Couplings of order six in the gauge field strength and the second fundamental form on D_p-branes at order α'^2

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Using the assumption that the independent gauge invariant couplings on the world-volume of the nonperturbative objects in the string theory are independent of the background, we find the four and the six gauge field strength and/or the second fundamental form couplings on the world volume of a D_p-brane in the superstring theory at order α'^2 in the normalization that F is dimensionless. We have found them by considering the particular background which has one circle and by imposing the corresponding T-duality constraint on the independent couplings. In particular, we find that there are $12 + 146$ independent gauge invariant couplings at this order, and the T-duality constraint can fix 150 of them. We show that these couplings are fully consistent with the partial results in the literature. This comparison also fixes the remaining 8 couplings.

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I. INTRODUCTION

The critical string theory is a quantum theory of gravity that reproduces the Einstein theory of general relativity at the low energy. As in Einstein theory, one expects string theory and its nonperturbative objects at the critical dimension to be background independent. In the low energy effective action, the background independence means the coefficients of the independent gauge invariant couplings at each order of α' should be independent of the background. If one could fix these coefficients in a particular background in which the effective action has some symmetries, then that coefficient would be valid for any other background that may have no symmetry.

The independent couplings at a given order of α' are given as all gauge invariant and covariant couplings at that order modulo the field redefinitions, the total derivative terms, and the Bianchi identities. The numbers of independent couplings in the bosonic string theory involving the metric, dilaton, and the B field at orders $\alpha', \alpha'^2, \alpha'^3$ are 8,60,872, respectively [1–3]. The number of independent world-volume couplings of O_p-plane in the superstring theory at order α'^2 involving only NS-NS fields is 48 [4], and involving linear R-R field and the NS-NS fields is 77 [5]. The background independent coefficients of all these couplings are fixed when one considers a particular background that includes one circle, and uses the corresponding T-duality constraints [6–10]. One may also use the background independence assumption to find the boundary couplings in the case that the background has boundary [4,11,12].

The world-volume gauge invariant couplings of a nonperturbative D_p-brane involving open string massless gauge fields/transverse scalars at long wavelength limit is given by the Dirac-Born-Infeld (DBI) action [13,14]

$$S_p = -T_p \int d^{p+1}\sigma \sqrt{-\det(\tilde{G}_{ab} + F_{ab})}, \qquad (1)$$

where T_p is the tension of the D_p-brane, F_{ab} is field strength of the gauge field A_a, and $\tilde{G}_{ab}$ is the pull back of the bulk metric onto the world volume,[1] i.e.,

$$\tilde{G}_{ab} = \frac{\partial X^\mu(\sigma)}{\partial\sigma^a}\frac{\partial X^\nu(\sigma)}{\partial\sigma^b}\eta_{\mu\nu} \equiv \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}, \qquad (2)$$

where $X^\mu(\sigma)$ is the spacetime coordinate which specifies the D_p-brane in the spacetime, and $\eta_{\mu\nu}$ is the spacetime metric which for simplicity we choose it to be the Minkowski metric. We have also chosen the B field to be zero and the dilaton to be a constant. We have

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[1]Our index convention is that the Greek letters $(\mu, \nu, \ldots)$ are the indices of the spacetime coordinates, the Latin letters $(a, d, c, \ldots)$ are the world-volume indices and the Latin letters $(i, j, k, \ldots)$ are the transverse indices.

normalized the gauge field A_a to have the same dimension as the world sheet field X^μ. With this normalization, the above action is at the leading order of α'. The above action includes all eve n-power of the gauge field strength F_{ab}. The transverse scalar fields Φ^i appear in the static gauge where $X^a = \sigma^a, X^i = \Phi^i(\sigma)$. In the static gauge and for $\Phi^i = 0$, the DBI action reduces to the Born-Infeld action. The α' corrections to the Born-Infeld action have been studied in [15–20].

In the superstring theory, the first correction to the DBI action is at order α'^2, which involves some contractions of the second fundamental form $\Omega_{ab}{}^\mu$, i.e.,

$$\Omega_{ab}{}^\mu = D_a \partial_b X^\mu, \tag{3}$$

the gauge field strength F_{ab} and their covariant derivatives, e.g.,

$$D_a F_{bc} = \partial_a F_{bc} - \tilde{\Gamma}_{ab}{}^d F_{dc} - \tilde{\Gamma}_{ac}{}^d F_{bd}, \tag{4}$$

where the Levi-Civita connection $\tilde{\Gamma}_{ab}{}^c$ is made of the pull-back metric (2). The world-volume indices of these gauge invariant tensors are contracted with the inverse of the pull-back metric $\tilde{G}^{ab}$, and the spacetime index in the second fundamental form are contracted with the spacetime metric $\eta_{\mu\nu}$. Even though the α'^2-order of the couplings constrains the independent couplings to have at most the first derivative of Ω and the second derivative of F, however, there are infinite towers of the gauge field strength, without derivative on it, in the couplings. Hence, for simplicity we consider only the couplings at order α'^2 that involve at most six gauge field strengths and/or the second fundamental form. Using the background independence assumption, we are going to find such couplings in this paper. That is, we first find the independent gauge invariant couplings and then consider a particular background that has one circle. For this background, the couplings should satisfy the T-duality constraint [21,22], i.e., the T-duality transformation of the world-volume reduction of the independent covariant couplings must be the same as the transverse reduction of the couplings, up to some total derivative terms and field redefinitions in the base space. This constraint may fix the coefficients of the independent couplings. This method has been used in [23] to find the corrections to the DBI action in the bosonic string theory at order α' which involve at most eight gauge field strengths and/or the second fundamental forms. The covariant approach has been used in [23] to find the independent couplings, however, the T-duality constraint has been used in the static gauge. In this paper, we are going to use the covariant approach for finding the independent couplings as well as for imposing the T-duality constraint.

The outline of the paper is as follows: In Sec. II, we find all independent covariant couplings at order α'^2 which involve at most six gauge fields and/or the second fundamental forms. We find there is no independent couplings at the level of two fields, there are 12 independent couplings at the level of four fields, and there are 146 couplings at the six-field level. The coefficients of these couplings are independent of the backgrounds in which the D_p-branes are placed. To fix these 158 background independent coefficients, in Sec. III, we consider a background that includes a circle. Then the independent couplings must satisfy the T-duality constraint. We find that the T-duality constraint fixes the 12 parameters of the four-field couplings up to five parameters. They are consistent with the couplings that are found in the literature by the S-matrix method. We use this comparison to fix the remaining 5 parameters. We then find that the T-duality constraint fixes 145 parameters of the six-field couplings. We show that the couplings which involve only the gauge field are consistent with the all-gauge-field couplings that are found by Wyllard in [18]. We also fix the remaining 3 parameters by this comparison. In Sec. IV, we extend the all-gauge-field couplings found in [18] to covariant form and found their corresponding four-field and six-field couplings involving the second fundamental form. In Sec. V, we briefly discuss our results.

II. INDEPENDENT COUPLINGS

In this section we are going to find the independent couplings at order α'^2 that involve at most six gauge fields and/or the second fundamental form. We apply the method used in [2] to find the independent couplings. The independent couplings are all gauge invariant couplings modulo the field redefinitions, the total derivative terms, the identities corresponding to the derivative of the second fundamental form, the Bianchi identity corresponding to the gauge field

$$\partial_{[a} F_{bc]} = 0 \tag{5}$$

and the following identity involving the second fundamental form and $\partial_a X^\mu$:

$$\Omega_{ab}{}^\mu \partial_c X^\nu \eta_{\mu\nu} = 0. \tag{6}$$

The above identity can easily be verified by using (3) and writing the covariant derivative in terms of the partial derivative and the Levi-Civita connection, and then writing the connection in terms of the pull-back metric (2). Using the above identity, one finds that there is a scheme in which ∂X can appear only through the pull-back metric (2) and its inverse. For example, the coupling $D\Omega\partial X$ can be written as $-\Omega\Omega$ which can easily be verified by taking the covariant derivative of the above identity. Hence, we use the scheme in which the couplings involve only the contractions of F, Ω and their covariant derivatives, i.e.,

$$S' = -\frac{(2\pi\alpha')^2}{96} T_p \int d^{p+1}\sigma \sqrt{-\det \tilde{G}_{ab}} \mathcal{L}' \times (F, DF, \ldots, \Omega, D\Omega, \ldots). \tag{7}$$

In principle, one can construct all contractions of the gauge-field strength and/or the second-fundamental form. We call the coefficients of these couplings $a'_1, a'_2, \ldots$. However, they are not independent couplings.

To remove the total derivative terms from the gauge invariant couplings in (7), we first construct a vector $\mathcal{I}^a$ at order $\alpha'^{3/2}$ from F, Ω and their covariant derivatives with arbitrary coefficients $z_1, z_2, \ldots$. Then one is free to add the following total derivative term to (7):

$$\mathcal{J} = -\alpha'^2 T_p \int d^{p+1}\sigma \sqrt{-\det \tilde{G}_{ab}} D_a \mathcal{I}^a. \tag{8}$$

The total derivative terms may remove some of the structures in (7) completely, e.g., $FDDDDF$ or $\Omega DD\Omega$, and may also remove only some of the couplings in a particular structure in (7). Hence, in writing the couplings in (7), we do not include the structures that are removed completely by the total derivative terms.

One is also free to change the field variables as[2]

$$A_a \to A_a + \alpha'^{3/2}\delta A_a, \qquad X^\mu \to X^\mu + \alpha'^{3/2}\delta X^\mu, \tag{9}$$

where the tensors δA_a and δX^μ are all contractions of F, Ω and their covariant derivatives at order $\alpha'^{3/2}$ with arbitrary coefficients $y_1, y_2, \ldots$. If one replaces this field redefinition into the leading order action (1), it would produce the following couplings at order α'^2:

$$\mathcal{K} = -\alpha'^2 T_p \int d^{p+1}\sigma \sqrt{-\det \tilde{G}_{ab}} \times [-D_a F^{ab}\delta A_b - \tilde{G}^{ab}\Omega_{ab}{}^{\nu}\delta X^\mu \eta_{\mu\nu} + \cdots], \tag{10}$$

where dots represent the terms that involve all higher orders of F resulting from the linear perturbation of the leading order action (1) around (9). If one uses the arbitrary parameters in δA_a and δX^μ to remove all couplings in (7) which have $D_a F^{ab}$ and $G^{ab}\Omega_{ab}{}^{\nu}$, then there would be no residual arbitrary parameters in δA_a and δX^μ to remove any couplings in (7) that have the same structure as the couplings in the dots above. Therefore, in the scheme that the field redefinitions remove the couplings that have $D_a F^{ab}$ or $G^{ab}\Omega_{ab}{}^{\nu}$, one must ignore the dots above.

If one adds $\mathcal{J}$, $\mathcal{K}$ to the action (7), they change only the coefficients of the gauge invariant couplings $a'_1, a'_2, \ldots$, i.e.,

$$S' + \mathcal{J} + \mathcal{K} = S, \tag{11}$$

where S is the same action as (7) in which the coefficients of the gauge invariant couplings are changed to $a_1, a_2, \ldots$. One can write the above equation as

$$\Delta S + \mathcal{J} + \mathcal{K} = 0, \tag{12}$$

where ΔS is the same as (7) in which the coefficients of the gauge invariant couplings are $\delta a_1, \delta a_2, \ldots$ where $\delta a_i = a'_i - a_i$. If one solves the above equation, one would find some relations between only $\delta a_1, \delta a_2, \ldots$. The number of these relations represents the number of couplings that are invariant under the field redefinitions and the total derivative terms.

However, to solve the equation (12), one has to impose the Bianchi identity (5) and the identities corresponding to the derivative of the second fundamental form, to write (12) in terms of independent couplings. To impose the latter identities automatically, one can write the covariant derivatives in terms of partial derivatives and the Levi-Civita connection. Moreover, one can go to the local frame in which the Levi-Civita connection is zero but its derivatives are not zero. Then, one can write the derivatives of the connection in terms of the pull-back metric (2). In the resulting expression, then one has to replace the two ∂X in which their spacetime indexes are contracted with each other, i.e., $\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$, by the pull-back metric (2). To impose the Bianchi identity (5), we write the terms that have partial derivative of the gauge field strength in terms of the gauge potential, e.g., $\partial_a F_{bc} = \partial_a\partial_b A_c - \partial_a\partial_c A_b$. The resulting terms have noncovariant expressions F, $\partial\partial A$, $\partial\partial\partial A, \ldots$, and ∂X, $\partial\partial X$, $\partial\partial\partial X, \ldots$. The world-volume indices are contracted with the inverse of the pull-back metric (2) and the spacetime indices are contracted with $\eta_{\mu\nu}$. In other words, the equation (12) is written in the local frame in terms of noncovariant but independent terms. The coefficients of the independent terms which involve $\delta a_1, \delta a_2, \ldots$, $z_1, z_2, \ldots, y_1, y_2, \ldots$ must be zero. The solution of the resulting linear algebraic equations gives $z_1, z_2, \ldots$, $z_n, y_1, y_2, \ldots, y_m$ in terms of $z_{n+1}, z_{n+2}, \ldots, y_{m+1}, y_{m+2}, \ldots$ and $\delta a_1, \delta a_2, \ldots$ in which we are not interested. The solution also gives some relations between only $\delta a_1, \delta a_2, \ldots$ in which we are interested. The number of the latter relations gives the number of independent couplings in (12).

Since there can be any number of gauge field strength F_{ab} in the couplings at any order of α', there are infinite number of independent couplings at each order of α'. Hence, we have to classify the independent couplings in substructures in which their couplings are independent. In

[2]One may also consider the change of variables at order $\alpha'^{1/2}$ and consider the second perturbation of the DBI action which also produces couplings at order α'^2. However, such field redefinition would also produce at the linear order, the couplings at order α' which is in conflict with the fact that there is no world-volume couplings in the superstring theory at order α'. Hence, there should be no such field redefinition.

our choice for the field redefinition that removes all terms that involve D_aF^{ab} and $G^{ab}\Omega_{ab}{}^{\nu}$, the field redefinition does not relate the terms which have different number of the gauge fields, to each other. The total derivative terms and the Bianchi identities do not relate the couplings with different number of gauge fields either. Hence, in our choice for the field redefinition, the number of independent couplings at each level of gauge field is fixed. Moreover, the couplings that involve only F, Ω modulo the trace of Ω, are not related to the other couplings by the field redefinitions, by the total derivative terms and by the Bianchi identity. Hence, we choose all such couplings at each level of gauge field as independent couplings. We use the above prescription to find all other independent couplings at each level of gauge field.

