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Entanglement dynamics in the atom-cavity system with atom quasi-random walk behavior

M. Mohammadi[1] · S. Jami[1] · M. Khazaei Nezhad[2]

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Abstract
Recently, it was demonstrated that an atom exhibits quasi-random walk behavior in an atom-cavity system when two longitudinal and transverse laser pumps are simultaneously excited by the system. The longitudinal pump irradiates the cavity, whereas the transverse pump irradiates an atom, directly. The model presented in this study contains a two-level atom in an electrodynamic cavity stimulated by two longitudinal and transverse laser pumps. The longitudinal laser frequency was tuned to excite the electrical (internal) state of an atom. We investigated the entanglement dynamics between the cavity field and internal atomic modes, and the von Neumann entropy measure was used to this end. The atomic quasi-random walk behavior and cavity dissipation were considered in this study. This study was conducted for different atomic states, two regimes of the strong, and the weak coupling. Our numerical results show that atomic random-walk motion can help us to enhance the amount of entanglement between the internal atomic modes and cavity fields for a long time.

Keywords Cavity QED · Quasi-random walk · Entanglement · Von Neumann entropy measure

1 Introduction

Quantum entanglement is a fascinating research topic in quantum mechanics that differs from classical physics (Chuang and Nielsen 2010; Mahon 2008; Bell 1964). Until 1990, quantum entanglement was considered as a strange curiosity with no practical applications. In 1991, Ekert introduced the first application of quantum entanglement (Ekert 1992). Later, Bennett and Wisner showed that two entangled parts can communicate between two classical bits by sending only one qubit (for instance, a two-dimensional quantum system in Hilbert space (Bennett and Wiesner 1992)). Entangled states are very important both conceptually and practically and are used in the processing of quantum information such as quantum cryptography, quantum teleportation, quantum computing and quantum

✉ S. Jami
safa.jami@iau.ac.ir

1 Department of Physics, Mashhad Branch, Islamic Azad University, Mashhad, Iran

2 Faculty of Science, Ferdowsi University of Mashhad, Mashhad, Iran

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communication (Bennett et al. 1996, 1993; Deutsch and Ekert 1998; Gisin et al. 2002; Imamog et al. 1999; Saharia et al. 2019; Ikram et al. 2017, 2015; Ul-Islam et al. 2016; Imran et al. 2016). In the presence of external fields, one can change the degrees of internal and external freedom of the atom. By adjusting the interaction time or by choosing the appropriate initial conditions for the atom or field one can generate the entangled or hyper-entangled states (Ali et al. 2022a, 2022b) which further can be utilized for teleportation (Nawaz et al. 2017; Poldy et al. 2008) and information distribution over the quantum networks (Leibrandt et al. 2009; Hinkel et al. 2015).

A random walk is a path involving random and sequential steps using a random simulation algorithm. The random walk includes a walker and a random value generator that walker follows its path in random steps. It is also frequently used and powerful for computational models that have been used in computer science, physics, economics, and psychology (Sames et al. 2014; Walther et al. 2006; Jaynes and Cummings 1963; Vedral et al. 1997; Baghshahi et al. 2015). Random walk is a basic model for stochastic processes over time and divides into the classical and quantum random walk.

One of the most investigated systems that are used to generate and maintain entanglement is the atom-cavity system. The atom-cavity system consists of one or atomic ensembles via tailored cavity modes (Uhlmann 2000; Vidal and Werner 2002). The atom-cavity system can be used for information processes and quantum computing [29–34]. Also one can create optimal states that are favorable for quantum information. For information processes and quantum computing, it is desirable to have more coherence between subsystems. In other words, one needs the entanglement with a long time and high maximum value, and high stability. In the presence of external fields, one can change the degrees of internal and external freedom of the atom. In most previous works, the electric (internal) levels of the atom were excited with an internal cavity field [35–38]. Meanwhile, by changing the atomic position in the cavity, the atomic motion (external mode) must be considered. It was demonstrated that the atom shows quasi-random walk (QRW) behavior in the atom-cavity system when two longitudinal and transverse laser pumps are excited the system, simultaneously [34]. The laser pumps light interference in the cavity that provides the time-depended electrical potential for the atom if two laser frequencies have small mismatched. The atomic QRW disappeared in the absence of a second transverse laser pump called as Non-Quasi-Random Walk (NQRW).

