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Robust ex-post Pareto efficiency and fairness in random assignments: Two impossibility results ☆

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ABSTRACT

A random assignment is *robust ex-post Pareto efficient* whenever for any of its lottery decomposition, each deterministic assignment in its support is Pareto efficient. We show that ordinal efficiency implies robust ex-post Pareto efficiency while the reverse does not hold. We know that strategy-proof and ordinal efficient mechanisms satisfy neither equal treatment of equals nor equal division lower bound. We prove that it is not possible to avoid these two impossibilities by weakening ordinal efficiency to robust ex-post Pareto efficiency.

1. Introduction

In an assignment problem, one allocates a finite set of indivisible objects, without monetary transfer, to a set of agents who have reported their ordinal preferences over objects. An assignment is Pareto efficient if one could not make an agent better off without making at least one other agent worse off. A random assignment is ex-post Pareto efficient whenever it has a lottery decomposition where each deterministic assignment in its support is Pareto efficient. Bogomolnaia and Moulin (2001), in their seminal paper, introduced ordinal efficiency, which is a stronger notion than ex-post Pareto efficiency. Robust ex-post Pareto efficiency is an intermediate notion of efficiency weaker than ordinal efficiency but stronger than ex-post Pareto efficiency (Aziz et al., 2015). An assignment is robust ex-post Pareto efficient whenever each deterministic assignment in support of any of its ex-post decomposition is Pareto efficient.

We already know that ex-post Pareto efficiency is possible with strategy-proofness and equal treatment of equals, as the well-known random priority mechanism of Zhou (1990) satisfies all these criteria. However, for more than four objects and agents, no ordinally efficient and strategy-proof mechanism satisfies equal treatment of equals (Bogomolnaia and Moulin, 2001) or equal division lower bound (Nesterov, 2017). The question arises whether there exists another notion of efficiency weaker than ordinal efficiency, which is possible with strategy-proofness and fairness. It is not clear in advance whether robust ex-post efficiency is closer to ex-post efficiency or ordinal efficiency in the sense of compatibility with other axioms.

The main contribution of this paper is to strengthen the impossibility results of Bogomolnaia and Moulin (2001) and Nesterov (2017) and show that they continue to hold if ordinal efficiency is replaced with robust ex-post Pareto efficiency.

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We show that a robust ex-post Pareto efficient random assignment might not be ordinal efficient and prove that there is no strategy-proof mechanism that is robust ex-post Pareto efficient and satisfies fairness, namely, equal treatment of equals or equal division lower bound.

The paper is organized as follows. In Section 2, we review the related literature. Section 3 recalls the standard model and axioms of random assignments. In Section 4, we study robust ex-post Pareto efficiency. Section 5 proves some impossibility results. Finally, Section 6 concludes, and omitted proofs are presented in the Appendix.

2. Related work

The compatibility of different axioms in random assignment mechanisms, namely efficiency, fairness, and strategy-proofness, has been studied in market design literature. *Strategy-proofness* is incompatible with Pareto efficiency and *equal treatment of equals* (Zhou, 1990). Moreover, no ex-post Pareto optimal and group-strategy-proof random matching mechanism treats equals equally (Bade, 2016). Zhang (2019) proved that a mechanism that is ex-post Pareto efficient and fair (in the sense of *equal treatment of equals*, *equal total assignment*,[1] and *uniform-head fairness*[2]) is strongly group manipulable. Zhang (2020) found that the results of Bade (2016) and Zhang (2019) still hold unless the domains are restricted to have a particular tier structure.

For a model of fractional matching, Alva and Manjunath (2020) showed that strategy-proofness, ex-post Pareto efficiency, and ex-ante individual rationality are incompatible when each agent's utility is a linear function of both their fractional assignment and money. Moreover, non-wasteful strategy-proof mechanisms are not dominated by strategy-proof mechanisms (Erdil, 2014). Martini (2016) proved that no strategy-proof assignment mechanism satisfies equal treatment of equals once we weaken ordinal efficiency to non-wastefulness, a weaker notion of efficiency.

Budish et al. (2013) generalized the randomized assignment problem to multi-unit allocation problems with *bihierarchy* constraints and developed new mechanisms that select ex-ante efficient and envy-free expected allocations through randomization. Nevertheless, their expansion of the previous works can rule out many real-world applications. Akbarpour and Nikzad (2020) generalized this approach to a much broader class of allocation problems and showed that by treating some of the constraints as goals, one could accommodate many more constraints.