When X^μ is constant, i.e., $\Omega = 0$, the independent couplings involve only the gauge field strength F_{ab} and its partial derivatives. The above prescription can be used to find the independent couplings in this case. In the case that X^μ is not constant, i.e., $\Omega \neq 0$, there is a scheme in which the independent couplings classify into two sets of couplings. One set of couplings is the same as the set of independent couplings in the case that X^μ is constant. The second set of couplings is the independent couplings which become zero when X^μ is constant. It has been shown in [2] that in fact there is such scheme for the independent couplings of the bosonic string theory for metric, B-field, and dilaton at order α'^2. In particular, it has been shown in [2] that there are 60 independent couplings at this order. In one particular scheme, the couplings have been written as two sets. One set, which has 20 couplings, includes the dilaton only as the overall factor $e^{-2\phi}$, and another set that has 40 couplings, includes the derivative of the dilaton. In this scheme, when the dilaton is constant, the couplings reduce to 20 couplings that are the independent couplings when the dilaton is constant [24]. It has been shown in [2], that there is also a scheme in which the dilaton appears as an overall factor in all 60 independent couplings. In this scheme, when the dilaton is constant, the number of couplings does not change, however, the 60 couplings are not independent any more when the dilaton is constant. In this paper we are going to use the scheme in which the independent couplings are such that when X^μ is constant, they reduce to the independent couplings of only the gauge field.

We begin with the couplings that have zero gauge field at order α'^2. There are 4 couplings involving $\Omega\Omega\Omega\Omega$ modulo the trace of Ω. Apart from this structure, the Lagrangian in (7) has one structure as

$$\mathcal{L}' \sim D\Omega D\Omega. \tag{13}$$

Using the package xAct [25], one finds there are 5 couplings in the above structure. The vector in the total derivative (8) has one structure as

$$\mathcal{I} \sim \Omega D\Omega. \tag{14}$$

The field redefinitions δA_a has no structure at this level and δX^μ has one structure as

$$\delta X^\mu \sim DD\Omega. \tag{15}$$

Using the package xAct, one can construct all possible contractions in Eqs. (14) and (15). Then we replace them in (12) and go to the local frame to write Eq. (12) in terms of the independent structures. However, the coefficients of all the resulting independent structures cannot be zero because we have already set aside some of the independent couplings. Since we have chosen the couplings in the structure $\Omega\Omega\Omega\Omega$ as independent couplings, we have to remove all independent structures that are reproduced also by $\Omega\Omega\Omega\Omega$, i.e., remove the terms that have four and more fields. One finds the resulting linear algebraic equations have no solution that involves only $\delta a_1, \ldots, \delta a_5$. It means there are no independent couplings at zero gauge field except the 4 couplings in $\Omega\Omega\Omega\Omega$, i.e.,

$$S \supset -\frac{(2\pi\alpha')^2}{96}T_p\int d^{p+1}\sigma\sqrt{-\det\tilde{G}_{ab}}[b_1\Omega_a{}^{c\nu}\Omega^{ab\mu}\Omega_b{}^d{}_\nu\Omega_{cd\mu} + b_2\Omega_a{}^c{}_\mu\Omega^{ab\mu}\Omega_b{}^{d\nu}\Omega_{cd\nu}$$
$$+ b_3\Omega_{ab}{}^\nu\Omega^{ab\mu}\Omega_{cd\nu}\Omega^{cd}{}_\mu + b_4\Omega_{ab\mu}\Omega^{ab\mu}\Omega_{cd\nu}\Omega^{cd\nu}], \tag{16}$$

where we have chosen the coefficients of the 4 independent couplings to be $b_1, \ldots, b_4$.

Next, we consider the couplings at the level of two gauge fields at order α'^2. There are 18 independent couplings in the structure $\Omega\Omega\Omega\Omega FF$ modulo the trace of Ω. Apart from this structure, the Lagrangian in (7) has 4 structures as

$$\mathcal{L}' \sim DDFDDF + DFDF\Omega\Omega + FDF\Omega D\Omega + FFD\Omega D\Omega. \tag{17}$$

Using the package xAct, one finds there are 118 gauge invariant couplings in these structures. The vector in the total derivative (8) has 4 structures as

$$\mathcal{I} \sim FF\Omega D\Omega + FDF\Omega\Omega + DFDDF + FDDDF. \tag{18}$$

The field redefinitions δA_a and δX^μ in (10) have 3 and 4 structures, respectively, as

$$\delta A_a \sim \Omega\Omega DF + F\Omega D\Omega + DDDF,$$
$$\delta X^\mu \sim \Omega DFDF + FDFD\Omega + F\Omega DDF + FFDD\Omega. \quad (19)$$

Using the package xAct, one can construct all possible contractions in (18) and (19). Then replacing them in (12), going to the local frame to write the equation (12) in terms of the independent structures, and removing the terms that have six and more fields which are reproduce also by the independent couplings in the structure $\Omega\Omega\Omega\Omega FF$, one finds the resulting linear algebraic equations has 4 solutions that involve only $\delta a_1, \ldots, \delta a_{118}$. It means there are 4 independent couplings at four gauge field level on top of the 18 couplings in the structure $\Omega\Omega\Omega\Omega FF$. One can set all of the coefficients in (7) to zero except 4 of them. However, one is not totally free to choose the 4 couplings. The correct choices must be such that when one replaces the nonzero couplings in (12), the linear algebraic equations produce 4 relations $\delta a_1 = \delta a_2 = \delta a_3 = \delta a_4 = 0$. For the wrong choices of the independent couplings, the algebraic equations, would produce less than 4 relations between only δa_i. There are different ways (schemes) to choose the 4 independent couplings. One can choose the 4 independent couplings in the structure $DFDF\Omega\Omega$. The couplings in a particular scheme are the following:

$$\begin{aligned}
S \supset -\frac{(2\pi\alpha')^2}{96} T_p \int d^{p+1}\sigma \sqrt{-\det \tilde{G}_{ab}} [& a_1 D_a F_{bc} D^a F^{bc} \Omega_{de\mu} \Omega^{de\mu} + a_2 D^a F^{bc} D^d F_b{}^e \Omega_{ae}{}^\mu \Omega_{cd\mu} \\
& + a_3 D^a F^{bc} D_b F_a{}^d \Omega_c{}^{e\mu} \Omega_{de\mu} + a_4 D^a F^{bc} D^d F_b{}^e \Omega_{ac}{}^\mu \Omega_{de\mu} \\
& + f_1 \Omega_a{}^{e\mu} \Omega_b{}^{f\nu} \Omega_{ce\nu} \Omega_{df\mu} F^{ab} F^{cd} + f_2 \Omega_a{}^{e\mu} \Omega_b{}^{f\nu} \Omega_{ce\mu} \Omega_{df\nu} F^{ab} F^{cd} \\
& + f_3 \Omega_a{}^{e\mu} \Omega_b{}^f{}_\mu \Omega_{ce}{}^\nu \Omega_{df\nu} F^{ab} F^{cd} + f_4 \Omega_a{}^{e\mu} \Omega_{be}{}^\nu \Omega_c{}^f{}_\mu \Omega_{df\nu} F^{ab} F^{cd} \\
& + f_5 \Omega_{ac}{}^\mu \Omega_b{}^{e\nu} \Omega_d{}^f{}_\nu \Omega_{ef\mu} F^{ab} F^{cd} + f_6 \Omega_b{}^{d\mu} \Omega_c{}^{e\nu} \Omega_d{}^f{}_\nu \Omega_{ef\mu} F_a{}^c F^{ab} \\
& + f_7 \Omega_c{}^{e\nu} \Omega^{cd\mu} \Omega_d{}^f{}_\nu \Omega_{ef\mu} F_{ab} F^{ab} + f_8 \Omega_b{}^{d\mu} \Omega_c{}^{e\nu} \Omega_d{}^f{}_\mu \Omega_{ef\nu} F_a{}^c F^{ab} \\
& + f_9 \Omega_{ac}{}^\mu \Omega_b{}^e{}_\mu \Omega_d{}^{f\nu} \Omega_{ef\nu} F^{ab} F^{cd} + f_{10} \Omega_b{}^{d\mu} \Omega_c{}^e{}_\mu \Omega_d{}^{f\nu} \Omega_{ef\nu} F_a{}^c F^{ab} \\
& + f_{11} \Omega_c{}^e{}_\mu \Omega^{cd\mu} \Omega_d{}^{f\nu} \Omega_{ef\nu} F_{ab} F^{ab} + f_{12} \Omega_{bc}{}^\mu \Omega_d{}^{f\nu} \Omega^{de}{}_\mu \Omega_{ef\nu} F_a{}^c F^{ab} \\
& + f_{13} \Omega_{ac}{}^\mu \Omega_{bd}{}^\nu \Omega_{ef\nu} \Omega^{ef}{}_\mu F^{ab} F^{cd} + f_{14} \Omega_b{}^{d\mu} \Omega_{cd}{}^\nu \Omega_{ef\nu} \Omega^{ef}{}_\mu F_a{}^c F^{ab} \\
& + f_{15} \Omega_{cd}{}^\nu \Omega^{cd\mu} \Omega_{ef\nu} \Omega^{ef}{}_\mu F_{ab} F^{ab} + f_{16} \Omega_{ac}{}^\mu \Omega_{bd\mu} \Omega_{ef\nu} \Omega^{ef\nu} F^{ab} F^{cd} \\
& + f_{17} \Omega_b{}^{d\mu} \Omega_{cd\mu} \Omega_{ef\nu} \Omega^{ef\nu} F_a{}^c F^{ab} + f_{18} \Omega_{cd\mu} \Omega^{cd\mu} \Omega_{ef\nu} \Omega^{ef\nu} F_{ab} F^{ab}],
\end{aligned} \quad (20)$$

where we have chosen the coefficients of the 4 independent couplings to be $a_1, \ldots, a_4$. We have also included in above couplings the 18 independent couplings in the structure $\Omega\Omega\Omega\Omega FF$ with coefficients $f_1, \ldots, f_{18}$. Note that the above independent couplings become zero when X^μ is constant, which is consistent with the fact that there is no independent couplings of two gauge fields at order α'^2.

We now consider the couplings that have four gauge fields at order α'^2. Apart from the structure $\Omega\Omega\Omega\Omega FFFF$, the Lagrangian in (7) has 6 structures as

$$\begin{aligned}
\mathcal{L}' \sim\ & DFDFDFDF + FDFDFDDF + FFDDFDDF \\
& + FFDFDF\Omega\Omega + FFFDF\Omega D\Omega + FFFFD\Omega D\Omega.
\end{aligned} \quad (21)$$

There are 1124 gauge invariant couplings in these structures. The vector in the total derivative (8) has 5 structures as

$$\begin{aligned}
\mathcal{I} \sim\ & FDFDFDF + FFDFDDF \\
& + FFFDDDF + FFFF\Omega D\Omega + FFFDF\Omega\Omega.
\end{aligned} \quad (22)$$

The field redefinitions δA_a and δX^μ in (10) each has 5 structures as

$$\begin{aligned}
\delta A_a &\sim DFDFDF + FDFDDF + FFDDDF + \Omega\Omega FFDF + FFF\Omega D\Omega, \\
\delta X^\mu &\sim FF\Omega DFDF + FFFDFD\Omega + \Omega FFFDDF + FFFFDD\Omega + \Omega\Omega\Omega FFFF.
\end{aligned} \quad (23)$$

In this case, after removing the eight and more fields from the independent structures in the local frame, one finds the resulting linear algebraic equations have 68 solutions that involve only $\delta a_1, \ldots, \delta a_{1124}$. It means there are 68 independent couplings at four gauge field on top of the independent couplings in $\Omega\Omega\Omega\Omega FFFF$. We find that there are at least 4 independent couplings in the structures in the first line of (21) and there are at most 64 independent couplings in the structures in the second line of (21). Since there are 4 independent couplings for only gauge field at order α'^2, we choose the 4 couplings in structure $DFDFDFDF$ and 64 couplings in the structures in the second line of (21). The couplings in a particular scheme are the following:

$$
\begin{aligned}
S \supset -\frac{(2\pi\alpha')^2}{96} T_p \int d^{p+1}\sigma \sqrt{-\det \tilde{G}_{ab}}[& c_1 D_a F_{bc} D^a F^{bc} D_d F_{ef} D^d F^{ef} + c_2 D_a F^{de} D^a F^{bc} D_f F_{de} D^f F_{bc} \\
& + c_3 D_a F^{de} D^a F^{bc} D_f F_{ce} D^f F_{bd} + c_4 D_a F_b{}^d D^a F^{bc} D_e F_{df} D^e F_c{}^f \\
& + d_1 D^a F^{bc} D^d F_b{}^e F_{fh} F^{fh} \Omega_{ad}{}^\mu \Omega_{ce\mu} + d_2 D^a F^{bc} D_a F^{de} F_{fh} F^{fh} \Omega_{bd}{}^\mu \Omega_{ce\mu} \\
& + d_3 D^a F^{bc} D^d F^{ef} F_b{}^h F_{eh} \Omega_{ad}{}^\mu \Omega_{cf\mu} + d_4 D^a F^{bc} D^d F^{ef} F_a{}^h F_{dh} \Omega_{be}{}^\mu \Omega_{cf\mu} \\
& + d_5 D^a F^{bc} D_b F^{de} F_d{}^f F_f{}^h \Omega_{ae}{}^\mu \Omega_{ch\mu} + d_6 D^a F^{bc} D^d F_b{}^e F_{fh} F^{fh} \Omega_{ac}{}^\mu \Omega_{de\mu} \\
& + d_7 D^a F^{bc} D^d F_b{}^e F_a{}^f F_f{}^h \Omega_{ch}{}^\mu \Omega_{de\mu} + d_8 D^a F^{bc} D^d F^{ef} F_b{}^h F_{eh} \Omega_{ac}{}^\mu \Omega_{df\mu} \\
& + d_9 D^a F^{bc} D^d F^{ef} F_a{}^h F_{be} \Omega_{ch}{}^\mu \Omega_{df\mu} + d_{10} D^a F^{bc} D^d F^{ef} F_{ae} F_b{}^h \Omega_{ch}{}^\mu \Omega_{df\mu} \\
& + d_{11} D^a F^{bc} D^d F^{ef} F_{ab} F_e{}^h \Omega_{ch}{}^\mu \Omega_{df\mu} + d_{12} D^a F^{bc} D^d F^{ef} F_a{}^h F_{be} \Omega_{cf}{}^\mu \Omega_{dh\mu} \\
& + d_{13} D^a F^{bc} D^d F^{ef} F_{ae} F_b{}^h \Omega_{cf}{}^\mu \Omega_{dh\mu} + d_{14} D^a F^{bc} D^d F^{ef} F_{ab} F_e{}^h \Omega_{cf}{}^\mu \Omega_{dh\mu} \\
& + d_{15} D^a F^{bc} D_a F_b{}^d F_e{}^h F^{ef} \Omega_{cf}{}^\mu \Omega_{dh\mu} + d_{16} D^a F^{bc} D_b F_a{}^d F_e{}^h F^{ef} \Omega_{cf}{}^\mu \Omega_{dh\mu} \\
& + d_{17} D^a F^{bc} D^d F^{ef} F_{ae} F_{bf} \Omega_c{}^{h\mu} \Omega_{dh\mu} + d_{18} D^a F^{bc} D^d F^{ef} F_{ab} F_{ef} \Omega_c{}^{h\mu} \Omega_{dh\mu} \\
& + d_{19} D^a F^{bc} D_a F_b{}^d F_{ef} F^{ef} \Omega_c{}^{h\mu} \Omega_{dh\mu} + d_{20} D^a F^{bc} D_b F_a{}^d F_{ef} F^{ef} \Omega_c{}^{h\mu} \Omega_{dh\mu} \\
& + d_{21} D^a F^{bc} D_b F^{de} F_a{}^f F_d{}^h \Omega_{ch}{}^\mu \Omega_{ef\mu} + d_{22} D^a F^{bc} D^d F_b{}^e F_a{}^f F_d{}^h \Omega_{ch}{}^\mu \Omega_{ef\mu} \\
& + d_{23} D^a F^{bc} D_a F^{de} F_b{}^f F_d{}^h \Omega_{ch}{}^\mu \Omega_{ef\mu} + d_{24} D^a F^{bc} D_b F^{de} F_d{}^f F_f{}^h \Omega_{ac}{}^\mu \Omega_{eh\mu} \\
& + d_{25} D^a F^{bc} D_b F^{de} F_a{}^f F_f{}^h \Omega_{cd}{}^\mu \Omega_{eh\mu} + d_{26} D^a F^{bc} D^d F_b{}^e F_a{}^f F_f{}^h \Omega_{cd}{}^\mu \Omega_{eh\mu} \\
& + d_{27} D^a F^{bc} D_a F^{de} F_b{}^f F_f{}^h \Omega_{cd}{}^\mu \Omega_{eh\mu} + d_{28} D^a F^{bc} D_b F^{de} F_a{}^f F_d{}^h \Omega_{cf}{}^\mu \Omega_{eh\mu} \\
& + d_{29} D^a F^{bc} D_a F^{de} F_b{}^f F_d{}^h \Omega_{cf}{}^\mu \Omega_{eh\mu} + d_{30} D^a F^{bc} D_b F^{de} F_{ad} F^{fh} \Omega_{cf}{}^\mu \Omega_{eh\mu} \\
& + d_{31} D^a F^{bc} D_a F^{de} F_{bd} F^{fh} \Omega_{cf}{}^\mu \Omega_{eh\mu} + d_{32} D^a F^{bc} D_b F^{de} F_a{}^f F_{df} \Omega_c{}^{h\mu} \Omega_{eh\mu} \\
& + d_{33} D^a F^{bc} D^d F_b{}^e F_a{}^f F_{df} \Omega_c{}^{h\mu} \Omega_{eh\mu} + d_{34} D^a F^{bc} D_a F^{de} F_b{}^f F_{df} \Omega_c{}^{h\mu} \Omega_{eh\mu} \\
& + d_{35} D^a F^{bc} D^d F_b{}^e F_a{}^f F_c{}^h \Omega_{df}{}^\mu \Omega_{eh\mu} + d_{36} D^a F^{bc} D_a F^{de} F_b{}^f F_c{}^h \Omega_{df}{}^\mu \Omega_{eh\mu} \\
& + d_{37} D^a F^{bc} D_a F^{de} F_{bc} F^{fh} \Omega_{df}{}^\mu \Omega_{eh\mu} + d_{38} D^a F^{bc} D_a F_b{}^d F_c{}^e F^{fh} \Omega_{df}{}^\mu \Omega_{eh\mu} \\
& + d_{39} D^a F^{bc} D_b F_a{}^d F_c{}^e F^{fh} \Omega_{df}{}^\mu \Omega_{eh\mu} + d_{40} D^a F^{bc} D_b F_{ac} F^{de} F^{fh} \Omega_{df}{}^\mu \Omega_{eh\mu} \\
& + d_{41} D^a F^{bc} D^d F_b{}^e F_a{}^f F_d{}^h \Omega_{ce}{}^\mu \Omega_{fh\mu} + d_{42} D^a F^{bc} D^d F^{ef} F_{ad} F_{be} \Omega_c{}^{h\mu} \Omega_{fh\mu} \\
& + d_{43} D^a F^{bc} D^d F_b{}^e F_a{}^f F_c{}^h \Omega_{de}{}^\mu \Omega_{fh\mu} + d_{44} D^a F^{bc} D_b F_a{}^d F_c{}^e F_e{}^f \Omega_d{}^{h\mu} \Omega_{fh\mu} \\
& + d_{45} D^a F^{bc} D^d F_b{}^e F_{ad} F_c{}^f \Omega_e{}^{h\mu} \Omega_{fh\mu} + d_{46} D^a F^{bc} D_a F^{de} F_{bd} F_c{}^f \Omega_e{}^{h\mu} \Omega_{fh\mu} \\
& + d_{47} D^a F^{bc} D^d F_b{}^e F_{ac} F_d{}^f \Omega_e{}^{h\mu} \Omega_{fh\mu} + d_{48} D^a F^{bc} D_a F^{de} F_{bc} F_d{}^f \Omega_e{}^{h\mu} \Omega_{fh\mu} \\
& + d_{49} D^a F^{bc} D_b F_{ac} F_d{}^f F^{de} \Omega_e{}^{h\mu} \Omega_{fh\mu} + d_{50} D^a F^{bc} D^d F_b{}^e F_{ad} F_{ce} \Omega_{fh\mu} \Omega^{fh\mu} \\
& + d_{51} D^a F^{bc} D_a F^{de} F_{bd} F_{ce} \Omega_{fh\mu} \Omega^{fh\mu} + d_{52} D^a F^{bc} D_a F^{de} F_{bc} F_{de} \Omega_{fh\mu} \Omega^{fh\mu} \\
& + d_{53} D^a F^{bc} D_a F_b{}^d F_c{}^e F_{de} \Omega_{fh\mu} \Omega^{fh\mu} + d_{54} D^a F^{bc} D_b F_a{}^d F_c{}^e F_{de} \Omega_{fh\mu} \Omega^{fh\mu} \\
& + d_{55} D^a F^{bc} D_b F_{ac} F_{de} F^{de} \Omega_{fh\mu} \Omega^{fh\mu} + d_{56} D^a F^{bc} D_a F_b{}^d F_e{}^h F^{ef} \Omega_{cd}{}^\mu \Omega_{fh\mu}
\end{aligned}
$$

$$
\begin{aligned}
&+ d_{57} D^a F^{bc} D_a F^{de} F_b{}^f F_d{}^h \Omega_{ce}{}^\mu \Omega_{fh\mu} + d_{58} D^a F^{bc} D^d F^{ef} F_{ab} F_c{}^h \Omega_{de}{}^\mu \Omega_{fh\mu} \\
&+ d_{59} D^a F^{bc} D^d F^{ef} F_a{}^h F_{bh} \Omega_{ce}{}^\mu \Omega_{df\mu} + d_{60} D^a F^{bc} D^d F_b{}^e F_a{}^f F_d{}^h \Omega_{cf}{}^\mu \Omega_{eh\mu} \\
&+ d_{61} D^a F^{bc} D^d F_b{}^e F_{ad} F^{fh} \Omega_{cf}{}^\mu \Omega_{eh\mu} + d_{62} D^a F^{bc} D^d F_b{}^e F_a{}^f F_{cf} \Omega_d{}^{h\mu} \Omega_{eh\mu} \\
&+ d_{63} D^a \Omega^{bc\mu} D^d \Omega^{ef}_\mu F_{ad} F_{be} F_c{}^h F_{fh} + d_{64} D^a F^{bc} D^d \Omega^{ef\mu} F_{ab} F_e{}^h F_{fh} \Omega_{cd\mu} + \Omega\Omega\Omega\Omega FFFF],
\end{aligned}
\tag{24}
$$

where we have chosen the coefficients of the 4 independent couplings to be $c_1, \ldots, c_4$, and the 64 couplings to be $d_1, \ldots, d_{64}$. Since the couplings in the structure $\Omega\Omega\Omega\Omega FFFF$ involve eight fields F, Ω in which we are not interested in this paper, we did not write the dependent couplings in this structure. Note that when Ω is zero, the above couplings reduce to the independent couplings of four gauge field at order α'^2.

We finally consider in this section the couplings which have six gauge fields. Apart from the structure $\Omega\Omega\Omega\Omega FFFFFF$, the Lagrangian in (7) has 6 structures as

$$
\begin{aligned}
\mathcal{L}' \sim\, & FFDFDFDFDF + FFFDFDFDDF \\
& + FFFFDDFDDF + FFFFDFDF\Omega\Omega \\
& + FFFFFDF\Omega D\Omega + FFFFFFD\Omega D\Omega.
\end{aligned}
\tag{25}
$$

In this case the couplings in the structures in the second line have eight gauge field or the second fundamental form in which we are not interested in this paper. On the other hand, in the scheme that we are using in this paper in which the independent couplings should be reduced to the independent couplings of only gauge field when X^μ is a constant, one can find the independent couplings in the first line by finding the independent couplings of only the gauge field. At the end, the partial derivatives are replaced by the covariant derivatives. So we consider only the six gauge field structures

$$
\begin{aligned}
\mathcal{L}'_F \sim\, & FFDFDFDFDF + FFFDFDFDDF \\
& + FFFFDDFDDF.
\end{aligned}
$$

There are 2836 gauge invariant couplings in these structures. The vector in the total derivative (8) has 3 structures as

$$
\mathcal{I} \sim FFFDFDFDF + FFFFDFDDF + FFFFFDDDF. \tag{26}
$$

The field redefinition δA_a has 3 structures as

$$
\delta A_a \sim FFDFDFDF + FFFDFDDF + FFFFDDDF. \tag{27}
$$

The derivatives are all partial derivatives. In this case, one needs only to impose the Bianchi identity (5) to find the corresponding independent structures in (12). One finds the linear algebraic equations have 64 solutions that involve only $\delta a_1, \ldots, \delta a_{2836}$. It means there are 64 independent couplings at six gauge fields when X^μ is constant. The couplings in a particular scheme are the following:

$$
\begin{aligned}
S \supset -\frac{(2\pi\alpha')^2}{96} T_p \int d^{p+1}\sigma \sqrt{-\det \tilde{G}_{ab}} [& e_1 D^a F^{bc} D_b F^{de} D^f F_{de} D^h F_f{}^u F_{au} F_{ch} \\
&+ e_2 D^a F^{bc} D_b F^{de} D_d F^{fh} D^u F_{ef} F_{au} F_{ch} + e_3 D^a F^{bc} D_b F^{de} D_d F^{fh} D^u F_{fh} F_{ae} F_{cu} \\
&+ e_4 D^a F^{bc} D_b F_c{}^d D^e F^{fh} D^u F_{fh} F_{ae} F_{du} + e_5 D^a F^{bc} D_b F_a{}^d D^e F^{fh} D_f F_h{}^u F_{ce} F_{du} \\
&+ e_6 D^a F^{bc} D_a F_b{}^d D^e F^{fh} D_f F_{eh} F_c{}^u F_{du} + e_7 D^a F^{bc} D_b F^{de} D^f F_{cd} D^h F_f{}^u F_{au} F_{eh} \\
&+ e_8 D^a F^{bc} D_b F^{de} D_d F^{fh} D^u F_{cf} F_{au} F_{eh} + e_9 D^a F^{bc} D_a F^{de} D_f F_d{}^u D^f F_b{}^h F_{cu} F_{eh} \\
&+ e_{10} D^a F^{bc} D_a F^{de} D_b F_d{}^f D^h F_f{}^u F_{cu} F_{eh} + e_{11} D^a F^{bc} D_b F_a{}^d D^e F_c{}^f D^h F_f{}^u F_{du} F_{eh} \\
&+ e_{12} D^a F^{bc} D_b F^{de} D_d F^{fh} D^u F_{fh} F_{ac} F_{eu} + e_{13} D^a F^{bc} D_b F^{de} D_c F^{fh} D_d F_f{}^u F_{ah} F_{eu} \\
&+ e_{14} D^a F^{bc} D_b F^{de} D_d F^{fh} D_f F_c{}^u F_{ah} F_{eu} + e_{15} D^a F^{bc} D_b F^{de} D_d F_c{}^f D_f F^{hu} F_{ah} F_{eu} \\
&+ e_{16} D^a F^{bc} D_b F_c{}^d D^e F_d{}^f D_f F^{hu} F_{ah} F_{eu} + e_{17} D^a F^{bc} D_b F^{de} D_d F_c{}^f D^h F_f{}^u F_{ah} F_{eu} \\
&+ e_{18} D^a F^{bc} D_b F_c{}^d D^e F_d{}^f D^h F_f{}^u F_{ah} F_{eu} + e_{19} D^a F^{bc} D_b F^{de} D^f F_{cd} D^h F_f{}^u F_{ah} F_{eu} \\
&+ e_{20} D^a F^{bc} D_b F^{de} D_d F^{fh} D^u F_{cf} F_{ah} F_{eu} + e_{21} D^a F^{bc} D_b F^{de} D^f F_c{}^h D^u F_{df} F_{ah} F_{eu}
\end{aligned}
$$