In this paper, first, we review the QRW of the atom in the cavity and then study the entanglement dynamics between internal atomic modes and the cavity field by considering QRW and dissipation in the cavity. We used the von Neumann entropy measure to this end. Our numerical results show that considering the QRW can help us to enhance the entanglement value between the internal atomic modes and cavity fields.

The paper is organized as follows: In the second section, we introduce the model and the entanglement measure that will be used in this study to investigate the entanglement dynamic in our system. In the third section, we present numerical evidence showing the occurrence of the QRW behavior. In the fourth section, we investigate numerical results and study the changes and find the optimal state of entanglement, we apply the electromagnetic field within the cavity in the Fock state and atom in-ground, excited, and the superposition of ground and excited states, as well as the initial entangled state between the atom and cavity field. We compared the results of the entanglement dynamic for two QRW and NQRW cases and two regimes of the strong and the weak coupling. In addition, we investigate the effect of changing the amplitude of the transverse pump on the amount of entanglement in two regimes of the strong and the weak coupling. The last section is devoted to the general results and discussions.

2 Model and entanglement dynamic

2.1 Model

The atom-cavity system, presented in this work, consists of an atom within the cavity that is stimulated by two laser pumps (See Fig. 1). We consider two energy levels for an atom. When an atom is excited with an external light field, the light may be absorbed with the electron in-ground ($|g\rangle$) state and jump to the excited ($|e\rangle$) state. In the full quantum model, the atom and external field are quantized and the interaction between fields and atom can be described with a complete quantum Jaynes-Cummings model. Therefore, the Hamiltonian of the supposed system can be written as follows [38]:

$$H_s = H_0 + H_{a-f} + H_P. \tag{1}$$

The Hamiltonians (under the rotating wave approximation) consist of the free part of the atom and cavity field H_0, the interaction between atom and cavity field H_{a-f}, and the exciting pump's Hamiltonian H_p:

$$H_0 = \frac{p^2}{2m} + \frac{\hbar\omega_a}{2}\hat{\sigma}^z + \hbar\omega_c\hat{a}^\dagger\hat{a}, \tag{2}$$

$$H_{a-f} = \hbar f(x)\left(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger\right), \tag{3}$$

$$H_p = \hbar\eta_L\left(\hat{a}^\dagger e^{-i\omega_L t} + \hat{a}e^{i\omega_L t}\right) + \hbar\eta_T\left(\hat{\sigma}^+ e^{-i\omega_T t} + \hat{\sigma}^- e^{i\omega_T t}\right). \tag{4}$$

In the above equations, p and m are the atomic momentum and mass, respectively. ω_a and ω_c are the atomic transition and the cavity field frequencies, respectively. We define the coupling function $f(\mathrm{x}) = \lambda\cos(\mathrm{kx})$ where λ is the maximum coupling rate between the atom and the electromagnetic field. It depends explicitly on the atomic position and is maximal when the atomic dipole moment and the linear light polarization are in parallel and k is the wave number. η_L and ω_L (η_T and ω_T) are the longitudinal (transverse) laser pump amplitude and frequency, respectively. $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators, respectively, $\hat{\sigma}^z$ is the z-component of the Pauli matrix, and $\hat{\sigma}^+$ and $\hat{\sigma}^-$ are the creation and the annihilation of the atomic transition.

