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The object allocation problem with favoring upper ranks

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Abstract

We introduce *(strict) favoring upper ranks*, which is an extension of *favoring higher ranks* for random assignments. We demonstrate that *ex post favoring ranks* implies (strict) favoring upper ranks, and envy-freeness implies favoring upper ranks. Moreover, for at least four agents, no mechanism satisfies strict favoring upper ranks and either equal division lower bound or equal treatment of equals and lower invariance. Finally, we proved that the (modified) eating algorithm provides a random assignment that is (strict) favoring upper ranks.

KEYWORDS

ex post favoring ranks, favoring higher ranks, favoring upper ranks, random assignment problem

JEL CLASSIFICATION

D61, D63, D82

1 | INTRODUCTION

An assignment problem allocates indivisible objects, without monetary transfer, to agents who have reported their ordinal preferences. One of the main properties of an assignment is fairness, mainly in the sense of envy-freeness. An assignment is envy-free when no agent prefers the object of another agent to hers. Nevertheless, it is almost impossible to assign indivisible objects, ex post, in a deterministic setting in a way that no agent envies another. Therefore, lotteries have been used to attain an envy-free assignment from an ex ante perspective.

There is a vast literature on random allocation of objects to agents that is usually attributed to Hylland and Zeckhauser (1979). Budish et al. (2013) extended the randomized assignment problem to multiunit allocation problems. Nonetheless, their generalization might rule out many real-world applications. Akbarpour and Nikzad (2020) generalized this approach to a considerably broader class of allocation problems and demonstrated that one could accommodate many more constraints by treating some of the constraints as goals.

There is some work on object allocation problems that respect agents with higher ranks for objects. *Favoring higher ranks*, à la Kojima and Ünver (2014) and Doğan and Klaus (2018), takes into account whether the envy is justified, in the sense that which agent is more rightful to get (has a higher rank for) a contested object. Ramezanian and Feizi (2021a) applied the idea for deterministic assignments and extended it to random assignments called *ex post favoring ranks*. Note that favoring higher ranks is not a theoretically founded fairness concept but rather a welfare criterion. Nevertheless, one can interpret it as some kind of fairness notion.

Featherstone (2020) incorporated the idea of respect for rank into a notion of efficiency. A random assignment rank dominates another one whenever the expected number of agents who received their first-best objects, first- and second-best objects, first-, second-, and third-best objects, and so on under the former is weakly higher than that of the latter. A random assignment is rank efficient if it is not rank-dominated by any other random assignment. Feizi (2022) introduced ex post rank efficiency, which is representable as a lottery over rank efficient deterministic assignments, and showed that rank efficiency implies ex post rank efficiency.

This paper presents another generalization of favoring higher ranks for random assignments, called *(strict) favoring upper ranks*, which is weaker than ex post favoring ranks. If a random assignment is not (strict) favoring upper ranks, then in any of its decompositions, there exists a deterministic assignment that is not favoring higher ranks. We assessed how (strict) favoring upper ranks is associated with notions of fairness and efficiency for random assignments. Moreover, we proved that the (modified) eating algorithm provides a (strict) favoring upper ranks random assignment.

The paper is organized as follows. In Section 2, we recall the standard model and axioms of random assignments. Section 3 introduces (strict) favoring higher ranks. In Section 4, we present our results. Section 5 introduces mechanisms that provide a (strict) favoring upper ranks random assignment. Finally, Section 6 concludes.

2 | MODEL

Let A be a finite set of objects which should be assigned to a finite set of agents, N, with $|A| = |N| = n$. Each agent $i \in N$ has a complete, transitive, and antisymmetric *strict preference relation* $\succ_i$ over A, while $a \succsim_i b$ means that agent i prefers a to b or she is indifferent between them. We denote a preference profile *by* $\succ \equiv (\succ_i)_{i\in N}$ and the domain of those preferences for each agent by F. Also, let $L(\succ_i, a) = \{b \in A | a \succ_i b\}$ be the strict lower contour set of a in $\succ_i$, $SU(\succ_i, a) = \{b \in A | b \succ_i a\}$ be the strict upper contour set of a in $\succ_i$, and $U(\succ_i, a) = \{b \in A | b \succsim_i a\}$ be the upper contour set of a in $\succ_i$. Each agent $i \in N$ has a ranking over any object $a \in A$, which we represent by $rank(a, \succ_i)$, where $rank(a, \succ_i) = |U(\succ_i, a)| + 1$.