$$
\begin{aligned}
&+ e_{22} D^a F^{bc} D_b F^{de} D_c F^{fh} D_d F_{fh} F_a{}^u F_{eu} + e_{23} D^a F^{bc} D_b F_a{}^d D^e F_c{}^f D^h F_d{}^u F_{eh} F_{fu} \\
&+ e_{24} D^a F^{bc} D_a F^{de} D_b F^{fh} D^u F_{de} F_{cf} F_{hu} + e_{25} D^a F^{bc} D_b F_a{}^d D^e F_c{}^f D^h F_e{}^u F_{df} F_{hu} \\
&+ e_{26} D^a F^{bc} D_b F_a{}^d D^e F_c{}^f D_f F_e{}^h F_d{}^u F_{hu} + e_{27} D^a F^{bc} D_b F_a{}^d D^e F_c{}^f D^h F_{ef} F_d{}^u F_{hu} \\
&+ e_{28} D^a F^{bc} D_b F^{de} D_c F^{fh} D_d F_a{}^u F_{ef} F_{hu} + e_{29} D^a F^{bc} D_a F_{bc} D_d F^{hu} D^d F^{ef} F_{ef} F_{hu} \\
&+ e_{30} D^a F^{bc} D_b F_a{}^d D_c F_d{}^e D^f F^{hu} F_{ef} F_{hu} + e_{31} D^a F^{bc} D_b F_c{}^d D_d F_a{}^e D^f F^{hu} F_{ef} F_{hu} \\
&+ e_{32} D^a F^{bc} D_a F^{de} D_b F_d{}^f D^h F_c{}^u F_{ef} F_{hu} + e_{33} D^a F^{bc} D_b F^{de} D_d F_a{}^f D^h F_c{}^u F_{ef} F_{hu} \\
&+ e_{34} D^a F^{bc} D_a F^{de} D^f F_{bd} D^h F_c{}^u F_{ef} F_{hu} + e_{35} D^a F^{bc} D_a F_b{}^d D^e F_c{}^f D^h F_d{}^u F_{ef} F_{hu} \\
&+ e_{36} D^a F^{bc} D_b F_a{}^d D^e F_c{}^f D^h F_d{}^u F_{ef} F_{hu} + e_{37} D^a F^{bc} D_a F^{de} D^f F_{bc} D^h F_d{}^u F_{ef} F_{hu} \\
&+ e_{38} D^a F^{bc} D_a F^{de} D_f F_c{}^h D^f F_{bd} F_e{}^u F_{hu} + e_{39} D^a F^{bc} D_a F^{de} D^f F_{bd} D^h F_{cf} F_e{}^u F_{hu} \\
&+ e_{40} D^a F^{bc} D_a F^{de} D_b F_d{}^f D_e F_c{}^h F_f{}^u F_{hu} + e_{41} D^a F^{bc} D_a F_b{}^d D_c F^{ef} D_e F_d{}^h F_f{}^u F_{hu} \\
&+ e_{42} D^a F^{bc} D_b F_a{}^d D_c F^{ef} D_e F_d{}^h F_f{}^u F_{hu} + e_{43} D^a F^{bc} D_b F_{ac} D^d F^{ef} D_e F_d{}^h F_f{}^u F_{hu} \\
&+ e_{44} D^a F^{bc} D_a F_b{}^d D_e F_d{}^h D^e F_c{}^f F_f{}^u F_{hu} + e_{45} D^a F^{bc} D_b F^{de} D_d F_a{}^f D^h F_{ce} F_f{}^u F_{hu} \\
&+ e_{46} D^a F^{bc} D_a F^{de} D^f F_{bd} D^h F_{ce} F_f{}^u F_{hu} + e_{47} D^a F^{bc} D_a F_b{}^d D_c F^{ef} D^h F_{de} F_f{}^u F_{hu} \\
&+ e_{48} D^a F^{bc} D_b F_a{}^d D_c F^{ef} D^h F_{de} F_f{}^u F_{hu} + e_{49} D^a F^{bc} D_a F^{de} D^f F_{bc} D^h F_{de} F_f{}^u F_{hu} \\
&+ e_{50} D^a F^{bc} D_b F_{ac} D^d F^{ef} D_e F_{df} F_{hu} F^{hu} + e_{51} D^a F^{bc} D_a F_b{}^d D^e F_c{}^f D_f F_{de} F_{hu} F^{hu} \\
&+ e_{52} D^a F^{bc} D_b F_a{}^d D^e F_c{}^f D_f F_{de} F_{hu} F^{hu} + e_{53} D^a F^{bc} D_a F^{de} D_f F_{de} D^f F_{bc} F_{hu} F^{hu} \\
&+ e_{54} D^a F^{bc} D_a F^{de} D_f F_{ce} D^f F_{bd} F_{hu} F^{hu} + e_{55} D^a F^{bc} D_a F^{de} D_f F_d{}^u D^f F_b{}^h F_{ce} F_{hu} \\
&+ e_{56} D_e D^h F_c{}^u D_f D_u F_{dh} F_a{}^c F^{ab} F_b{}^d F^{ef} + e_{57} D_a D_c F_{eh} D_b D_f F_{du} F^{ab} F^{cd} F^{ef} F^{hu} \\
&+ e_{58} D^a F^{bc} D_a F^{de} D_b F^{fh} D^u F_{df} F_{ce} F_{hu} + e_{59} D^a F^{bc} D_b F_a{}^d D^e F^{fh} D_f F_e{}^u F_{ch} F_{du} \\
&+ e_{60} D^a F^{bc} D_b F_a{}^d D_c F_d{}^e D^f F_e{}^h F_f{}^u F_{hu} + e_{61} D^a F^{bc} D_b F_c{}^d D_d F_a{}^e D^f F_e{}^h F_f{}^u F_{hu} \\
&+ e_{62} D^a F^{bc} D_b F^{de} D^f F_c{}^h D^u F_{dh} F_{af} F_{eu} + e_{63} D_b D^h F_e{}^u D_f D_u F_{ch} F_a{}^c F^{ab} F_d{}^f F_{de} \\
&+ e_{64} D_e D_h F_{fu} D^a F^{bc} D_b F_{ac} F_d{}^f F^{de} F^{hu} + \Omega\Omega\Omega\Omega FFFFFF \\
&+ FFFFDFDF\Omega\Omega + FFFFFDF\Omega D\Omega + FFFFFFD\Omega D\Omega],
\end{aligned}
\tag{28}
$$

where $e_1, \ldots, e_{64}$ are some parameters. The couplings in the structures in the last line above and in the structure $\Omega\Omega\Omega\Omega FFFFFF$ involve more than six gauge field and/or the second fundamental forms in which we are not interested in this paper. When Ω is zero, the above couplings reduce to the independent couplings of four gauge fields at order α'^2. Hence, the derivatives in the above independent couplings are now covariant derivatives.

The parameters of the independent couplings in (16), (20), (24), and (28) are background independent parameters which may be found by the appropriate S-matrix elements in flat spacetime. The couplings of four gauge field and/or the second fundamental form have been found by the S-matrix element of four open string vertex operators [17,26]. They are

$$
\begin{aligned}
&a_1 = \frac{1}{2}, \qquad a_2 = a_3 = a_4 = -2; \qquad b_1 = 0, \qquad b_2 = 2, \qquad b_3 = -2, \qquad b_4 = 0; \\
&c_1 = \frac{1}{8}, \qquad c_2 = \frac{1}{4}, \qquad c_3 = -\frac{1}{2}, \qquad c_4 = -1.
\end{aligned}
\tag{29}
$$

However, we are going to find the parameters in the next section by imposing the T-duality constraint.

III. T-DUALITY CONSTRAINT

We now try to fix the parameters in the actions (16), (20), (24), and (28). The assumption that the world-volume effective action at the critical dimension is background independent, means the parameters in these actions are independent of the background. Hence, to fix them we consider a specific background that has a circle. That is, the manifold has the structure

$M^{(10)} = M^{(9)} \times S^{(1)}$. The manifold $M^{(10)}$ has coordinates $x^\mu = (x^{\tilde\mu}, y)$, where $x^{\tilde\mu}$ is the coordinate of the manifold $M^{(9)}$, and y is the coordinate of the circle $S^{(1)}$. The world-volume action has two reductions on the circle. When the D_p-brane is along the circle, i.e., $a = (\tilde a, y)$, the reduction is called S_p^w, and when the D_p-brane is orthogonal to the circle, i.e., $a = \tilde a$, the reduction is called S_p^t. These two actions are not identical. However, the transformation of S_p^w under the following T-duality transformations

$$\begin{aligned} A_y &\to X^y, \\ A_{\tilde a} &\to A_{\tilde a}, \\ X^{\tilde\mu} &\to X^{\tilde\mu}, \end{aligned} \tag{30}$$

which is called S_{p-1}^{wT}, should be the same as S_{p-1}^t, up to some total derivative terms and field redefinitions in the base space, i.e.,

$$\Delta\tilde S + \tilde{\mathcal{J}} + \tilde{\mathcal{K}} = 0, \tag{31}$$

where $\Delta\tilde S = S_{p-1}^{wT} - S_{p-1}^t$, the total derivative term $\tilde{\mathcal{J}}$ and the field redefinition contributions are

$$\begin{aligned} \tilde{\mathcal{J}} &= \alpha'^2 T_{p-1} \int d^p\sigma \sqrt{-\det \tilde g_{\tilde a\tilde b}}\,\tilde D_{\tilde a}\tilde{\mathcal{I}}^{\tilde a}, \\ \tilde{\mathcal{K}} &= \alpha'^2 T_{p-1} \int d^p\sigma \sqrt{-\det \tilde g_{\tilde a\tilde b}}[-\tilde D_{\tilde a}F^{\tilde a\tilde b}\delta A_{\tilde b} - \tilde g^{\tilde a\tilde b}\tilde\Omega_{\tilde a\tilde b}^{\tilde\nu}\delta X^{\tilde\mu}\eta_{\tilde\mu\tilde\nu} - \tilde g^{\tilde a\tilde b}\tilde\Omega_{\tilde a\tilde b}^{y}\delta X^y\eta_{yy} + \cdots], \end{aligned} \tag{32}$$

where $\tilde{\mathcal{I}}^{\tilde a}$ is a vector which is made of the base space fields $F, \tilde\Omega^{\tilde\mu}, \partial X^y$ and their covariant derivatives at order $\alpha'^{3/2}$ with coefficients $j_1, j_2, \ldots$. In the above equations, the world-volume indices are contracted with the inverse of the pull back of the base space metric onto the world volume of D_{p-1}-brane, i.e.,

$$\tilde g_{\tilde a\tilde b} = \partial_{\tilde a}X^{\tilde\mu}\partial_{\tilde b}X^{\tilde\nu}\eta_{\tilde\mu\tilde\nu}, \tag{33}$$

and the dots in $\tilde{\mathcal{K}}$ represent the terms that involve all higher orders of F and ∂X^y that are resulted from inserting in the world-volume reduction of (1), the following field redefinitions:

$$\begin{aligned} A_{\tilde a} &\to A_{\tilde a} + \alpha'^{3/2}\delta A_{\tilde a}, \\ X^{\tilde\mu} &\to X^{\tilde\mu} + \alpha'^{3/2}\delta X^{\tilde\mu}, \\ X^y &\to X^y + \alpha'^{3/2}\delta X^y, \end{aligned} \tag{34}$$

and using integration by part. The coefficients of the gauge invariant terms in $\delta A_{\tilde a}, \delta X^{\tilde\mu}, \delta X^y$ at order $\alpha'^{3/2}$ are $k_1, k_2, \ldots$. Unlike in $\mathcal{K}$, the dots in $\tilde{\mathcal{K}}$ cannot be ignored because they have contribution with some fixed parameters in some of the structures in the constraint (31), i.e., if one ignores them, then one would find the world-volume actions (16), (20), (24), and (28) satisfy the constraint (31) when all parameters in the actions are zero, which is not true.

For the world-volume reduction, $a = (\tilde a, y)$ and $\mu = (\tilde\mu, y)$. Using the fact that the second fundamental form is zero when μ is a world-volume index, and the fact that in the dimensional reduction one assumes field are independent of the y coordinate, i.e., the Kaluza-Klein modes are ignored, one finds the following nonzero world-volume reductions:

$$\begin{aligned} G_{\tilde a\tilde b} &= \tilde g_{\tilde a\tilde b}, \\ \Omega_{\tilde a\tilde b}{}^{\mu} &= \tilde\Omega_{\tilde a\tilde b}{}^{\tilde\mu}, \\ D_{\tilde a}\Omega_{\tilde b\tilde c}{}^{\mu} &= \tilde D_{\tilde a}\tilde\Omega_{\tilde b\tilde c}{}^{\tilde\mu}, \\ F_{\tilde a\tilde b} &= F_{\tilde a\tilde b}, \\ F_{\tilde a y} &= F_{\tilde a y}, \\ F_{y\tilde a} &= F_{y\tilde a}, \\ D_{\tilde a}F_{\tilde b\tilde c} &= \tilde D_{\tilde a}F_{\tilde a\tilde b}, \\ D_{\tilde a}F_{\tilde b y} &= \tilde D_{\tilde a}F_{\tilde b y}, \\ D_{\tilde a}F_{y\tilde b} &= \tilde D_{\tilde a}F_{y\tilde b}, \\ D_{\tilde a}D_{\tilde b}F_{\tilde c\tilde d} &= \tilde D_{\tilde a}\tilde D_{\tilde b}F_{\tilde c\tilde d}, \\ D_{\tilde a}D_{\tilde b}F_{\tilde c y} &= \tilde D_{\tilde a}\tilde D_{\tilde b}F_{\tilde c y}, \\ D_{\tilde a}D_{\tilde b}F_{y\tilde c} &= \tilde D_{\tilde a}\tilde D_{\tilde b}F_{y\tilde c}, \end{aligned} \tag{35}$$

where $\tilde\Omega_{\tilde a\tilde b}{}^{\tilde\mu} = \tilde D_{\tilde a}\partial_{\tilde b}X^{\tilde\mu}$, and the covariant derivatives on the right-hand side are made of the pull-back metric (33).