In order to get rid of the explicit time dependence, we transform the Hamiltonian to a frame rotating with ω_L. The Hamiltonian now reads:

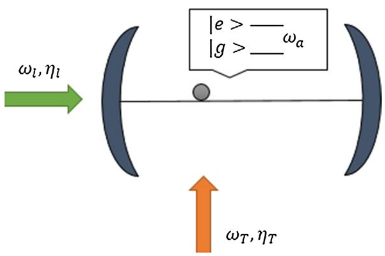

Fig. 1 A schematic view of the atom-cavity system

$$
\begin{aligned}
H =& \frac{p^2}{2m} + \frac{\hbar\Delta_a}{2}\hat{\sigma}^z + \hbar\Delta_c \hat{a}^\dagger \hat{a} + \hbar\lambda\cos(kx)\left(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger\right) \\
&+ \hbar\eta_L\left(\hat{a}^\dagger + \hat{a}\right) + \hbar\eta_T\left(\hat{\sigma}^+ e^{-i\delta_T t} + \hat{\sigma}^- e^{i\delta_T t}\right)
\end{aligned}
\tag{5}
$$

where, $\Delta_c = \omega_L - \omega_c$ $\Delta_c = \omega_L - \omega_c$, $\Delta_a = \omega_L - \omega_a$ and$\delta_T = \omega_L - \omega_T$.

One needs to find the evolution of the system by solving the Lindblad equation as (Chuang and Nielsen 2010):

$$
\begin{aligned}
\dot{\rho} =& -\frac{i}{\hbar}[H, \rho] + \frac{\sqrt{\kappa}}{2}\left(2a\rho\hat{a}^\dagger - \hat{a}^\dagger a\rho - \rho\hat{a}^\dagger a\right) \\
&+ \frac{\sqrt{\gamma}}{2}\left(2\hat{\sigma}^-\rho\hat{\sigma}^+ - \hat{\sigma}^+\hat{\sigma}^-\rho - \rho\hat{\sigma}^+\hat{\sigma}^-\right),
\end{aligned}
\tag{6}
$$

where, γ and κ are the atomic and cavity decay rates, respectively.

2.2 Entanglement dynamics

There are different measures to calculate the entanglement dynamics such as von Neumann entropy [39, 40], concurrence [41], negativity [42], and entanglement of formation [43, 44]. Using von Neumann entropy, one can calculate the entanglement between the atom and the field within the cavity. Meanwhile, the reduced von Neumann entropy, as a measure of entanglement between subsystems, satisfies the general conditions consisting of Schmidt decomposition, local invariance, continuity, and additivity (Chuang and Nielsen 2010). To this end, after solving the Lindblad equation to find the density matrix of the system at each time, according to the von Neumann entropy, the reduced entropy of the atom is calculated through the corresponding reduced density operator by:

$$
S_A(t) = -Tr\left(\rho_A(t)\log(\rho_A(t))\right), \tag{7}
$$

where, ρ_A(t)=$Tr_F\rho$(t) is the reduced density matrix of the atom and ρ(t) is the density matrix of the system at each time. Next, the entanglement is achieved using the following equation:

$$
S_A(t) = -\sum_{i=1}^{2}\xi_i(t)\log\xi_i(t). \tag{8}
$$

The quantities ξ_i are eigenvalues of the matrix ρ_A(t). In the following, the entanglement value is calculated in both the QRW and the NQRW cases.

As mentioned, in the NQRW case, the internal mode of the atom interacts with the cavity field, which is excited with a longitudinal laser pump, only. But in the QRW case, both laser pumps are presented and the atom is excited with a transverse laser pump, directly. In the next section, we use the QUTIP package to solve the Lindblad equation and obtain the entanglement dynamics, numerically [45, 46].