We represent a random assignment by a *bistochastic matrix* $P = [p_{ia}]_{i\in N, a\in A}$, with agents on rows and objects on columns, where p_{ia} is the probability of assigning object a to agent i. We denote the domain of random assignments by R. A random allocation for some agent $i \in N$, P_i, is a probabilistic distribution over all objects in A where the sum of probabilities of assigning objects to the agent i equals 1. A deterministic assignment, $\Pi = [\Pi_{ia}]_{i \in N, a \in A}$, is a particular case of random assignment where its entries are all either 0 or 1.

Upon enumerating objects in A for agent i from best to worst according to $a_{i,1} \succ_i a_{i,2} \succ_i \cdots \succ_i a_{i,n}$, where $a_{i,k}$ is the kth best object of agent i, we define $u_{ir}^P = \sum_{k=1}^{r} p_{ia_{i,k}}$ to be the summation of probabilities of receiving the first r best objects of agent i in the random assignment P. Given a preference relation $\succ_i$ on A, and random assignments $P, Q \in R$, the allocation of agent i in P stochasticlly dominates her allocation in Q, that is, $P_i \succ_i^{sd} Q_i$ (where the stochastic dominance relation is denoted by $\succ_i^{sd}$), if and only if $u_{ir}^P \geq u_{ir}^Q$ for $r = 1, \ldots, n$. A random assignment is ordinally efficient if it is not stochastically dominated.

A mechanism is a function from F^n into R that provides a procedure to associate each preference profile with some random assignment. A mechanism $\mu(\cdot)$ is strategy-proof whenever for any preference profile $\succ \equiv (\succ_j)_{j \in N}$, and for each $i \in N, \mu_i(\succ_i, \succ_{-i}) \succ_i^{sd} \mu_i(\succ_i', \succ_{-i})$ for all $\succ_i' \neq \succ_i$, where $\succ_{-i} \equiv (\succ_j)_{i \in N \setminus \{i\}}$. For each pair $\succ_i, \succ_i' \in F, \succ_i'$ is adjacent to $\succ_i$ if $\succ_i'$ is attained from $\succ_i$ by swapping two sequentially ranked objects without changing the rank of any other objects, that is, there exist $a, b \in A$ such that $rank(b, \succ_i) = rank(a, \succ_i) + 1, rank(a, \succ_i') = rank(b, \succ_i') + 1 = rank(b, \succ_i)$, and $rank(c, \succ_i) = rank(c, \succ_i')$ for all $c \in A \setminus \{a, b\}$. Lastly, a mechanism meets lower invariance if an agent switches her preference to another adjacent one, the probabilities of receiving any object in the strict lower-contour set of the two swapping objects should not be altered. More formally, for $\succ_i \in \Gamma$, each $i \in N$, each $\succ_i' \in \Gamma$, and each $a, b \in A$, if $\succ_i'$ is adjacent to $\succ_i$, that is, $a \succ_i b$, and $b \succ_i' a$, then $\mu_{ic}(\succ_i', \succ_{-i}) = \mu_{ic}(\succ_i, \succ_{-i})$ for each $c \in L(\succ_i, b)$.

For each $\succ \in \Gamma^n$ and all $i, j \in N$, a random assignment $P \in R$ is envy-free (EF) if $P_i \succ_i^{sd} P_j$. For each $\succ \in \Gamma^n$ and all $i, j \in N$, a random assignment $P \in R$ is weakly envy-free (WEF) if $P_j \succ_i^{sd} P_i$ then $P_i = P_j$. However, there are some other notions for fairness as well in the literature. Equal treatment of equals requires that for each $i, j \in N$ with $\succ_i = \succ_j$ the allocation of both agents i and j are identical: $P_i = P_j$. If a random assignment P first-order stochastically dominates the random assignment with equal division, then it satisfies equal division lower bound, that is, if $\forall\, i \in N, P_i \succ_i^{sd} 1/n$. A deterministic assignment Π is favoring higher ranks (FHR), à la Kojima and Ünver (2014), when for every agent j and object b, Π assigns b to $j, \Pi_{jb} = 1$, while there is another agent i who likes b more, $rank(b, \succ_i) < rank(b, \succ_j)$, then Π does not assign to i any inferior object, such as c that $b \succ_i c, \Pi_{ic} = 0$. A random assignment is ex post favoring ranks (EFR), à la Ramezanian and Feizi (2021a), if it can be represented as a lottery over deterministic assignments that each favors higher ranks.