3. Model

Let A be a finite set of objects which should be assigned to a finite set of agents, N, with $|A| = |N| = n$. Each agent $i \in N$ has a complete, transitive, and anti-symmetric *strict preference relation* $\succ_i$ over A. We denote a preference profile by $\succ \equiv (\succ_i)_{i\in N}$ and the domain of those preferences by $\mathcal{F}$. To simplify our notation, for each $i \in N$, we write $\succ_i : abcd$ instead of $a \succ_i b \succ_i c \succ_i d$. We represent a **random assignment** by a *bistochastic matrix* $P = [p_{ia}]_{i\in N, a\in A}$, with agents on rows and objects on columns, where p_{ia} is the probability of assigning object a to agent i. We denote the domain of random assignments by R. A **random allocation** for some agent $i \in N$, P_i, is a probabilistic distribution over all objects in A where the sum of probabilities of assigning objects to the agent i equals to 1. A **deterministic assignment**, $\Pi = [p_{ia}]_{i\in N, a\in A}$, is a particular case of random assignment where its entries are all either 0 or 1. The Birkhoff–von Neumann theorem states that all random assignments can be decomposed as a probability distribution over deterministic assignments and so can be implemented in practice.

Upon enumerating objects in A for agent i from best to worst according to $a_{i,1} \succ_i a_{i,2} \succ_i \ldots \succ_i a_{i,n}$, where $a_{i,k}$ is the k^{th} best object of agent i, we define $u^P_{ir} = \sum_{k=1}^{r} p_{ia_{i,k}}$ to be the summation of probabilities of receiving the first r best objects of agent i in the random assignment P. Given a preference ordering $\succ_i$ on A, the stochastic dominance relation is denoted by $\succ^{sd}_i$, where $P_i \succ^{sd}_i Q_i$ if and only if $u^P_{ir} \geq u^Q_{ir}$ for $r = 1, \ldots, n$. Given a preference profile $\succ_i$, we say that the random assignment P, is stochastically dominated by another random assignment $Q \neq P$, if $Q_i \succ^{sd}_i P_i$ for two allocations of all $i \in N$. A random assignment is **ordinally efficient** if it is not stochastically dominated.

A mechanism, given the preferences of all agents, provides us with a procedure to assign objects to agents. More formally, a **mechanism** $\mu(.)$ is a function from $\mathcal{F}^n$ into R, that associates each preference profile with some random assignment. A mechanism is strategy-proof when no agent can beneficially misreport her preference. More formally, a mechanism $\mu(.)$ is **strategy-proof** whenever for each profile $\succ \equiv \left(\succ_j\right)_{j\in N}$, and for each $i \in N$, $\mu_i(\succ_i, \succ_{-i}) \succ^{sd}_i \mu_i(\succ'_i, \succ_{-i})$ for all $\succ'_i \neq \succ_i$.

Equal treatment of equals requires that agents with the same preference should have the same allocation, i.e., for each $i, j \in N$ with $\succ_i = \succ_j$ we have $P_i = P_j$. Another approach to define fairness is to compare the allocations of agents with the equal share where each agent receives each object with equal probability, i.e., $1/n$. If a random assignment P stochastically dominates the random assignment with equal division, i.e., $\forall i \in N$, $P_i \succ^{sd}_i 1/n$, then it satisfies **equal division lower bound**.

[1] Equal total assignment requires that agents obtain equal total probability shares of objects.

[2] Uniform-head fairness requires that if all agents have equal preferences over a subset of objects and prefer the objects in that subset to the remaining objects, they obtain equal probability shares of the objects in that subset.

4. Robust ex-post Pareto efficiency

Every ex-post Pareto efficient random assignment can be decomposed into a lottery over Pareto efficient deterministic assignments. However, it is also possible that it has a decomposition with non-Pareto efficient deterministic assignment in its support, as illustrated in Example 1.

Example 1. For the preference profile

$$\succ_1 = adbc, \ \succ_2 = cbda, \ \succ_3 = cdba, \text{ and } \succ_4 = abcd, \tag{1}$$

the random assignment

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix} = \frac{1}{2}\overbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}}^{\text{Pareto Efficient}} + \frac{1}{2}\overbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}^{\text{Pareto Efficient}}$$
$$= \frac{1}{2}\overbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}^{\Pi_1:\text{ Non-Pareto Efficient}} + \frac{1}{2}\overbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}}^{\text{Pareto Efficient}},$$

is ex-post Pareto efficient but it has a decomposition that admits a non-Pareto efficient deterministic assignment in its support.