3 Evidence for quasi-random walk

In this section, according to the equations mentioned for the motion of the atom, we study the changes in the position of the atom. The complete equations of motion for atom and field are (including the dissipative dynamics of the field mode at rate κ and the atomic de coherence at rate γ) [35]:

$$\dot{\alpha} = \left(-\kappa + i\Delta_c\right)\alpha - \lambda \cos(kx)\beta + \eta_L, \tag{9}$$

$$\dot{\beta} = \left(-\gamma + i\Delta_a\right)\beta + \lambda \cos(kx)\alpha + \eta_T e^{i\delta_T t}, \tag{10}$$

$$\dot{x} = 2\omega_r p, \tag{11}$$

$$\dot{p} = 2\lambda k \sin(kx)\mathrm{Im}(\alpha^* \beta). \tag{12}$$

here, $\alpha(\beta)$ is an average of the cavity field (atomic polarization). Also, $\omega_r = \frac{\hbar k^2}{2m}$ and called the recoil frequency of the atom. We present numerical evidence showing the occurrence of a QRW behavior of the atom (see Fig. 2). This is achieved by fixing the set of parameters (all of these parameters were used in next section): $\Delta_a = -15\kappa$, $\Delta_c = -15\kappa$, $\eta_L = 10\kappa$, $\eta_T = (5.5, 10, 19)\,\kappa$, $\delta_T = 0.1\pi\,\kappa$, $\gamma = (0.2, 0.9)\,\kappa$, $\lambda = (0.6, 3)\,\kappa$, $k = 2\pi$, $x_0 = 3.5 \times 10^{-3}$, $p_0 = 0$ and recoil frequency $\omega_r = 1\kappa$. In the following, we set $\kappa = 0.1$ for numerical simulations and normalize the time in units ofκ^{-1}.

4 Numerical results and discussion

In this section, we investigate entanglement in two regimes of the strong and the weak coupling. As known, in the strong coupling regime, the coupling rate between the atom and the cavity field is larger than any dissipation rate in the system ($\lambda \gg \gamma, \kappa$). Also, the effect of changes in the transverse pump amplitude on the amount of entanglement was investigated. The initial state of the atom can be tailored in the ground ($|g\rangle$), excited ($|e\rangle$), the

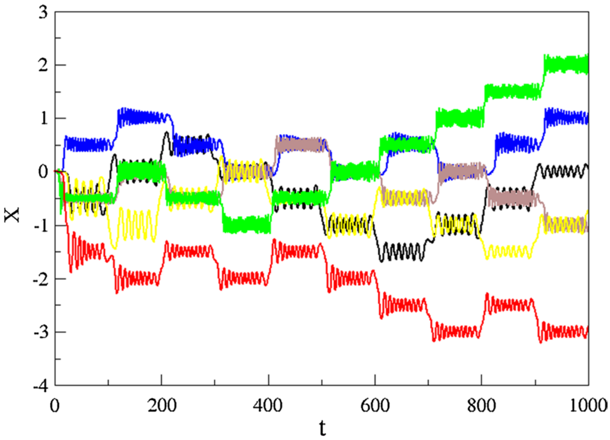

Fig. 2 Quasi-random walk trajectories: (Blue curve) $\lambda = 3\kappa$, $\gamma = 0.2\kappa$, $\kappa = 0.1$,$\eta_L = 10\kappa$,$\eta_T = 5.5\kappa$, (green curve) $\lambda = 3\kappa$, $\gamma = 0.2\kappa$, $\eta_L = 10\kappa$, $\eta_T = 19\kappa$ (brown curve) $\lambda = 3\kappa$, $\gamma = 0.2\,\kappa$, $\eta_L = \eta_T = 10\kappa$, (black curve) $\lambda = 0.6\,\kappa$, $\gamma = 0.9\,\kappa$,$\eta_L = 10\kappa$,$\eta_T = 5.5\kappa$ (yellow curve) $\lambda = 0.6\kappa$, $\gamma = 0.9\kappa$, $\eta_L = \eta_T = 10\kappa$, (red curve) $\lambda = 0.6\kappa$, $\gamma = 0.9\kappa$,,$\eta_L = 10\kappa$,$\eta_T = 19\kappa$