3 | (STRICT) FAVORING UPPER RANKS

(Strict) favoring upper ranks is, to some extent, based on a stability concept suggested by Afacan (2018)[1] for the priority-based object allocation problem and Aziz and Klaus (2019) for random matching: An agent $i \in N$ has a claim against an agent $j \in N$, if there exists an object a such that $i \succ_a j$ and $p_{ja} > \sum_{a': a' \succ_i a} p'_{ia}$. A random matching p is claimwise stable if it does not admit any claim, that is, for each pair $(i, a) \in N \times A$ and each $j \in N$ such that $i \succ_a j, \sum_{a': a' \succ_i a} p_{ia'} \geq p_{ja}$.

[1]Though it is for a model with probabilistic priorities, we focus on its strict priority part.

(Strict) favoring upper ranks allows any agent to acquire as much share of an object as up to the total allotment of more preferred objects of agents who rank the object more favorable than the former. More formally, an agent i has an upper contour claim against agent j for object a if the assigned share of object a to agent j, who has a lower rank for a, is greater than agent i's total share of his more preferred objects. More precisely, once we read probabilities as a share of receiving objects, the inequality encodes that agent i envies agent j, with a lower rank for object a, for consuming more of object a than what she consumes from her strict upper contour set of a.

Definition 1. An agent i has a strict upper contour claim against agent j for object a under random assignment P, if $rank(a, \succ_i) < rank(a, \succ_j)$, while agent j has a more chance to get a than agent i receiving objects in her strict upper contour, that is, $p_{ja} > \sum_{a':a' \succ_i a} p_{ia'}$.

Definition 1 implies that agent i receives some fraction of an object in her strict lower contour set at a (otherwise, we have $p_{ia} + p_{ja} > p_{ia} + \sum_{a':a' \succ_i a} p_{ia'} = 1$, which is a contradiction). Therefore, we could interpret this definition as an envy notion: as long as agent i receives objects better than a, she does not envy lower priority agent j for getting some fractions of a. However, once agent j obtains the same fraction of a as agent i's strict upper contour set at a, agent i starts envying him for any additional a. We could also define a less strict definition once we consider upper contour claim instead of strict upper contour claim:

Definition 2. An agent i has an upper contour claim against agent j for object a under random assignment P, if $rank(a, \succ_i) < rank(a, \succ_j)$, while agent j has a more chance to get a than agent i receiving objects in her upper contour, that is, $p_{ja} > \sum_{a':a' \succsim_i a} p_{ia'}$.

Definition 2 states that agent i has an upper contour claim against agent j if the excess probability that agent j has on object a is not *paid up* by additional probability in the strict upper contour set of a for agent i, that is, $p_{ja} - p_{ia} > \sum_{a':a' \succ_i a} p_{ia'}$. A random assignment that is (strictly) favoring agents with a better rank for an object and does not allow any agent to have an upper contour claim is called *(strict) favoring upper ranks* random assignment.

Definition 3. A random assignment P is strict favoring upper ranks (SFUR) if it does not admit any strict upper contour claim, that is, for each pair $(i, a) \in N \times A$ and each $j \in N$ such that $rank(a, \succ_i) < rank(a, \succ_j)$, agent i does not have a less chance to get objects in her strict upper contour than agent j receiving a, that is, $\sum_{a':a' \succ_i a} p_{ia'} \geq p_{ja}$.

Definition 4. A random assignment P is favoring upper ranks (FUR) if it does not admit any upper contour claim, that is, for each pair $(i, a) \in N \times A$ and each $j \in N$ such that $rank(a, \succ_i) < rank(a, \succ_j)$, agent i does not have a less chance to get objects in her upper contour than agent j receiving a, that is, $\sum_{a':a' \succsim_i a} p_{ia'} \geq p_{ja}$.

It is trivial, by definition, to show that any SFURs random assignment is favoring upper ranks, while the reverse does not necessarily hold.

4 | RESULTS

In Lemma 1, we show that an SFURs random assignment is *first-choice maximal*, in the sense of Dur et al. (2018), as it matches a maximal number of agents to their reported first best objects.