For the transverse reduction, $a = \tilde a$ and $\mu = (\tilde\mu, y)$. Since the index y is a transverse index, one finds the following nonzero transverse reductions:

$$\begin{aligned}
\tilde{G}_{\tilde{a}\tilde{b}} &\to \tilde{g}_{\tilde{a}\tilde{b}} + \partial_{\tilde{a}}X^y\partial_{\tilde{b}}X^y,\\
\Omega_{\tilde{a}\tilde{b}}{}^{\mu} &\to \tilde{\Omega}_{\tilde{a}\tilde{b}}{}^{\tilde{\mu}} - \partial_{\tilde{c}}X^y\partial^{\tilde{c}}X^{\tilde{\mu}}\tilde{\Omega}_{\tilde{a}\tilde{b}}{}^{y}\left(\frac{1}{1-\partial_{\tilde{e}}X^y\partial^{\tilde{e}}X^y}\right),\\
\Omega_{\tilde{a}\tilde{b}}{}^{y} &\to \tilde{\Omega}_{\tilde{a}\tilde{b}}{}^{y}\left(\frac{1}{1-\partial_{\tilde{e}}X^y\partial^{\tilde{e}}X^y}\right),\\
D_{\tilde{a}}\Omega_{\tilde{b}\tilde{c}}{}^{\mu} &\to \tilde{D}_{\tilde{a}}\Omega_{\tilde{b}\tilde{c}}{}^{\tilde{\mu}} - \partial^{\tilde{d}}X^y(\Omega_{\tilde{c}\tilde{d}}{}^{\tilde{\mu}}\Omega_{\tilde{a}\tilde{b}}{}^{y} + \Omega_{\tilde{b}\tilde{d}}{}^{\tilde{\mu}}\Omega_{\tilde{a}\tilde{c}}{}^{y}),\\
D_{\tilde{a}}\Omega_{\tilde{b}\tilde{c}}{}^{y} &\to \tilde{D}_{\tilde{a}}\Omega_{\tilde{b}\tilde{c}}{}^{y} - \partial^{\tilde{d}}X^y(\Omega_{\tilde{c}\tilde{d}}{}^{y}\Omega_{\tilde{a}\tilde{b}}{}^{y} + \Omega_{\tilde{b}\tilde{d}}{}^{y}\Omega_{\tilde{a}\tilde{c}}{}^{y}),\\
F_{\tilde{a}\tilde{b}} &\to F_{\tilde{a}\tilde{b}},\\
D_{\tilde{a}}F_{\tilde{b}\tilde{c}} &\to \tilde{D}_{\tilde{a}}F_{\tilde{b}\tilde{c}} + \partial^{\tilde{d}}X^y(F_{\tilde{c}\tilde{d}}\Omega_{\tilde{a}\tilde{b}}{}^{y} - F_{\tilde{b}\tilde{d}}\Omega_{\tilde{a}\tilde{c}}{}^{y}),\\
D_{\tilde{a}}D_{\tilde{b}}F_{\tilde{c}\tilde{d}} &\to \tilde{D}_{\tilde{a}}D_{\tilde{b}}F_{\tilde{c}\tilde{d}} - \partial^{\tilde{e}}X^y(D_{\tilde{e}}F_{\tilde{c}\tilde{d}}\Omega_{\tilde{a}\tilde{b}}{}^{y} - D_{\tilde{b}}F_{\tilde{d}\tilde{e}}\Omega_{\tilde{a}\tilde{c}}{}^{y} + D_{\tilde{b}}F_{\tilde{c}\tilde{e}}\Omega_{\tilde{a}\tilde{d}}{}^{y}),
\end{aligned}\tag{36}$$

where, for the simplicity in writing, on the right-hand side of the transverse reductions of $D\Omega$ and DDF, we have not written the results completely in terms of the base space tensors $\tilde{\Omega}$ and $\tilde{D}$. One can easily replace them from the reductions of Ω and DF. In both world-volume and transverse reductions, one observers that the identity (6) reduces to the corresponding identity in the base space, i.e.,

$$\tilde{\Omega}_{\tilde{a}\tilde{b}}{}^{\tilde{\mu}}\partial_{\tilde{c}}X^{\tilde{\nu}}\eta_{\tilde{\mu}\tilde{\nu}} = 0 \tag{37}$$

and the reductions satisfies the Bianchi identity (5). Note that the gauge field in the base space satisfies its corresponding Bianchi identity

$$\partial_{[\tilde{a}}F_{\tilde{b}\tilde{c}]} = 0. \tag{38}$$

Note also that there is no relation corresponding to (6) for $\mu, \nu = y$. Hence, one cannot remove the term $\partial_{\tilde{a}}X^y$ from the independent covariant couplings in the base space.

Using the reductions (35) and (36), one can calculate $\Delta\tilde{S}$ in (31). To solve the T-duality constraint (31), one has to write it in terms of independent couplings in the base space, i.e., the Bianchi identity (38) and the identities corresponding to the second fundamental forms must be imposed into it. As in the previous section, we write the covariant derivatives in the base space in terms of partial derivatives and the Levi-Civita connection which is made of the pull-back metric (33). Moreover, one can go to the local frame in which the Levi-Civita connection is zero but its partial derivatives are not zero. Then, one can write the derivatives of the connection in terms of the pull-back metric (33). In the resulting expression, then one has to replace the two $\partial_{\tilde{a}}X^{\tilde{\mu}}$ in which their spacetime index are contracted with each other, i.e., $\partial_{\tilde{a}}X^{\tilde{\mu}}\partial_{\tilde{b}}X^{\tilde{\nu}}\eta_{\tilde{\mu}\tilde{\nu}}$, by the pull-back metric (33). One also has to write the partial derivatives of the gauge field strength in terms of the gauge field potential. The final resulting noncovariant expression involves independent structures made of $F_{\tilde{a}\tilde{b}}, \partial_{\tilde{a}}\partial_{\tilde{b}}A_{\tilde{c}}, \ldots, \partial_{\tilde{a}}X^{\tilde{\mu}}, \partial_{\tilde{a}}\partial_{\tilde{b}}X^{\tilde{\mu}}, \ldots$ and $\partial_{\tilde{a}}X^y, \partial_{\tilde{a}}\partial_{\tilde{b}}X^y, \ldots$. The coefficients of these independent structures which involve the parameters in the effective action found in the previous section, the parameters in the total derivative terms (32) and the parameters in the field redefinitions (34), must be zero. They produces some linear algebraic equations for these parameters. Solving them, one finds some relations involving only the parameters of the independent couplings found in the previous section in which we are interested in this paper. The solution also produces some relations for $j_1, j_2, \ldots, j_{n_j}, k_1, k_2, \ldots, k_{n_k}$ in terms of the parameters of the effective action and $j_{n_j+1}, j_{n_j+2}, \ldots, k_{n_k+1}, k_{n_k+2}, \ldots$ in which we are not interested.

Since the T-duality constraint (31) at order α'^2 involve all orders of fields $F_{\tilde{a}\tilde{b}}, \partial_{\tilde{a}}X^y$, it relates the coefficients of all infinite number of independent couplings at order α'^2. However, to solve this constraint one has to truncate the independent couplings in the effective action to a fixed number of F, Ω. In the previous section we have found the couplings up to six F, Ω. To find the parameters of these truncated couplings, one has to truncate also the independent structures in (31). If one considers an action at a given order of α', and at the level of m fields F, Ω, then the independent structures in the constraint (31) which have more than m fields in the local frame must be ignored. The coefficients of the remaining independent structures must be zero. The resulting linear algebraic equation should be solved to find some relations between the parameters of the independent couplings in the action.

The independent couplings that we have found in the previous section have 12 couplings at the level of four F, Ω, i.e., the couplings with coefficients $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4$. To find the T-duality constraint on these couplings, we consider the following structures for the vector of the total derivative terms:

$$
\begin{aligned}
\tilde{\mathcal{I}} \sim\ & \tilde{D}\tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + \tilde{\Omega}^y\tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y + \tilde{\Omega}^y\tilde{\Omega}^y\tilde{\Omega}^y\tilde{D}X^y + F\tilde{D}\tilde{D}\tilde{D}F\tilde{D}X^y\tilde{D}X^y + \tilde{D}\tilde{D}F\tilde{D}F\tilde{D}X^y\tilde{D}X^y \\
& + F\tilde{D}\tilde{D}F\tilde{\Omega}^y\tilde{D}X^y + \tilde{D}F\tilde{D}F\tilde{\Omega}^y\tilde{D}X^y + FF\tilde{\Omega}^y\tilde{D}\tilde{\Omega}^y + F\tilde{D}F\tilde{\Omega}^y\tilde{\Omega}^y + F\tilde{D}F\tilde{D}\tilde{\Omega}^y\tilde{D}X^y \\
& + FF\tilde{D}\tilde{D}\tilde{\Omega}^y\tilde{D}X^y + \tilde{D}X^y\tilde{D}X^y\tilde{\Omega}^{\tilde{\mu}}\tilde{D}\tilde{\Omega}^{\tilde{\nu}} + \tilde{D}X^y\tilde{\Omega}^y\tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^{\tilde{\nu}}.
\end{aligned} \tag{39}
$$

For the field redefinitions, we consider the following structures:

$$
\begin{aligned}
\delta X^y \sim\ & FF\tilde{D}\tilde{D}\tilde{\Omega}^y + F\tilde{D}\tilde{D}\tilde{D}F\tilde{D}X^y + F\tilde{D}F\tilde{D}\tilde{\Omega}^y + F\tilde{D}\tilde{D}F\tilde{\Omega}^y + \tilde{D}F\tilde{D}\tilde{D}F\tilde{D}X^y + \tilde{D}F\tilde{D}F\tilde{\Omega}^y \\
& + \tilde{D}\tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y + \tilde{\Omega}^y\tilde{D}\tilde{\Omega}^y\tilde{D}X^y + \tilde{\Omega}^y\tilde{\Omega}^y\tilde{\Omega}^y + \tilde{\Omega}^{\tilde{\mu}}\tilde{D}\tilde{\Omega}^{\tilde{\nu}}\tilde{D}X^y + \tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^{\tilde{\nu}}\tilde{\Omega}^y, \\
\delta A^{\tilde{a}} \sim\ & \tilde{D}\tilde{D}\tilde{D}F\tilde{D}X^y\tilde{D}X^y + \tilde{D}\tilde{D}\tilde{\Omega}^y\tilde{D}X^yF + \tilde{\Omega}^y\tilde{D}X^y\tilde{D}\tilde{D}F + \tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}F + \tilde{\Omega}^y\tilde{D}\tilde{\Omega}^yF + \tilde{\Omega}^y\tilde{\Omega}^y\tilde{D}F, \\
\delta X^{\tilde{\mu}} \sim\ & \tilde{D}\tilde{D}\tilde{\Omega}^{\tilde{\mu}}\tilde{D}X^y\tilde{D}X^y + \tilde{\Omega}^y\tilde{D}\tilde{\Omega}^{\tilde{\mu}}\tilde{D}X^y + \tilde{\Omega}^{\tilde{\mu}}\tilde{D}\tilde{\Omega}^y\tilde{D}X^y + \tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^y\tilde{\Omega}^y.
\end{aligned} \tag{40}
$$

Note that all terms in the reduction $\Delta\tilde{S}$ in the constraint (31) involve, among other things, ∂X^y and/or $\tilde{\Omega}^y$. Hence the total derivative terms and the field redefinitions must include these fields as well. Using the package xAct, one can construct all possible contractions in (39) and (40). Then replacing them in (31), going to the local frame to write the equation (31) in terms of the independent structures, and removing the terms that have six and more fields, one finds the resulting linear algebraic equations have the following solution that involves only the parameters of the effective action

$$
\begin{aligned}
& c_1 \to \frac{a_1}{4} + \frac{a_3 - a_4}{16}, \qquad c_2 \to -\frac{3}{8}a_3 + \frac{a_4}{4}, \qquad c_3 \to \frac{a_2 - a_3 + a_4}{4}, \qquad c_4 \to \frac{a_3}{2}, \\
& b_2 \to -a_2 + 2b_1, \qquad b_3 \to \frac{a_2 + a_4}{4} - b_1, \qquad b_4 \to \frac{1}{4}(4a_1 + a_2 - 2b_1).
\end{aligned} \tag{41}
$$

The above parameters are consistent with the results from the S-matrix method (29). It turns out that the unfixed parameters above can not be fixed by studying the T-duality constraint at the level of six F, Ω. So for studying the constraint (31) at level of six F, Ω, we consider the parameters (29) for the four F, Ω couplings.

We have found the independent couplings at the level of six F, Ω in the previous section, i.e., the couplings with coefficients $f_1, f_2, \ldots, f_{18}$, $d_1, d_2, \ldots, d_{64}$ and $e_1, e_2, \ldots, e_{64}$. To find the T-duality constraint on these couplings, we consider the following structures for the vector of the total derivative terms:

$$
\begin{aligned}
\tilde{\mathcal{I}} \sim\ & \tilde{D}\tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + \tilde{D}\tilde{\Omega}^y\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + \tilde{\Omega}^y\tilde{\Omega}^y\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + F\tilde{D}\tilde{D}\tilde{D}F\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y \\
& + FF\tilde{D}\tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + \tilde{D}\tilde{D}F\tilde{D}F\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + F\tilde{D}\tilde{D}F\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + F\tilde{D}F\tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y \\
& + FF\tilde{\Omega}^y\tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y + \tilde{D}F\tilde{D}F\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + F\tilde{D}F\tilde{\Omega}^y\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y + FF\tilde{\Omega}^y\tilde{\Omega}^y\tilde{\Omega}^y\tilde{D}X^y \\
& + FFF\tilde{D}\tilde{D}\tilde{D}F\tilde{D}X^y\tilde{D}X^y + FFFF\tilde{D}\tilde{D}\tilde{\Omega}^y\tilde{D}X^y + FF\tilde{D}\tilde{D}F\tilde{D}F\tilde{D}X^y\tilde{D}X^y + FFF\tilde{D}\tilde{D}F\tilde{\Omega}^y\tilde{D}X^y \\
& + F\tilde{D}F\tilde{D}F\tilde{D}F\tilde{D}X^y\tilde{D}X^y + FF\tilde{D}F\tilde{D}F\tilde{\Omega}^y\tilde{D}X^y + FFFF\tilde{\Omega}^y\tilde{D}\tilde{\Omega}^y + FFFF\tilde{D}F\tilde{D}\tilde{\Omega}^y\tilde{D}X^y \\
& + FFF\tilde{D}F\tilde{\Omega}^y\tilde{\Omega}^y + \tilde{\Omega}^{\tilde{\mu}}\tilde{D}\tilde{\Omega}^{\tilde{\nu}}\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + \tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^{\tilde{\nu}}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + FF\tilde{\Omega}^{\tilde{\mu}}\tilde{D}\tilde{\Omega}^{\tilde{\nu}}\tilde{D}X^y\tilde{D}X^y \\
& + F\tilde{D}F\tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^{\tilde{\nu}}\tilde{D}X^y\tilde{D}X^y + FF\tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^{\tilde{\nu}}\tilde{\Omega}^y\tilde{D}X^y.
\end{aligned} \tag{42}
$$