superposition of ground and excited as well as the initial entangled state between the atom and the cavity fields ($\frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$). In addition, the cavity fields can be expanded of Fock, and we cut off the field dimensional in 16 because the results are similar and higher dimensional would increase the computational time. First, we compare the amount of entanglement with the reduced von Neumann entropy measurement in QRW and NQRW cases in the strong coupling regime. This is achieved by fixing the set of parameters to: $\Delta_a = -15\kappa$, $\Delta_c = -15\kappa$, $\eta_L = 10\kappa$, $\eta_T = 19\kappa$, $\delta_T = 0.1\pi\kappa$, $\gamma = (0.2, 0.9)\kappa$, $\lambda = (0.6, 3)\kappa$, $\kappa = 0.1$, $k = 2\pi$, $x_0 = 3.5 \times 10^{-3}$, $p_0 = 0$ and recoil frequency $\omega_r = 1\kappa$.

The result of the entanglement for the strong coupling regime is shown in Fig. 3. It is obvious from this figure that, by adding QRW atomic motion, the maximum entanglement is increased. In addition, over time, the amount of entanglement tends to be a constant value, which this constant value is higher in the QRW case with respect to the NQRW case.

Also, when the atom is in the ground state, the maximum amount of entanglement is higher between these different initial states. In the initial entangled state, the amount of entanglement decreases to a constant value of 0.8. The amount of entanglement in the QRW case retains for a long time but in the NQRW, the entanglement decreases and then tends to zero value.

In the following, we investigate the amount of entanglement in QRW and NQRW cases in the weak coupling regime. The results are shown in Fig. 4.

It can be seen from this figure that, when the time goes up, the amount of entanglement is increased, and the maximum amount of entanglement in the QRW case is about 1. In both cases, the amount of entanglement tends to be a constant value and this constant value is higher in the QRW case with respect to the NQRW case. Also, when the atom is in the ground state, the maximum amount of entanglement is higher, compared to other states. In the initial entangled state, the amount of entanglement decreases to a constant value of 0.7.

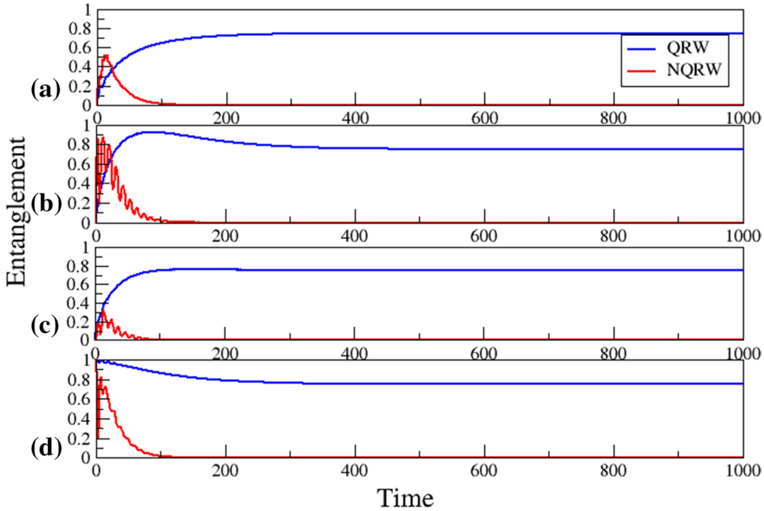

Fig. 3 Comparison of entanglement with the reduced von Neumann entropy measurement in the QRW (blue) and the NQRW (red) cases in the strong coupling regime, $\lambda = 3\kappa$, $\gamma = 0.2\kappa$, $\kappa = 0.1$when the atomic initial states are in **a**) the superposition of ground and excited states **b**) the ground state **c**) the excited state **d**) the entangled state