Lemma 1. *Any SFURs random assignment should not assign an object which is the first best of an agent to another agent for whom it is not the first best object.*

Proof. If $rank(a, \succ_i) = 1 < rank(a, \succ_j)$ and in a random assignment P which is SFURs we have $p_{ja} > 0$, then agent i has an upper contour claim against agent j since $p_{ja} > \sum_{a':a' \succ_i a} p_{ia}^{'} = 0$. Therefore, by definition, the random assignment P is not SFURs. □

Remark 1. (Strict) favoring upper ranks reduces to favoring higher ranks in the deterministic assignments.

Given Remark 1, (strict) favoring upper ranks and ex post favoring ranks, in the sense of Ramezanian and Feizi (2021a), are both extensions of favoring higher ranks into random assignments. The question arises how these two notions are associated with each other. Proposition 1 shows that ex post favoring ranks implies (strict) favoring upper ranks, but not vice versa.

Proposition 1. *An ex post favoring ranks random assignment is SFURs, but the reverse does not hold necessarily.*

Proof. In order to show the direct claim, we show that if a random assignment P is not SFURs, it could not be ex post favoring ranks as well. We know from the Birkhoff–von Neumann theorem (Birkhoff, 1946; Von Neumann, 1953), and its extension (Davis, 1961) that any bistochastic matrix can be written as a convex combination of deterministic matrices. Since P is not SFURs, there exists a pair $(i, a) \in N \times A$ for which, $\sum_{a':a' \succ_i a} p_{ia'} < p_{ja}$. Therefore, there is a deterministic assignment in the decomposition of P which gives object a to agent j with $rank(a, \succ_j) > rank(a, \succ_i)$, while agent i gets neither a nor any preferred object. This deterministic assignment does not favor higher ranks, and hence P is not ex post favoring ranks as well.

We show the other direction by a counterexample. For the set of agents $N = \{1, 2, 3, 4\}$, the set of objects $A = \{a, b, c, d\}$, and the preference profile

$$
\begin{array}{ll}
1: & a \succ_1 b \succ_1 c \succ_1 d \\
2: & a \succ_2 c \succ_2 b \succ_2 d \\
3: & b \succ_3 c \succ_3 d \succ_3 a' \\
4: & b \succ_4 c \succ_4 a \succ_4 d
\end{array}
$$

let us consider the random assignment

$$
P = \begin{pmatrix} 3/4 & 0 & 1/4 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 0 & 1/4 & 0 & 3/4 \end{pmatrix} = \frac{3}{4}\overbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\text{FHR}} + \frac{1}{4}\overbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}}^{\Pi:\text{Not FHR}}.
$$

We show that P is SFURs. As objects a and b have been given only to those agents who prefer them as their first best, there is no claim on them. Agents $i = 2, 3, 4$ prefer object c more than agent $j = 1$. Nevertheless, we have $\sum_{c':c' \succ_i c} p_{ic'} \geq p_{jc}$. Agent $i = 3$ prefers object d more than others $j = 1, 2, 4$, while $\sum_{d':d' \succ_i d} p_{id'} \geq p_{jd}$. However, the random assignment P is not ex post favoring ranks since in its unique decomposition, there is a deterministic matrix which does not favor higher ranks: In Π, agent 3 prefers object c more than agent 1 while it has been given to agent 1 and agent 3 gets a lower rank object d. □

Ramezanian and Feizi (2021a) showed that ex post favoring ranks and (weak) envy-free are logically unrelated. The question arises how envy-freeness and SFURs related to each other. Example 1 depicts an envy-free random assignment that is not SFURs.

Example 1. For the set of agents $N = \{1, 2, 3, 4\}$, the set of objects $A = \{a, b, c, d\}$, and the preference profile

$$\begin{array}{lccccccc} 1: & a & \succ_1 & b & \succ_1 & c & \succ_1 & d \\ 2: & a & \succ_2 & d & \succ_2 & b & \succ_2 & c \\ 3: & a & \succ_3 & d & \succ_3 & c & \succ_3 & b, \\ 4: & a & \succ_4 & b & \succ_4 & d & \succ_4 & c \end{array}$$

the random assignment

$$P = \begin{pmatrix} 1/4 & 5/16 & 3/8 & 1/16 \\ 1/4 & 5/16 & 1/8 & 5/16 \\ 1/4 & 1/16 & 3/8 & 5/16 \\ 1/4 & 5/16 & 1/8 & 5/16 \end{pmatrix}$$

is envy-free but not SFURs as $rank(b, \succ_1) < rank(b, \succ_2)$ while $p_{2b} = 5/16 > \sum_{b':b' \succ_1 b} p'_{1b'} = p_{1a} = 1/4$.