For the field redefinitions, we consider the following structures:

$$
\begin{aligned}
\delta X^y \sim\ & \tilde{D}\tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + \tilde{\Omega}^y\tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + \tilde{\Omega}^y\tilde{\Omega}^y\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y + \tilde{D}\tilde{D}\tilde{\Omega}^yFFFF \\
& + FFF\tilde{D}\tilde{D}\tilde{D}F\tilde{D}X^y + \tilde{D}\tilde{\Omega}^yFFF\tilde{D}F + FF\tilde{D}F\tilde{D}\tilde{D}F\tilde{D}X^y + \tilde{\Omega}^yFFF\tilde{D}\tilde{D}F + \tilde{\Omega}^yFF\tilde{D}F\tilde{D}F \\
& + F\tilde{D}F\tilde{D}F\tilde{D}F\tilde{D}X^y + \tilde{D}\tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^yFF + F\tilde{D}\tilde{D}\tilde{D}F\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y \\
& + \tilde{\Omega}^y\tilde{D}\tilde{\Omega}^y\tilde{D}X^yFF + \tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^yF\tilde{D}F + \tilde{D}F\tilde{D}\tilde{D}F\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y \\
& + \tilde{\Omega}^yF\tilde{D}\tilde{D}F\tilde{D}X^y\tilde{D}X^y + \tilde{\Omega}^y\tilde{\Omega}^y\tilde{\Omega}^yFF + \tilde{\Omega}^y\tilde{\Omega}^yF\tilde{D}F\tilde{D}X^y \\
& + \tilde{\Omega}^y\tilde{D}F\tilde{D}F\tilde{D}X^y\tilde{D}X^y + \tilde{\Omega}^{\tilde{\mu}}\tilde{D}\tilde{\Omega}^{\tilde{\nu}}\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + \tilde{\Omega}^y\tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^{\tilde{\nu}}\tilde{D}X^y\tilde{D}X^y \\
& + FF\tilde{\Omega}^{\tilde{\mu}}\tilde{D}\tilde{\Omega}^{\tilde{\nu}}\tilde{D}X^y + \tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^{\tilde{\nu}}\tilde{D}F\tilde{D}F\tilde{D}X^y + \tilde{\Omega}^yF\tilde{D}F\tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^{\tilde{\nu}},
\end{aligned}
$$

$$\begin{aligned}\delta A^{\tilde{a}} \sim\, & FF\tilde{D}\,\tilde{D}\,\tilde{D}\,F\tilde{D}X^y\tilde{D}X^y + FFF\tilde{D}\,\tilde{D}\,\tilde{\Omega}^y\tilde{D}X^y + F\tilde{D}F\tilde{D}\,\tilde{D}\,F\tilde{D}X^y\tilde{D}X^y + FF\tilde{D}\,\tilde{D}\,F\tilde{\Omega}^y\tilde{D}X^y + FFF\tilde{\Omega}^y\tilde{D}\tilde{\Omega}^y \\ & + FF\tilde{D}F\tilde{D}\tilde{\Omega}^y\tilde{D}X^y + \tilde{D}F\tilde{D}F\tilde{D}F\tilde{D}X^y\tilde{D}X^y + F\tilde{D}F\tilde{D}F\tilde{\Omega}^y\tilde{D}X^y + FF\tilde{D}F\tilde{\Omega}^y\tilde{\Omega}^y + \tilde{D}\,\tilde{D}\,\tilde{D}\,F\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y \\ & + F\tilde{D}\,\tilde{D}\,\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + \tilde{D}\,\tilde{D}\,F\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + \tilde{D}F\tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + F\tilde{\Omega}^y\tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y \\ & + \tilde{D}F\tilde{\Omega}^y\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y + F\tilde{\Omega}^y\tilde{\Omega}^y\tilde{\Omega}^y\tilde{D}X^y + F\tilde{\Omega}^{\tilde{\mu}}\tilde{D}\tilde{\Omega}^{\tilde{\nu}}\tilde{D}X^y\tilde{D}X^y + F\tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^{\tilde{\nu}}\tilde{\Omega}^y\tilde{D}X^y + \tilde{D}F\tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^{\tilde{\nu}}\tilde{D}X^y\tilde{D}X^y, \\ \delta X^{\tilde{\mu}} \sim\, & \tilde{D}\,\tilde{D}\,\tilde{\Omega}^{\tilde{\mu}}\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + \tilde{D}\tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + \tilde{\Omega}^{\tilde{\mu}}\tilde{D}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y\tilde{D}X^y + \tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^y\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y \\ & + \tilde{D}\,\tilde{D}\,\tilde{\Omega}^{\tilde{\mu}}\tilde{D}X^y\tilde{D}X^yFF + \tilde{D}\tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^y\tilde{D}X^yFF + \tilde{D}\tilde{\Omega}^{\tilde{\mu}}\tilde{D}X^y\tilde{D}X^yF\tilde{D}F + \tilde{\Omega}^{\tilde{\mu}}\tilde{D}\tilde{\Omega}^y\tilde{D}X^yFF \\ & + \tilde{\Omega}^{\tilde{\mu}}\tilde{D}X^y\tilde{D}X^yF\tilde{D}\,\tilde{D}\,F + \tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^y\tilde{\Omega}^yFF + \tilde{\Omega}^{\tilde{\mu}}\tilde{D}X^y\tilde{D}X^y\tilde{D}F\tilde{D}F + \tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^y\tilde{D}X^yF\tilde{D}F + \tilde{\Omega}^{\tilde{\mu}}\tilde{\Omega}^{\tilde{\nu}}\tilde{\Omega}^y\tilde{D}X^y\tilde{D}X^y.\end{aligned} \tag{43}$$

Using the package xAct, one can construct all possible contractions in (42) and (43). Then replacing them in (31), using the parameters (29) for the four F, Ω couplings, going to the local frame to write the equation (31) in terms of the independent structures, and removing the terms that have eight and more fields, one finds the resulting linear algebraic equations have the following solution that involves only the 146 parameters of the effective action:

$$\begin{aligned}
& d_2\to\frac{1}{4}(8+6d_1-d_{16}),\quad d_3\to\frac{5}{2}-2d_1,\quad d_4\to-\frac{17}{4}-d_1-d_{16},\quad d_5\to1-4d_1,\\
& d_6\to-1-d_1,\quad d_7\to-8,\quad d_8\to\frac{3}{2}+2d_1,\quad d_9\to-4,\quad d_{10}\to-1-4d_1,\\
& d_{11}\to1+4d_1,\quad d_{12}\to4,\quad d_{13}\to1+4d_1,\quad d_{14}\to-1-4d_1,\quad d_{15}\to-1,\\
& d_{17}\to1+4d_1,\quad d_{18}\to-\frac{1}{2}-2d_1,\quad d_{19}\to1+2d_1,\quad d_{20}\to\frac{2-d_{16}}{4},\\
& d_{21}\to-1+4d_1+2d_{16},\quad d_{22}\to\frac{27}{2}+6d_1-5d_{16},\quad d_{23}\to-\frac{3}{2}-2d_1,\\
& d_{24}\to7+4d_1,\quad d_{25}\to13+8d_1-2d_{16},\quad d_{26}\to-8,\quad d_{27}\to-21-12d_1+4d_{16},\\
& d_{28}\to-7-4d_1+2d_{16},\quad d_{29}\to\frac{3}{2}+2d_1,\quad d_{30}\to2(6+6d_1-d_{16}),\quad d_{31}\to-4(1+d_1),\\
& d_{32}\to2(-2+d_{16}),\quad d_{33}\to1,\quad d_{34}\to1-4d_1,\\
& d_{35}\to-13-4d_1+4d_{16},\quad d_{36}\to\frac{1}{2}-6d_1-d_{16},\quad d_{37}\to\frac{17}{4}+6d_1-\frac{d_{16}}{2},\\
& d_{38}\to-9-20d_1,\quad d_{39}\to4+12d_1,\quad d_{40}\to2+4d_1-\frac{d_{16}}{2},\\
& d_{41}\to-17-4d_1+6d_{16},\quad d_{42}\to5+4d_1,\quad d_{43}\to5+4d_1,\quad d_{44}\to6+8d_1-2d_{16},\\
& d_{45}\to0,\quad d_{46}\to-4-8d_1,\quad d_{47}\to0,\quad d_{48}\to2+4d_1,\quad d_{49}\to-\frac{3}{2},\\
& d_{50}\to-\frac{5}{4}-d_1,\quad d_{51}\to\frac{5}{8}+\frac{3}{2}d_1,\quad d_{52}\to-\frac{5}{16}-\frac{3}{4}d_1,\quad d_{53}\to-\frac{5}{2},\quad d_{54}\to\frac{5}{4}+d_1,\\
& d_{55}\to-\frac{d_1}{2},\quad d_{56}\to1,\quad d_{57}\to0,\quad d_{58}\to5+4d_1,\\
& d_{59}\to-6-4d_1+2d_{16},\quad d_{61}\to-\frac{1}{2}-2d_1-d_{16}+d_{60},\quad d_{62}\to-1,\\
& d_{63}\to-2+d_{16},\quad d_{64}\to0,\quad e_1\to\frac{1}{16}(-7+4d_1+2d_{16}+2d_{60}),\quad e_2\to\frac{3}{8}(1+4d_1-2d_{16}+2d_{60}),\\
& e_3\to\frac{1}{8}(7-4d_1-2d_{16}-2d_{60}),\quad e_4\to-d_1+\frac{5}{4}(-3+d_{16}),\\
& e_5\to\frac{1}{8}(-7+4d_1+2d_{16}+2d_{60}),\quad e_6\to\frac{1}{16}(19+12d_1-14d_{16}+6d_{60}),
\end{aligned}$$

$$e_7 \to 0, \quad e_8 \to 0, \quad e_9 \to \frac{1}{4}(-13 + -4d_1 + 8d_{16} - 2d_{60}),$$
$$e_{10} \to \frac{1}{4}(-7 + 4d_1 + 2d_{16} + 2d_{60}), \quad e_{11} \to 2(-2 + d_{16}),$$
$$e_{12} \to -2 + d_{16}, \quad e_{13} \to 0, \quad e_{14} \to \frac{1}{8}(-67 - 12d_1 + 38d_{16} - 6d_{60}), \quad e_{15} \to 0,$$
$$e_{16} \to d_1 + \frac{1}{4}(-23 + 10d_{16} + 2d_{60}), \quad e_{17} \to \frac{1}{8}(-15 + 4d_1 + 6d_{16} + 2d_{60}),$$
$$e_{18} \to -6(-2 + d_{16}), \quad e_{19} \to \frac{1}{4}(-9 - 4d_1 + 6d_{16} - 2d_{60}),$$
$$e_{20} \to \frac{1}{8}(1 + 4d_1 - 2d_{16} + 2d_{60}), \quad e_{21} \to 2(-2 + d_{16}),$$
$$e_{22} \to \frac{1}{8}(17 + 4d_1 - 2d_{16} + 2d_{60}), \quad e_{23} \to 2(-2 + d_{16}), \quad e_{24} \to 0,$$
$$e_{25} \to \frac{1}{8}(15 - 4d_1 - 6d_{16} - 2d_{60}), \quad e_{26} \to \frac{1}{8}(-49 - 4d_1 + 10d_{16} - 2d_{60}),$$
$$e_{27} \to 6 - d_{16}, \quad e_{28} \to 2(-2 + d_{16}), \quad e_{29} \to \frac{1}{128}(1 + 4d_1 - 2d_{16} + 2d_{60}),$$
$$e_{30} \to \frac{1}{8}(9 + 4d_1 - 6d_{16} + 2d_{60}), \quad e_{31} \to \frac{1}{16}(1 + 4d_1 - 2d_{16} + 2d_{60}),$$
$$e_{32} \to \frac{1}{2} + 2d_1 - d_{16} + d_{60}, \quad e_{33} \to \frac{1}{8}(-11 + 20d_1 - 2d_{16} + 10d_{60}),$$
$$e_{34} \to \frac{1}{8}(15 - 4d_1 - 6d_{16} - 2d_{60}), \quad e_{35} \to \frac{1}{2}(-2 + d_{16}),$$
$$e_{36} \to \frac{5}{4} + d_1 - d_{16} + \frac{d_{60}}{2}, \quad e_{37} \to \frac{1}{16}(33 + 4d_1 - 18d_{16} + 2d_{60}),$$
$$e_{38} \to \frac{1}{8}(-17 - 4d_1 + 10d_{16} - 2d_{60}), \quad e_{39} \to \frac{1}{8}(-33 - 4d_1 + 18d_{16} - 2d_{60}),$$
$$e_{40} \to -2 + d_{16}, \quad e_{41} \to \frac{1}{8}(-149 - 20d_1 + 34d_{16} - 10d_{60}),$$
$$e_{42} \to d_1 + \frac{1}{4}(41 - 10d_{16} + 2d_{60}), \quad e_{43} \to \frac{1}{8}(-1 - 4d_1 + 2d_{16} - 2d_{60}),$$
$$e_{44} \to \frac{1}{8}(83 + 12d_1 - 14d_{16} + 6d_{60}), \quad e_{45} \to \frac{1}{8}(-63 + 4d_1 + 30d_{16} + 2d_{60}),$$
$$e_{46} \to \frac{1}{16}(15 - 4d_1 - 6d_{16} - 2d_{60}), \quad e_{47} \to \frac{1}{8}(51 + 12d_1 - 14d_{16} + 6d_{60}),$$
$$e_{48} \to \frac{1}{4}(-33 - 4d_1 + 6d_{16} - 2d_{60}), \quad e_{49} \to \frac{1}{32}(-51 - 12d_1 + 14d_{16} - 6d_{60}),$$
$$e_{50} \to 0, \quad e_{51} \to -\frac{1}{2}, \quad e_{52} \to 0, \quad e_{53} \to \frac{1}{8}, \quad e_{54} \to 0,$$
$$e_{55} \to 0, \quad e_{56} \to -2 + d_{16}, \quad e_{57} \to -2 + d_{16}, \quad e_{58} \to 0, \quad e_{59} \to 0,$$
$$e_{60} \to \frac{1}{8}(-1 - 4d_1 + 2d_{16} - 2d_{60}), \quad e_{61} \to \frac{1}{8}(51 + 12d_1 - 14d_{16} + 6d_{60}),$$
$$e_{62} \to -\frac{1}{2} - 2d_1 + d_{16} - d_{60}, \quad e_{63} \to 1 - \frac{d_{16}}{2}, \quad e_{64} \to 0,$$
$$f_1 \to -4 + 3d_{16}, \quad f_2 \to \frac{1}{2}(7 + 4d_1 - d_{16}),$$
$$f_3 \to \frac{1}{4}(9 + 4d_1 - 6d_{16} + 2d_{60}), \quad f_4 \to \frac{1}{4}(1 + 4d_1 + 2d_{16} + 2d_{60}),$$

$$f_5 \to 4(-2+d_{16}), \quad f_6 \to 2, \quad f_7 \to 0, \quad f_8 \to -2, \quad f_9 \to 8-2d_{16},$$
$$f_{10} \to -2, \quad f_{11} \to \frac{1}{2}, \quad f_{12} \to -2, \quad f_{13} \to -2+d_{16}, \quad f_{14} \to 6,$$
$$f_{15} \to -\frac{1}{2}, \quad f_{16} \to -1, \quad f_{17} \to -1, \quad f_{18} \to 0, \tag{44}$$

where the parameters, d_1, d_{16}, d_{60} remain unfixed. We have also imposed the T-duality constraint (31) in the static gauge [23] and found exactly the same result.