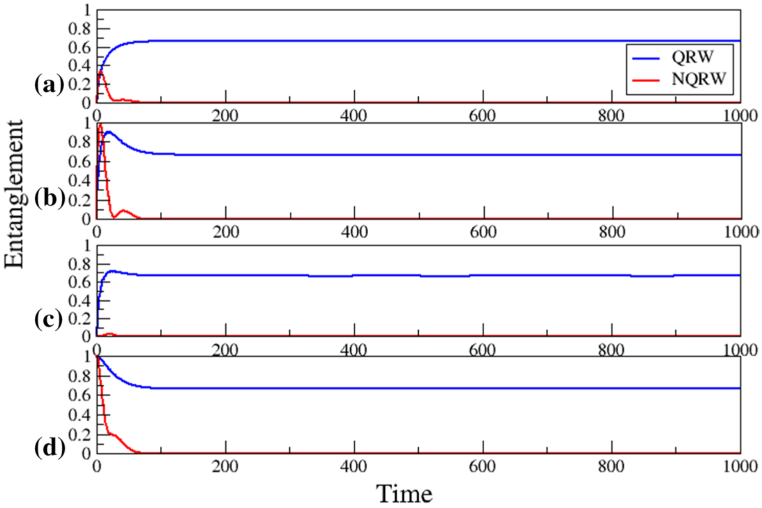

Fig. 4 Comparison of entanglement with the reduced von Neumann entropy measurement in the QRW (blue) and the NQRW (red) cases in the weak coupling regime, $\lambda = 0.6\kappa$, $\gamma = 0.9\kappa$, $\kappa = 0.1$, when the atomic initial states are in **a**) the superposition of ground and excited states **b**) the ground state **c**) the excited state **d**) the entangled state

The amount of entanglement in the QRW case retains this value for a long time but in the NQRW, the entanglement decreases and then tends to zero value.

As shown in the previous section, the presence of the transverse laser pump (via small frequency mismatched with respect to the longitudinal pump) led to atomic QRW motion in the cavity. In the next step, we want to investigate the effect of changing the amplitude of the transverse pump on the entanglement dynamics in two regimes of strong and weak coupling. For this purpose, we consider three values η_T=(5.5κ, 10κ, 19κ) for the transverse pump amplitude. The results are depicted in Fig. 5. This shows that with increased transverse pump amplitude, the amount of entanglement increases and reaches the maximum value and decreases to a constant value that preserves this amount for a long time. When the atom is in the ground state, the maximum amount of entanglement is higher with respect to the other states.

The results for the entanglement value for the QRW case and the weak coupling regime, in which we change the transverse pump amplitude, are depicted in Fig. 6. These results confirm the results of Fig. 5, which is that with the increase of the transverse pump amplitude, the amount of entanglement increases and then tends to a constant value, with the difference that the constant value of entanglement is lower than in the case of the strong coupling regime.

5 Conclusion

We studied the entanglement in an atom-cavity system by considering the atomic quasi-random walk motion. We investigated different states to find the optimal state. The results show that considering the QRW enhances the amount of entanglement between the internal atomic modes and cavity fields. In addition, over time, the amount of entanglement tends to be a constant value, which this constant value is higher in the QRW case. We show that