Proposition 2 proves that, all envy-free random assignments are favoring upper ranks.

Proposition 2. *Envy-freeness implies favoring upper ranks, but not vice versa.*

Proof. We prove by contradiction. Assume that P is not a favoring upper ranks random assignment. Therefore, by definition, it admits an upper contour claim, that is, there exists a pair $(i, a) \in N \times A$ and $j \in N$ such that $r = rank(a, \succ_i) < rank(a, \succ_j)$, and $\sum_{a':a' \succsim_i a} p_{ia'} < p_{ja}$. Thus, $u_{ir}^P < u_{jr}^P$, and since the allocation of agent i does not stochastically dominate the allocation of agent j, the random assignment P is not envy-free.

We show the other direction by a counter-example. For the set of agents $N = \{1, 2, 3, 4\}$, the set of objects $A = \{a, b, c, d\}$, and the preference profile

$$
\begin{array}{l}
1: \; a \succ_1 b \succ_1 c \succ_1 d \\
2: \; c \succ_2 b \succ_2 d \succ_2 a \\
3: \; c \succ_3 d \succ_3 b \succ_3 a \\
4: \; a \succ_4 d \succ_4 b \succ_4 c
\end{array}, \tag{1}
$$

the random assignment

$$
\begin{pmatrix}
1/2 & 1/2 & 0 & 0 \\
0 & 0 & 1/2 & 1/2 \\
0 & 1/2 & 1/2 & 0 \\
1/2 & 0 & 0 & 1/2
\end{pmatrix}
$$

is favoring upper ranks but not envy-free as agent 2 and agent 3 envy each other. □

Since we know that envy-freeness implies weak envy-freeness, the question emerges how favoring upper ranks, and weak envy-freeness are associated. Example 2 assesses this relationship.

Example 2. For the preference profile (1), the random assignment

$$
\begin{pmatrix}
3/10 & 0 & 4/10 & 3/10 \\
2/10 & 4/10 & 4/10 & 0 \\
1/10 & 4/10 & 2/10 & 3/10 \\
4/10 & 2/10 & 0 & 4/10
\end{pmatrix}
$$

is weak envy-free but not favoring upper ranks since $rank(b, \succ_1) < rank(b, \succ_3)$, while $p_{3b} = 4/10 > \sum_{b':b' \succsim_1 b} p_{1b'} = p_{1a} + p_{1b} = 3/10$.

We can summarize the relationship between (strict) favoring upper ranks and different notions of fairness in Figure 1.

We prove that SFURs is possible with neither (lower invariance, a weaker notion of strategy-proofness à la, Mennle & Seuken, 2021, and) equal treatment of equals, nor equal division lower bound, two weaker notions of envy-freeness.

Theorem 1. *For at least four agents, there is no mechanism that satisfies SFURs and either*

(*i*) *lower invariance and equal treatment of equals*, *or*
(*ii*) *equal division lower bound.*

Proof. First, for $n = 4$, let $N = \{1, 2, 3, 4\}$ be the set of agents, and $A = \{a, b, c, d\}$ be the set of objects.

(i) Suppose by a contradictory argument that μ is a lower invariant mechanism satisfying SFURs and equal treatment of equals. For profile

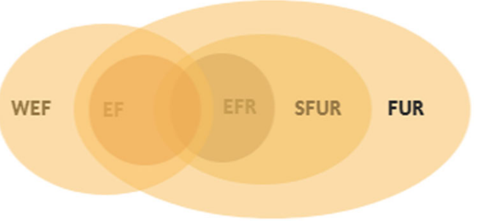

FIGURE 1 Relationship between favoring upper ranks (FUR), ex post favoring ranks (EFR), and ex post favoring (EF) [Color figure can be viewed at wileyonlinelibrary.com]

$$P1: \begin{array}{l} 1: \; a \succ_1 b \succ_1 c \succ_1 d \\ 2: \; a \succ_2 b \succ_2 c \succ_2 d \\ 3: \; a \succ_3 b \succ_3 c \succ_3 d, \\ 4: \; b \succ_4 a \succ_4 c \succ_4 d \end{array} \tag{2}$$

given Lemma 1, we should have $\mu_{4b}(P1) = 1$, and given the equal treatment of equals, we get

$$\mu(P1) = \begin{pmatrix} 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

