omega;\mathcal{H})} \leq C\big(h^{\beta} + \Delta t^{\frac{\beta}{2}}\big), \tag{1.5}$$

where C is a constant independent of h and Δt.

2. Numerical test

In this subsection, we present a numerical test to illustrate the convergence analysis. We consider the following stochastic problem

$$\begin{aligned} &\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) + f(x,t) = \dot{W} \\ &u(x,0) = 10x^2(1-x)^2,\ \ x \in [0,1], \\ &u(0,t) = u(1,t) = 0,\ \ t \in [0,1]. \end{aligned} \tag{2.1}$$

where

$$f(x,t) = 15e^t x^2 (1-x)^2 - 10e^t(2 - 12x + 12x^2).$$

we use a piecewise linear finite element method for the spatial discretization and an implicit Euler method for the temporal discretization. Let u_h^n be the approximate solution of $u(t)$ in finite element space S_h at $t_n = n\Delta t$. The implicit Euler method is to find $u_h^n \in S_h$ such that, for all $\phi \in S_h$,

$$\big(\frac{u_h^{n+1} - u_h^n}{\Delta t}, \phi\big) + (A_h u_h^n, \phi) = \Big(\frac{1}{\Delta t} P_h\big(W_Q(t_n) - W_Q(t_{n-1})\big), \phi\Big) \tag{2.2}$$

$$= \frac{1}{\Delta t} \sum_{i=1}^{\infty} \sqrt{q}_i \big(\beta_i(t_n) - \beta_i(t_{n-1})\big)(e_i, \phi), \tag{2.3}$$

where $\frac{1}{\Delta t}\big(\beta_i(t_n) - \beta_i(t_{n-1})\big) = \mathcal{N}(0,1)$. We choose two types of covariance operators, $Q = I$ and the other operator, $Qe_1 = 0$ and $Qe_i = \frac{1}{i \log i^2} e_i$ for $i \geq 2$. In Figure 1, we plot one realization of the stochastic problem (2.1) for he two types of the covariance operators. We also plot in Figure 2 the corresponding profiles at times

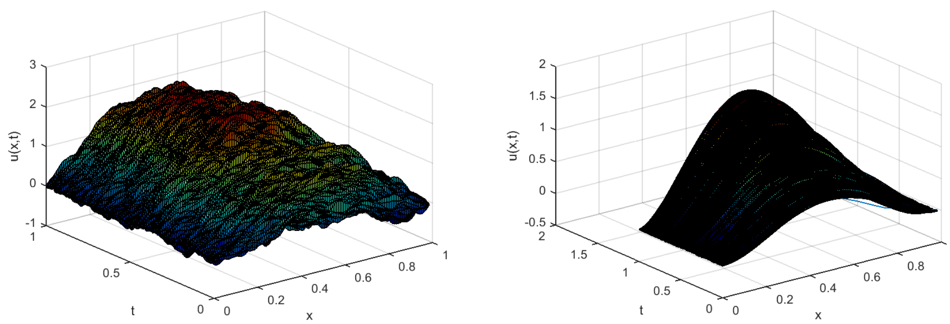

FIGURE 1. Samples of realization of SPDE (2.1) (left $Q = I$, $Tr(Q) < \infty$)

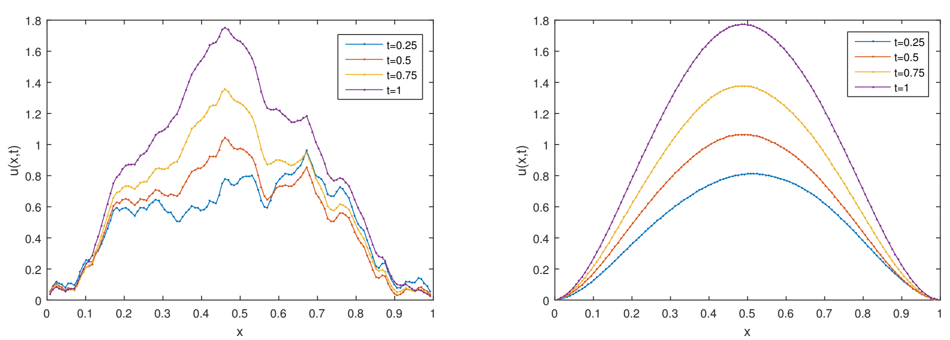

FIGURE 2. Solution profile at different times (left $Q = I$, $Tr(Q) < \infty$)

$t = 0.25, 0.5, 0.75$ and final time $T = 1$. In Figure 3, we present the convergence curves for the strong error for the covariance operator $Q = I$. At first, we demonstrate the convergence rates for the temporal discretization. To do this, we compute the reference solution with the small timstep $\Delta t_{ref} = 2^{-11}$ and $h_{ref} = 2^{-7}$. We perform our numerical simulation with different time step sizes $\Delta t_{ref} = 2^{-i}$, $i = 3, \ldots, 9$ and present the mean square errors in Figure 3 (left). As expected, we observe the convergence rate of order $\frac{1}{4}$, this is consistent with the strong convergence estimates of Theorem 1.1. Next, we turn to spatial error approximation. To this aim, we compute the reference solution using fixed small $h_{ref} = 2^{-10}$ and $\Delta t_{ref} = 2^{-6}$. We plot in Figure 3 (right) the mean square errors due to the spatial discretization using the step sizes $h = 2^{-i}$, $i = 2, \ldots, 8$.

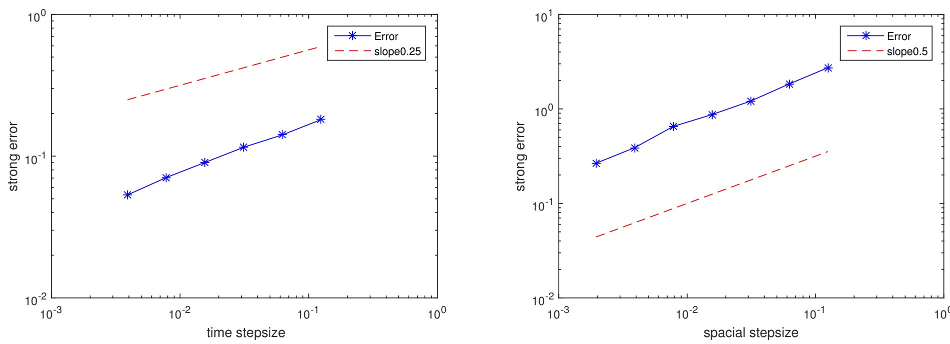

FIGURE 3. Error versus time stepsize (left) and space stepsize (right)

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