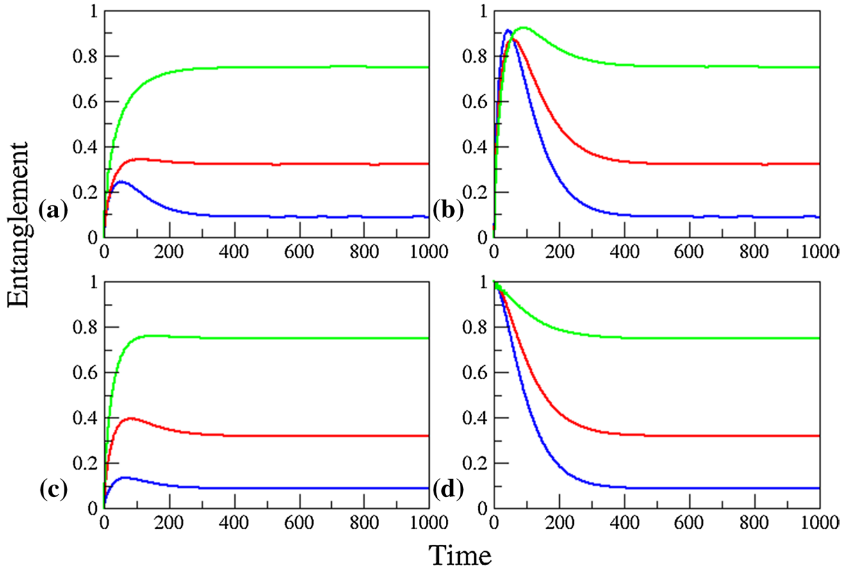

Fig. 5 Comparison of entanglement for different values of transverse pump amplitude in the QRW case and the strong coupling regime. $\lambda = 3\kappa$, $\gamma = 0.2\kappa$, $\eta_L = 10\kappa$ for all diagrams and $\eta_T = 5.5\kappa$ (blue) $\eta_T = 10\kappa$ (red), $\eta_T = 19\kappa$ (green). **a**) The superposition of ground and excited states **b**) the ground state **c**) the excited state **d**) the entangled state

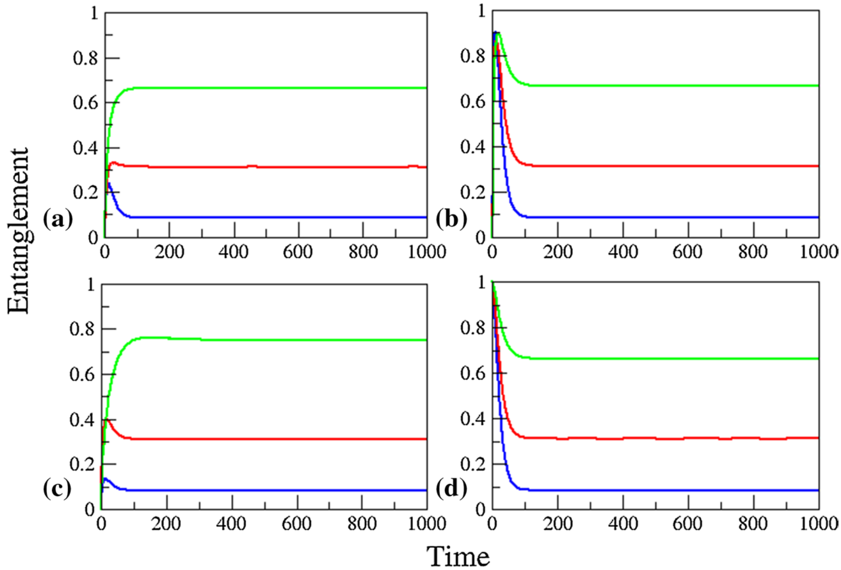

Fig. 6 Comparison of entanglement for different values of transverse pump amplitude in the QRW case and the weak coupling regime. $\lambda = 0.6\kappa$, $\gamma = 0.9\kappa$, $\eta_L = 10\kappa$ for all diagrams and $\eta_T = 5.5\kappa$ (blue) $\eta_T = 10\kappa$ (red), $\eta_T = 19\kappa$ (green): **a**) The superposition of ground and excited states **b**) the ground state **c**) the excited state **d**) the entangled state

with increased transverse pump amplitude, the amount of entanglement reaches the maximum value and decreases to a constant value, and almost preserves this amount for a long time. This maximum value and constant value in the strong coupling regime and η_T= 19κ, are higher. It seems that the atomic initial ground in the strong coupling regime has more favorable results compared to other states since it can better show distinctions between the QRW and the NQRW states. As known, in quantum information and quantum computation, it is desirable to have more coherence between the subsystems. In other words, one needs to create and preserve entanglement with a long time, high maximum value, and more stability. Furthermore, in high temperatures (with respect to Absolute zero Kelvin temperature) the atomic motion must be considered and cannot be ignored. Therefore, we hope our results help to tailor more efficient initial quantum states and open up new research topics in high-temperature quantum information processing.

Author contributions All authors reviewed the manuscript. They work on it with together as the same.

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Declarations

Competing interests The author declares no competing interests.

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