For profile

$$P2: \begin{array}{l} 1: \; a \succ_1 b \succ_1 c \succ_1 d \\ 2: \; a \succ_2 b \succ_2 c \succ_2 d \\ 3: \; b \succ_3 a \succ_3 c \succ_3 d, \\ 4: \; b \succ_4 a \succ_4 c \succ_4 d \end{array} \tag{3}$$

by lower invariance, $\mu_{3c}(P2) = \mu_{3c}(P1) = 1/3$ and $\mu_{3d}(P2) = \mu_{3d}(P1) = 1/3$. By equal treatment of equals, $\mu_{4c}(P2) = \mu_{4d}(P2) = 1/3$. Again, by equal treatment of equals, $\mu_{1c}(P2) = \mu_{2c}(P2) = 1/6$, and $\mu_{1d}(P2) = \mu_{2d}(P2) = 1/6$. Moreover, given Lemma 1, we should have $\mu_{1a}(P2) = \mu_{2a}(P2) = \mu_{3b}(P2) = \mu_{4b}(P2) = 0$. By equal treatment of equals, we should have $\mu_{1b}(P2) = \mu_{2b}(P2) = 1/2$, and $\mu_{1a}(P2) = \mu_{2a}(P2) = 1/2$.

$$\mu(P2) = \begin{pmatrix} 0 & 1/2 & 1/6 & 1/6 \\ 0 & 1/2 & 1/6 & 1/6 \\ 1/2 & 0 & 1/3 & 1/3 \\ 1/2 & 0 & 1/3 & 1/3 \end{pmatrix}.$$

However, $\mu(P2)$ is not a bistochastic matrix as the summation of each row is not one.

(ii) Suppose by a contradictory argument that μ is a lower invariant mechanism satisfying SFURs and equal division lower bound. For profile ($P1$), given Lemma 1, we should have $\mu_{4b}(P1) = 1$, and therefore $\mu_{1b}(P1) = \mu_{2b}(P1) = \mu_{3b}(P1) = 0$. Since $\mu(P1)$

satisfies equal division lower bound, that is, $\mu_i(P1) \succ_i^{sd} 1/4$ for $i = 1, 2, 3$, we should have $\mu_{ia}(P1) = 1/2$, and $\mu_{ic}(P1) = \mu_{id}(P1) = 1/4$.

$$\mu(P1) = \begin{pmatrix} 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

However, $\mu(P1)$ is not a bistochastic matrix as the summation of each column is not 1.

For $n > 4$, let $N_1 = \{1, 2, 3, 4\}, N_2 = \{5, 6, ..., n\}, A_1 = \{a, b, c, d\}$, and $A_2 = \{o_5, o_6, ..., o_n\}$. We extend $P1$ and $P2$ to $P'1$ and $P'2$ in the way that o_j is the first-best of only agent $j \in N_2$. For $i \in N_1$, the preference of agent i in profile $P'1$ or $P'2'$over objects in A_1 is the same as her preference in profile $P1$ and $P2$, respectively. Moreover, for all $x \in A_1$ and all $y \in A_2$, we have $x \succ_i' y$. Now, for all $j \in N_2$, given Lemma 1, we have $\mu_{jo_j}(P'1) = 1$ and $\mu_{jo_j}(P'2) = 1$. Therefore, the same arguments for $n = 4$ in (i) and (ii) work for the general case where $n \geq 4$. □

Ramezanian and Feizi (2021a) proved that ex post favoring ranks implies ex post Pareto efficiency. Example 3 shows that it is not the case for SFURs.