To check the above results, we compare them with the correction at order α'^2 to the Born-Infeld action that has been found by Wyllard by the boundary state method [18]. This correction, which involves all levels of the gauge field strength is the following:

$$S_{BI}^{(2)} = -T_p \int d^{p+1}\sigma \sqrt{-\det h_{ab}}\left[1+\frac{(2\pi\alpha')^2}{96} \times \left(-h^{da}h^{bc}h^{fe}h^{uz}S_{euab}S_{zfcd}+\frac{1}{2}h^{fe}h^{uz}S_{eu}S_{zf}\right)\right], \tag{45}$$

where

$$S_{euab} = \partial_e\partial_u F_{ab} + 2h^{cd}\partial_e F_{[a|c}\partial_{u|}F_{b]d},$$
$$S_{eu} = h^{ab}S_{euab},$$
$$h_{ab} = \eta_{ab} + F_{ab}, \tag{46}$$

and h^{ab} is the inverse of h_{ab}. The above action can be expanded to find the gauge field couplings at all levels of gauge field strength. The couplings at the level of four and six gauge field should be related to the corresponding couplings in (24) and (28) with the parameters (29) and (44), up to some total derivative terms and field redefinitions. Note that, since the parameters in the two actions are fixed, in order to compare the two actions, one should use all terms in the field redefintion $\mathcal{K}$, i.e., the dots in (10) should not be ignored. It has been verified in [18] that up to field redefinitions and total derivative terms, the two gauge field couplings are zero and the four gauge field couplings are the same as the corresponding couplings in (24) with parameters (29). We have compared the six gauge fields in the action (45) with the couplings in (28) with the parameters in (44). We have found they are the same up to some total derivative terms and field redefinitions provided that the unfixed parameter to be the following:

$$d_1 = \frac{1}{4}, \quad d_{16} = 4, \quad d_{60} = -1. \tag{47}$$

To perform this calculation, we insert the six gauge field coupling of (45) on the right hand side of (11), and the six independent gauge field couplings of (28) with the parameters (44) on the left hand side of (11). Then after imposing the Bianchi identity, one finds for some total derivative terms in $\mathcal{J}$ and field redefinitions in $\mathcal{K}$, the two sets of couplings are exactly the same if there is the above values for the unfixed parameters. This confirms that the couplings involving F, Ω that are fixed by the T-duality constraint (31), are consistent with the couplings involving F that are fixed by the boundary state method. Note that, neither the couplings in the action (45) nor the independent couplings in our scheme, include terms that have D_aF^{ab}. However, since the total derivative terms include terms that have D_aF^{ab}, in the comparison (11), one must include the field redefinition $\mathcal{K}$.

Hence, the six gauge field strengths and/or the second fundamental forms are fixed in the particular scheme that we have chosen in the previous section for the following parameters:

$$d_2 \to \frac{11}{8}, \quad d_3 \to 2, \quad d_4 \to -\frac{1}{2}, \quad d_5 \to 0, \quad d_6 \to -\frac{5}{4},$$
$$d_7 \to -8, \quad d_8 \to 2, \quad d_9 \to -4, \quad d_{10} \to -2, \quad d_{11} \to 2,$$
$$d_{12} \to 4, \quad d_{13} \to 2, \quad d_{14} \to -2, \quad d_{15} \to -1, \quad d_{17} \to 2,$$
$$d_{18} \to -1 d_{19} \to \frac{3}{2}, \quad d_{20} \to -\frac{1}{2}, \quad d_{21} \to 8, \quad d_{22} \to -5,$$
$$d_{23} \to -2, \quad d_{24} \to 8, \quad d_{25} \to 7, \quad d_{26} \to -8, \quad d_{27} \to -8,$$
$$d_{28} \to 0, \quad d_{29} \to 2, \quad d_{30} \to 7, \quad d_{31} \to -5, \quad d_{32} \to 4,$$

$$
\begin{aligned}
&d_{33}\to 1,\quad d_{34}\to 0,\quad d_{35}\to 2,\quad d_{36}\to -5,\quad d_{37}\to \frac{15}{4},\\
&d_{38}\to -14,\quad d_{39}\to 7,\quad d_{40}\to 1,\quad d_{41}\to 6,\quad d_{42}\to 6,\\
&d_{43}\to 6,\quad d_{44}\to 0,\quad d_{45}\to 0,\quad d_{46}\to -6,\quad d_{47}\to 0,\\
&d_{48}\to 3,\quad d_{49}\to -\frac{3}{2},\quad d_{50}\to -\frac{3}{2},\quad d_{51}\to 1,\quad d_{52}\to -\frac{1}{2},\\
&d_{53}\to -\frac{5}{2},\quad d_{54}\to \frac{3}{2},\quad d_{55}\to -\frac{1}{8},\quad d_{56}\to 1,\quad d_{57}\to 0,\\
&d_{58}\to 6,\quad d_{59}\to 1,\quad d_{61}\to -6,\quad d_{62}\to -1,\quad d_{63}\to 2,\quad d_{64}\to 0,\\
&e_{1}\to 0,\quad e_{2}\to -3,\quad e_{3}\to 0,\quad e_{4}\to 1,\quad e_{5}\to 0,\\
&e_{6}\to -\frac{5}{2},\quad e_{7}\to 0,\quad e_{8}\to 0,\quad e_{9}\to 5,\quad e_{10}\to 0,\\
&e_{11}\to 4,\quad e_{12}\to 2,\quad e_{13}\to 0,\quad e_{14}\to 11,\quad e_{15}\to 0,\\
&e_{16}\to 4,\quad e_{17}\to 1,\quad e_{18}\to -12,\quad e_{19}\to 4,\quad e_{20}\to -1,\\
&e_{21}\to 4,\quad e_{22}\to 1,\quad e_{23}\to 4,\quad e_{24}\to 0,\quad e_{25}\to -1,\\
&e_{26}\to -1,\quad e_{27}\to 2,\quad e_{28}\to 4,\quad e_{29}\to -\frac{1}{16},\quad e_{30}\to -2,\\
&e_{31}\to -\frac{1}{2},\quad e_{32}\to -4,\quad e_{33}\to -3,\quad e_{34}\to -1,\quad e_{35}\to 1,\\
&e_{36}\to -3,\quad e_{37}\to -\frac{5}{2},\quad e_{38}\to 3,\quad e_{39}\to 5,\quad e_{40}\to 2,\\
&e_{41}\to -1,\quad e_{42}\to 0,\quad e_{43}\to 1,\quad e_{44}\to 3,\quad e_{45}\to 7,\\
&e_{46}\to -\frac{1}{2},\quad e_{47}\to -1,\quad e_{48}\to -2,\quad e_{49}\to \frac{1}{4},\quad e_{50}\to 0,\\
&e_{51}\to -\frac{1}{2},\quad e_{52}\to 0,\quad e_{53}\to \frac{1}{8},\quad e_{54}\to 0,\quad e_{55}\to 0,\\
&e_{56}\to 2,\quad e_{57}\to 2,\quad e_{58}\to 0,\quad e_{59}\to 0,\quad e_{60}\to 1,\\
&e_{61}\to -1,\quad e_{62}\to 4,\quad e_{63}\to -1,\quad e_{64}\to 0,\\
&f_{1}\to 8,\quad f_{2}\to 2,\quad f_{3}\to -4,\quad f_{4}\to 2,\\
&f_{5}\to 8,\quad f_{6}\to 2,\quad f_{7}\to 0,\quad f_{8}\to -2,\quad f_{9}\to 0,\\
&f_{10}\to -2,\quad f_{11}\to \frac{1}{2},\quad f_{12}\to -2,\quad f_{13}\to 2,\quad f_{14}\to 6,\\
&f_{15}\to -\frac{1}{2},\quad f_{16}\to -1,\quad f_{17}\to -1,\quad f_{18}\to 0.
\end{aligned}
\tag{48}
$$

The independent couplings at order α'^2 in the previous section with the parameters (29) and (48) include only four and six gauge field strengths and/or the second fundamental forms. It would be interesting to extend these couplings to all levels of the gauge field strength. In the next section we study this extension.

IV. TOWARDS ALL GAUGE FIELD COUPLINGS

Using the fact that the corrections to Born-Infeld action at order α'^2 and at all levels of F_{ab} are known [18], one can easily extend these couplings to the covariant form by extending the partial derivatives in these couplings to the covariant derivatives and by extending the world-volume flat metric to the pull back of the bulk flat metric (2), i.e.,

$$
\begin{aligned}
S_p \supset -T_p\int d^{p+1}\sigma\sqrt{\det h_{ab}}\Bigg[1+\frac{(2\pi\alpha')^2}{96}\\
\times\left(-h^{da}h^{bc}h^{fe}h^{uz}S_{euab}S_{zfcd}+\frac{1}{2}h^{fe}h^{uz}S_{eu}S_{zf}\right)\Bigg],
\end{aligned}
\tag{49}
$$

where

$$
\begin{aligned}
S_{euab} &= D_e D_u F_{ab} + 2h^{cd} D_e F_{[a|c} D_{u|} F_{b]d}, \\
S_{eu} &= h^{ab} S_{euab}, \\
h_{ab} &= \tilde{G}_{ab} + F_{ab},
\end{aligned} \tag{50}
$$

and h^{ab} is inverse of h_{ab}.

One can expand the above action to find the four and six gauge field couplings with known coefficients. Then one can use them, and the couplings which involve Ω with the unknown coefficients that we have found them in a particular scheme in Sec. II, as the starting point for imposing the T-duality constraint (31), to find the unknown coefficients. We have done this calculation and found the following couplings for $\Omega\Omega\Omega\Omega$ and $\Omega\Omega\Omega\Omega FF$:

$$
\begin{aligned}
S_p \supset -\frac{(2\pi\alpha')^2}{96} T_p \int d^{p+1}\sigma \sqrt{-\det \tilde{G}_{ab}} [& 2\Omega_a{}^c{}_\mu \Omega^{ab\mu} \Omega_b{}^{d\nu} \Omega_{cd\nu} - 2\Omega_{ab}{}^\nu \Omega^{ab\mu} \Omega_{cd\nu} \Omega^{cd}{}_\mu \\
& + 9\Omega_a{}^{e\mu} \Omega_b{}^{f\nu} \Omega_{ce\nu} \Omega_{df\mu} F^{ab} F^{cd} - 8\Omega_a{}^{e\mu} \Omega_b{}^f{}_\mu \Omega_{ce}{}^\nu \Omega_{df\nu} F^{ab} F^{cd} \\
& + 3\Omega_a{}^{e\mu} \Omega_{be}{}^\nu \Omega_c{}^f{}_\mu \Omega_{df\nu} F^{ab} F^{cd} + 8\Omega_{ac}{}^\mu \Omega_b{}^{e\nu} \Omega_d{}^f{}_\nu \Omega_{ef\mu} F^{ab} F^{cd} \\
& + 3\Omega_b{}^{d\mu} \Omega_c{}^{e\nu} \Omega_d{}^f{}_\nu \Omega_{ef\mu} F_a{}^c F^{ab} - 5\Omega_b{}^{d\mu} \Omega_c{}^{e\nu} \Omega_d{}^f{}_\mu \Omega_{ef\nu} F_a{}^c F^{ab} \\
& - 10\Omega_{ac}{}^\mu \Omega_b{}^e{}_\mu \Omega_d{}^{f\nu} \Omega_{ef\nu} F^{ab} F^{cd} - 3\Omega_b{}^{d\mu} \Omega_c{}^e{}_\mu \Omega_d{}^{f\nu} \Omega_{ef\nu} F_a{}^c F^{ab} \\
& + \frac{1}{2}\Omega_c{}^e{}_\mu \Omega^{cd\mu} \Omega_d{}^{f\nu} \Omega_{ef\nu} F_{ab} F^{ab} - \Omega_{bc}{}^\mu \Omega_d{}^{f\nu} \Omega^{de}{}_\mu \Omega_{ef\nu} F_a{}^c F^{ab} \\
& + 2\Omega_{ac}{}^\mu \Omega_{bd}{}^\nu \Omega_{ef\nu} \Omega^{ef}{}_\mu F^{ab} F^{cd} + 5\Omega_b{}^{d\mu} \Omega_{cd}{}^\nu \Omega_{ef\nu} \Omega^{ef}{}_\mu F_a{}^c F^{ab} \\
& - \frac{1}{2}\Omega_{cd}{}^\nu \Omega^{cd\mu} \Omega_{ef\nu} \Omega^{ef}{}_\mu F_{ab} F^{ab} - \frac{1}{2}\Omega_{ac}{}^\mu \Omega_{bd\mu} \Omega_{ef\nu} \Omega^{ef\nu} F^{ab} F^{cd} - \frac{1}{2}\Omega_b{}^{d\mu} \Omega_{cd\mu} \Omega_{ef\nu} \Omega^{ef\nu} F_a{}^c F^{ab}].
\end{aligned} \tag{51}
$$

Note that the $\Omega\Omega\Omega\Omega$ terms above are exactly the second fundamental form correction that has been found in [27] up to terms that involve the trace of the second fundamental form, (see [20]), which are removed in our scheme.