Example 3. For the set of agents $N = \{1, 2, 3, 4\}$, the set of objects $A = \{a, b, c, d\}$, and the preference profile

$$\begin{array}{llllllll} 1: & a & \succ_1 & b & \succ_1 & c & \succ_1 & d \\ 2: & a & \succ_2 & d & \succ_2 & b & \succ_2 & c \\ 3: & a & \succ_3 & b & \succ_3 & d & \succ_3 & c, \\ 4: & a & \succ_4 & c & \succ_4 & b & \succ_4 & d \end{array}$$

the random assignment

$$\begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/2 & 0 \end{pmatrix}$$

$$= \frac{1}{4}\overbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}^{\text{Not Pareto Efficient}} + \frac{1}{4}\overbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}^{\text{Not Pareto Efficient}} + \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{4}\overbrace{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}^{\text{Not Pareto Efficient}} + \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is SFURs. However, it is not ex post Pareto efficient since in its decompositions into deterministic assignments, there always exists a deterministic assignment that is not Pareto efficient.

Moreover, Ramezanian and Feizi (2021a) showed there exist ex post favoring ranks assignments that fail to be ordinal efficient and ordinal efficient random assignments that fail to be ex post favoring ranks. Since ex post favoring ranks implies SFURs, there are SFURs random assignments that fail to be ordinal efficient. Example 4 shows that favoring upper ranks and ordinal efficiency do not imply each other.

Example 4. For the set of agents $N = \{1, 2, 3, 4\}$, the set of objects $A = \{a, b, c, d\}$, and the preference profile

$$\begin{array}{lccccccc} 1: & a & \succ_1 & b & \succ_1 & c & \succ_1 & d \\ 2: & a & \succ_2 & c & \succ_2 & b & \succ_2 & d \\ 3: & b & \succ_3 & c & \succ_3 & d & \succ_3 & a, \\ 4: & b & \succ_4 & d & \succ_4 & a & \succ_4 & c \end{array} \tag{4}$$

the random assignment

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}, \tag{5}$$

is ordinal efficient. However, it is not favoring upper ranks since $rank(c, \succ_3) < rank(c, \succ_1)$, while $p_{1c} = 1 > \sum_{c':c' \succsim_1 c} p_{3c'} = p_{3b} + p_{3c} = 1/2$.

We can summarize the relationship between (strict) favoring upper ranks and different efficiency notions for random assignments in Figure 2.

5 | MECHANISM

Bogomolnaia and Hervé (2001), in their seminal paper, showed that their proposed eating algorithm satisfies ordinal efficiency. In this algorithm, each agent eats her most favored object (among the remaining ones), with a similar speed as others, until all objects are exhausted. The fraction of each object eaten by each agent is the probability of assigning the object to her. These probabilities make a bistochastic random assignment as the output of the mechanism. We show that this algorithm provides a random assignment that favors upper ranks.

Proposition 3. *The eating algorithm provides a favoring upper ranks random assignment.*

Proof. We show that for any agents $i, j \in N$, and an object a, such that $rank(a, \succ_i) < rank(a, \succ_j)$, agent i does not have an upper contour claim against agent

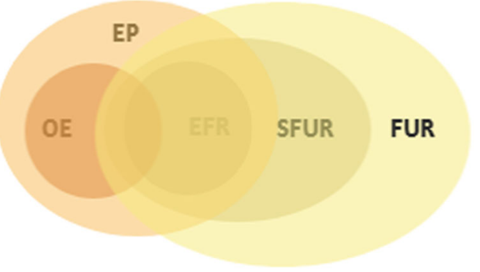

FIGURE 2 Relationship between FUR and different notions of efficiency [Color figure can be viewed at wileyonlinelibrary.com]

j, that is, $\sum_{a':a' \succsim_i a} p_{ia'} \geq p_{ja}$, in the output of the eating algorithm. In the algorithm, either agents j get to eat object a earlier than agent i or not. In the latter case, we have either

(i) agents i and j continue to eat object a until it is exhausted, that means $p_{ia} \geq p_{ja}$ which in turn leads to $\sum_{a':a' \succsim_i a} p_{ia'} \geq p_{ja}$, or

(ii) one of them gets full. If agent j gets full, then agent i continues to eat object a in the next step of the algorithm, which gives $p_{ia} \geq p_{ja}$. Otherwise, if agent i gets full, by definition, we have $\sum_{a':a' \succsim_i a} p_{ia'} = 1 \geq p_{ja}$.

In the former case, while agent j is starting to eat object a, agent i is continuing to eat better objects a', where $a' \succ_i a$, until she joins agent j in eating object a. Thereafter, they have the same share from object a (unless one of them gets full, which brings us back to II). Therefore, all in all, $\sum_{a':a' \succsim_i a} p_{ia'} \geq p_{ja}$. □

However, Example 5 illustrates that the output of the two best-known mechanisms for the assignment problem does not satisfy strict favoring upper ranks.

Example 5. For the set of agents $N = \{1, 2, 3\}$, the set of objects $A = \{a, b, c\}$, and the preference profile

$$\begin{array}{lccccc} 1: & a & \succ_1 & b & \succ_1 & c \\ 2: & b & \succ_2 & a & \succ_2 & c, \\ 3: & b & \succ_3 & c & \succ_3 & a \end{array}$$

the output of the eating algorithm is

$$\begin{pmatrix} 3/4 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

In the random priority (RP) mechanism of Zhou (1990), agents are randomly ordered and choose their most favored objects (among the remaining ones) in that order, one by one. The output of the RP mechanism, for the given preference profile, is

$$\begin{pmatrix} 5/6 & 0 & 1/6 \\ 1/6 & 1/2 & 1/3 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

Both these two random assignments are not strict favoring upper ranks since, in contrast to Lemma 1, they assigns a, which is the first best-of agent 1, to agent 2 for whom it is not the first best object. However,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

for the same preference profile satisfies strict favoring upper ranks.

As illustrated in Example 5, neither the eating algorithm nor the RP necessarily provides a strict favoring upper ranks random assignment. Nevertheless, we show that a modified eating algorithm, à la Ramezanian and Feizi (2021b), where no agent can eat an object unless all the others ranking the object better are full, provides a random assignment that is strict favoring upper ranks.

The main difference between this modified eating algorithm and the typical one is that we do not allow agents with a lower rank for an object to eat it unless all agents with a higher rank are full. This difference makes agents finish their share of objects in each round of eating, not necessarily at the same time as others. We continue sharing anything that remained from the object equally among those still hungry agents with the highest rank for those objects, as long as no agent is willing to eat anymore because there is nothing left from the object to share or all agents are full.

Ramezanian and Feizi (2021b), in their Proposition 5, proved that the outcome of the modified eating algorithm is *stepwise ordinal efficient*, which itself is a stronger notion than ordinal efficiency. Moreover, in their Lemma 1, it is proved that at every step s of the algorithm, any object a, which is the sth-best object of an agent i, get exhausted or agent i gets full.

Proposition 4. *The modified eating algorithm provides a strict favoring upper ranks random assignment.*

Proof. Given Lemma 1 of Ramezanian and Feizi (2021b), at each step s of the modified eating algorithm, given an object a that is a sth-best of an agent i, that is, $rank(a, \succ_i) = s$, we have two possible cases: either object a get exhausted, or agent i gets full at step s. In the former case, for all other agents such as j that $rank(a, \succ_j) > s$ we have $\sum_{a':a' \succ_i a} p_{ia'} \geq p_{ja} = 0$. In the latter case, from $\sum_{a':a' \succ_i a} p_{ia'} + p_{ia} = \sum_{j \neq i} p_{ja} + p_{ia} = 1$, we get $\sum_{a':a' \succ_i a} p_{ia'} = \sum_{j \neq i} p_{ja} \geq p_{ja}$. In both cases, no agent has a upper contour claim, and we are done. □

6 | CONCLUSION

This paper introduced *(strict) favoring upper ranks* for the random assignment. A random assignment is (strict) favoring upper ranks if an agent has a higher rank for an object than another agent, the total probability that she gets (strictly) more preferred objects is not less than the probability that the latter agent receives the object. *Ex post favoring ranks* and (strict) favoring upper ranks are extensions of *favoring higher ranks* for random assignments. We demonstrated that ex post favoring ranks implies (strict) favoring upper ranks but not vice versa. We showed that while strict favoring upper ranks and envy-free are logically unrelated, envy-freeness implies favoring upper ranks. Moreover, for at least four agents, no mechanism satisfies strict favoring upper ranks and either equal division lower bound or equal treatment of equals, and lower invariance. Furthermore, while favoring upper ranks and ordinal efficiency do not imply each other, we proved that the eating algorithm provides a favoring upper ranks random assignment, and the modified eating algorithm outputs a random assignment that is strict favoring upper ranks.

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CONFLICT OF INTEREST

The author declares no conflict of interest.

DATA AVAILABILITY STATEMENT

Due to the nature of this research, data availability is not applicable.

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