We have also found the following couplings for the structures that include DF or $D\Omega$:

$$
\begin{aligned}
S_p \supset -\frac{(2\pi\alpha')^2}{96} T_p \int d^{p+1}\sigma \sqrt{-\det(\tilde{G}_{ab})} \Big[& \frac{1}{4} D_a F_{bc} D^a F^{bc} \Omega_{de\mu} \Omega^{de\mu} + D^a F^{bc} D^d F_b{}^e \Omega_{ae}{}^\mu \Omega_{cd\mu} \\
& - 2D^a F^{bc} D_b F_a{}^d \Omega_c{}^{e\mu} \Omega_{de\mu} - 5D^a F^{bc} D^d F_b{}^e \Omega_{ac}{}^\mu \Omega_{de\mu} \\
& - D^a F^{bc} D^d F_b{}^e F_{fh} F^{fh} \Omega_{ad}{}^\mu \Omega_{ce\mu} + \frac{3}{2} D^a F^{bc} D_a F^{de} F_{fh} F^{fh} \Omega_{bd}{}^\mu \Omega_{ce\mu} \\
& - D^a F^{bc} D^d F^{ef} F_b{}^h F_{eh} \Omega_{ad}{}^\mu \Omega_{cf\mu} - \frac{1}{2} D^a F^{bc} D^d F^{ef} F_a{}^h F_{dh} \Omega_{be}{}^\mu \Omega_{cf\mu} \\
& - 2D^a F^{bc} D_b F^{de} F_d{}^f F_f{}^h \Omega_{ae}{}^\mu \Omega_{ch\mu} - 16 D^a F^{bc} D^d F_b{}^e F_a{}^f F_f{}^h \Omega_{ch}{}^\mu \Omega_{de\mu} \\
& + 5D^a F^{bc} D^d F^{ef} F_b{}^h F_{eh} \Omega_{ac}{}^\mu \Omega_{df\mu} + 4D^a F^{bc} D^d F^{ef} F_a{}^h F_{be} \Omega_{ch}{}^\mu \Omega_{df\mu} \\
& - 6D^a F^{bc} D^d F^{ef} F_{ae} F_b{}^h \Omega_{ch}{}^\mu \Omega_{df\mu} - 2D^a F^{bc} D^d F^{ef} F_{ab} F_e{}^h \Omega_{ch}{}^\mu \Omega_{df\mu} \\
& - 4D^a F^{bc} D^d F^{ef} F_a{}^h F_{be} \Omega_{cf}{}^\mu \Omega_{dh\mu} + 6D^a F^{bc} D^d F^{ef} F_{ae} F_b{}^h \Omega_{cf}{}^\mu \Omega_{dh\mu} \\
& + 2D^a F^{bc} D^d F^{ef} F_{ab} F_e{}^h \Omega_{cf}{}^\mu \Omega_{dh\mu} + 6D^a F^{bc} D_a F_b{}^d F_e{}^h F^{ef} \Omega_{cf}{}^\mu \Omega_{dh\mu} \\
& + 4D^a F^{bc} D_b F_a{}^d F_e{}^h F^{ef} \Omega_{cf}{}^\mu \Omega_{dh\mu} - \frac{5}{2} D^a F^{bc} D_a F_b{}^d F_{ef} F^{ef} \Omega_c{}^{h\mu} \Omega_{dh\mu} \\
& + 3D^a F^{bc} D_b F_a{}^d F_{ef} F^{ef} \Omega_c{}^{h\mu} \Omega_{dh\mu} + 10 D^a F^{bc} D_b F^{de} F_a{}^f F_d{}^h \Omega_{ch}{}^\mu \Omega_{ef\mu} \\
& - 7D^a F^{bc} D^d F_b{}^e F_a{}^f F_d{}^h \Omega_{ch}{}^\mu \Omega_{ef\mu} + 3D^a F^{bc} D_a F^{de} F_b{}^f F_d{}^h \Omega_{ch}{}^\mu \Omega_{ef\mu} \\
& + 10 D^a F^{bc} D_b F^{de} F_d{}^f F_f{}^h \Omega_{ac}{}^\mu \Omega_{eh\mu} + 4D^a F^{bc} D_b F^{de} F_a{}^f F_f{}^h \Omega_{cd}{}^\mu \Omega_{eh\mu} \\
& - 2D^a F^{bc} D_a F^{de} F_b{}^f F_f{}^h \Omega_{cd}{}^\mu \Omega_{eh\mu} - 2D^a F^{bc} D_b F^{de} F_a{}^f F_d{}^h \Omega_{cf}{}^\mu \Omega_{eh\mu} \\
& - 11 D^a F^{bc} D_a F^{de} F_b{}^f F_d{}^h \Omega_{cf}{}^\mu \Omega_{eh\mu} + 6D^a F^{bc} D_b F^{de} F_{ad} F^{fh} \Omega_{cf}{}^\mu \Omega_{eh\mu}
\end{aligned}
$$

$$
\begin{aligned}
&-2D^aF^{bc}D_aF^{de}F_{bd}F^{fh}\Omega_{cf}{}^{\mu}\Omega_{eh\mu}-2D^aF^{bc}D^dF_b{}^eF_a{}^fF_{df}\Omega_c{}^{h\mu}\Omega_{eh\mu}\\
&+2D^aF^{bc}D^dF_b{}^eF_a{}^fF_c{}^h\Omega_{df}{}^{\mu}\Omega_{eh\mu}-5D^aF^{bc}D_aF^{de}F_b{}^fF_c{}^h\Omega_{df}{}^{\mu}\Omega_{eh\mu}\\
&+4D^aF^{bc}D_aF^{de}F_{bc}F^{fh}\Omega_{df}{}^{\mu}\Omega_{eh\mu}-14D^aF^{bc}D_aF_b{}^dF_c{}^eF^{fh}\Omega_{df}{}^{\mu}\Omega_{eh\mu}\\
&+6D^aF^{bc}D_bF_a{}^dF_c{}^eF^{fh}\Omega_{df}{}^{\mu}\Omega_{eh\mu}+3D^aF^{bc}D_bF_{ac}F^{de}F^{fh}\Omega_{df}{}^{\mu}\Omega_{eh\mu}\\
&+6D^aF^{bc}D^dF_b{}^eF_a{}^fF_d{}^h\Omega_{ce}{}^{\mu}\Omega_{fh\mu}+6D^aF^{bc}D^dF_b{}^eF_a{}^fF_c{}^h\Omega_{de}{}^{\mu}\Omega_{fh\mu}\\
&-4D^aF^{bc}D^dF_b{}^eF_{ad}F_c{}^f\Omega_e{}^{h\mu}\Omega_{fh\mu}+4D^aF^{bc}D_aF^{de}F_{bd}F_c{}^f\Omega_e{}^{h\mu}\Omega_{fh\mu}\\
&+12D^aF^{bc}D^dF_b{}^eF_{ac}F_d{}^f\Omega_e{}^{h\mu}\Omega_{fh\mu}-10D^aF^{bc}D_aF^{de}F_{bc}F_d{}^f\Omega_e{}^{h\mu}\Omega_{fh\mu}\\
&-4D^aF^{bc}D_bF_{ac}F_d{}^fF^{de}\Omega_e{}^{h\mu}\Omega_{fh\mu}+\frac{5}{2}D^aF^{bc}D^dF_b{}^eF_{ad}F_{ce}\Omega_{fh\mu}\Omega^{fh\mu}\\
&-\frac{11}{4}D^aF^{bc}D_aF^{de}F_{bd}F_{ce}\Omega_{fh\mu}\Omega^{fh\mu}+\frac{11}{8}D^aF^{bc}D_aF^{de}F_{bc}F_{de}\Omega_{fh\mu}\Omega^{fh\mu}\\
&+2D^aF^{bc}D_aF_b{}^dF_c{}^eF_{de}\Omega_{fh\mu}\Omega^{fh\mu}-\frac{5}{2}D^aF^{bc}D_bF_a{}^dF_c{}^eF_{de}\Omega_{fh\mu}\Omega^{fh\mu}\\
&+\frac{3}{4}D^aF^{bc}D_bF_{ac}F_{de}F^{de}\Omega_{fh\mu}\Omega^{fh\mu}-6D^aF^{bc}D_aF_b{}^dF_e{}^hF^{ef}\Omega_{cd}{}^{\mu}\Omega_{fh\mu}\\
&+8D^aF^{bc}D_aF^{de}F_b{}^fF_d{}^h\Omega_{ce}{}^{\mu}\Omega_{fh\mu}+2D^aF^{bc}D^dF^{ef}F_{ab}F_c{}^h\Omega_{de}{}^{\mu}\Omega_{fh\mu}\\
&-10D^aF^{bc}D^dF^{ef}F_a{}^hF_{bh}\Omega_{ce}{}^{\mu}\Omega_{df\mu}+D^aF^{bc}D^dF_b{}^eF_a{}^fF_d{}^h\Omega_{cf}{}^{\mu}\Omega_{eh\mu}\\
&-8D^aF^{bc}D^dF_b{}^eF_{ad}F^{fh}\Omega_{cf}{}^{\mu}\Omega_{eh\mu}+8D^aF^{bc}D^dF_b{}^eF_a{}^fF_{cf}\Omega_d{}^{h\mu}\Omega_{eh\mu}\\
&+2D^a\Omega^{bc\mu}D^d\Omega^{ef}{}_{\mu}F_{ad}F_{be}F_c{}^hF_{fh}-4D^aF^{bc}D^d\Omega^{ef\mu}F_{ab}F_e{}^hF_{fh}\Omega_{cd\mu}\bigg].
\end{aligned}
\tag{52}
$$

One may extend the above calculations to find the covariant couplings involving Ω at the higher levels of gauge fields that correspond to the action (49). Then the question arises, is it possible to find a compact expression for the covariant couplings involving Ω in terms of h^{ab}, as in (49)? We have checked that the couplings in (51) cannot be written in terms of $\Omega\Omega\Omega\Omega hhhh$. The reason may be related to the particular scheme that we have used for the independent couplings in Sec. II. Even though the couplings with the structure $\Omega\Omega\Omega\Omega$ and $\Omega\Omega\Omega\Omega FF$ are independent of the scheme, but their coefficients that are fixed by the T-duality are scheme dependent because the T-duality relates these parameters to the parameters of the couplings involving DF or $D\Omega$, which are scheme dependent.

V. CONCLUSION

In this paper we have found the independent world-volume couplings at order α'^2 involving four and six F, Ω and their covariant derivatives, in the normalization that F is dimensionless. We have found that there are 12 couplings at the four-field level and 146 couplings at the six-field level. The assumption that the effective action of the D_p-brane at the critical dimension is background independent is then used to find the parameters of the above independent couplings. That is, we have considered a particular background that has one circle. In this background, the effective action should satisfy the T-duality constraint (31). This constraint fixes all parameters in terms of only 8 parameters. We have shown that these parameters are consistent with the all-gauge-field corrections to the Born-Infeld action that have been found in [18]. This comparison also fixes the remaining 8 parameters. We have found the couplings in a particular scheme that is different from the scheme that has been used in [18].

We then considered the couplings that have no second fundamental form to be the same as the couplings found by Wyllard in [18] in which the partial derivatives are extended to the covariant derivatives and the flat world-volume metric to the pull back of the bulk flat metric. We have found the covariant couplings involving four and six F, Ω that are consistent with these couplings under the T-duality, i.e., (51) and (52). We could not succeed to extend them to all levels of F. The independent couplings (51) and (52) are in the particular scheme that we have considered in Sec. II. That may be the reason the covariant couplings (51) and (52) could not be written in a closed form in terms of h^{ab} to include all levels of F.

To find the covariant couplings at all levels of F, one may first need to find the independent covariant couplings involving Ω, in terms of h^{ab}. That is, one should consider all gauge invariant couplings involving DF, Ω and their covariant derivatives at order α'^2

$$S' = -\frac{(2\pi\alpha')^2}{96}T_p\int d^{p+1}\sigma\sqrt{-\det h_{ab}}\mathcal{L}'(DF,\ldots,\Omega,D\Omega,\ldots), \tag{53}$$

where the spacetime index of Ω is contracted with the spacetime metric $\eta_{\mu\nu}$ and the world-volume indices are contracted with h^{ab}. Then one adds to this action the total derivative terms and field redefinitions

$$\begin{aligned}\mathcal{J} &= -\alpha'^2T_p\int d^{p+1}\sigma D_a[\sqrt{-\det h_{ab}}\mathcal{I}^a]\\ \mathcal{K} &= -\alpha'^2T_p\int d^{p+1}\sigma\sqrt{-\det h_{ab}}\left[-\frac{1}{2}(h^{ab}-h^{ba})D_a\delta A_b+\frac{1}{2}(h^{ab}+h^{ba})\partial_aX^\mu D_b\delta X^\nu\eta_{\mu\nu}\right].\end{aligned} \tag{54}$$

In this case, one has to use the identity (6) to write the field redefinition terms produced by the last terms above, in terms of Ω. Doing the same steps as in Sec. II, one can find the independent couplings in terms of h^{ab}. Then one may expand them and impose the T-duality constraint (31) to find the parameters of the independent couplings. It would be interesting to find these covariant couplings in terms of h^{ab}, if they exist. We have done this calculation for the covariant couplings at order α' in the bosonic string theory that have been found in [23] up to the eight-field level, and found that there is no such covariant couplings in terms of h^{ab}. It is in accord with the observation that in the bosonic string theory, the world-volume gravity couplings on the D_p-brane in the presence of constant B field, in terms of inverse of $h_{ab}=\eta_{ab}+B_{ab}+F_{ab}$ cannot be written in a covariant form at order α' [28].

The D_p-brane action in type II superstring theory has also the Wess-Zumino coupling that at the lowest order of α' involves the R-R potential and F_{ab} [29]. The α'^2 corrections to this action involving only F and its covariant derivatives, have been found in [18]. The corrections that involve only Ω have been found in [27]. It would be interesting to use the T-duality constraint to find the correction that involves F, Ω and their covariant derivatives at order α'^2, as we have done in this paper for the DBI action.

We have used the field redefinitions to remove the couplings that have D_aF^{ab} or $G^{ab}\Omega_{ab}{}^{\nu}$. These couplings are not produced by the disc-level S-matrix elements of massless open string vertex operators either. However, the second fundamental form in nontrivial bulk background has gravity contribution as well as the transverse scalar contributions. If one uses the bulk field redefinitions to write the bulk effective action in a fixed scheme, then one would not be allowed to use the field redefinitions to remove the trace of the second fundamental form from the world-volume effective action of the D$_p$-brane. Hence, the world-volume couplings involving the trace of the second fundamental form should be reproduced by the disc-level S-matrix elements of massless closed string vertex operators. In fact, such couplings have been found in the bosonic string theory at order α' in [30] and in the superstring theories at order α'^2 in [27]. Similarly, the couplings involving D_aF^{ab} have closed string contribution through the standard replacement of F by $F+B$ in the world-volume effective actions. If one uses the bulk field redefinitions to write the bulk effective action in a fixed scheme, then one would not be allowed to use the field redefinitions to remove D_aB^{ab} from the world-volume effective action of the D$_p$-brane. Hence such couplings may be reproduced by the disc-level S-matrix elements of massless closed string vertex operators. Alternatively, one may find the couplings involving D_aB^{ab} and the second fundamental form by studying the T-duality constraint on the world-volume couplings of massless closed string states [31].

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