An assignment is robust ex-post Pareto efficient whenever *for all* of its lottery decomposition, *each* deterministic assignment in its support is Pareto efficient. More formally,

Definition 1. A random assignment P for a preference profile $\succ$, is robust ex-post Pareto efficient whenever for every decomposition of P to deterministic assignments, i.e., $P = \lambda_1\Pi_1 + \lambda_2\Pi_2 + \ldots + \lambda_k\Pi_k$ for some k, each Π_j with $\lambda_j \neq 0$, is Pareto efficient, where $j = 1, \ldots, k$.

The particular appeal of robust ex-post Pareto efficiency comes from a computational complexity argument: Given an ex-post Pareto efficient assignment P, it is NP-hard to find a decomposition where all the deterministic assignments in its support are Pareto efficient (Aziz et al., 2015). However, if we know that P is robust ex-post Pareto efficient, we could arbitrarily decompose it in polynomial time and ensure that the decomposition is a lottery over Pareto efficient deterministic assignments. Example 3 of Abdulkadiroğlu and Sönmez (2003) showed a (robust) ex-post Pareto efficient random assignment that is not ordinally efficient.

To study the relationship between robust ex-post Pareto efficiency and ordinal efficiency in the general case, we need to introduce some new concepts, namely *ex-ante/ex-post feasible trading cycles* for random assignments. A trading cycle is a sequence of agent/object pairs, where each agent prefers the object of the next pair to her. For the very last pair, the next pair is defined to be the very first one. More formally, let $r = \langle (a_1, i_1), (a_2, i_2), \ldots, (a_m, i_m) \rangle$ be a sequence of agent/object pairs. Given a preference profile $\langle \succ_i \rangle_{i \in N}$, we say r is a trading cycle whenever $a_{(k+1 \mod m)} \succ_{i_k} a_k$, for all $k = 1 \ldots m$. We refer to the set of all trading cycles by $CL^{\succ}$.

The existence of a trading cycle in an assignment matters as it makes the assignment inefficient. However, given an assignment, some potential trading cycles might not be feasible because they do not occur. Given a random assignment, depending on whether the trading cycles happen ex-ante or ex-post, we could define ex-ante/ex-post feasible trading cycles: we say that a trading cycle is ex-ante feasible whenever each agent/object pair in the cycle happens with a positive probability, and ex-post feasible whenever there exists a decomposition of the random assignment such that the cycle happens in one of the deterministic assignments in its support.

Definition 2. Let P be a random assignment and $r = \langle (a_1, i_1), (a_2, i_2), \ldots, (a_m, i_m) \rangle$ be a trading cycle regarding a preference profile $\succ$. We say that the trading cycle r is

- **ex-ante P-feasible** whenever $p_{i_t a_t} > 0$, for all $1 \leq t \leq m$.
- **ex-post P-feasible** whenever there exists a decomposition of P to deterministic assignments, i.e., $P = \lambda_1\Pi_1 + \lambda_2\Pi_2 + \ldots + \lambda_k\Pi_k$ for some k, such that for some Π_j with $\lambda_j \neq 0$, the cycle r happens in Π_j, i.e., $(\Pi_j)_{i_t a_t} = 1$, for all $1 \leq t \leq m$.

Example 2 illustrates definition 2 for a given preference profile and a random assignment.

Example 2. For the preference profile (1) and random assignment P (in Example 1), $r_1 = \langle(b,1),(d,2)\rangle$ is an ex-post P-feasible trading cycle as r_1 happens in Π_1. For the same preference profile, and random assignment

$$Q = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix} = \frac{1}{2}\overbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}^{\text{Pareto Efficient}} + \frac{1}{2}\overbrace{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}^{\text{Pareto Efficient}},$$

$r_2 = \langle(b,3),(d,2)\rangle$ is ex-ante Q-feasible since $q_{3b} > 0$ and $q_{2d} > 0$. However, it is not ex-post Q-feasible, as r_2 does not happen in any of deterministic assignments in its support.

Lemma 3 of Bogomolnaia and Moulin (2001) stated that a random assignment, given a preference profile, is ordinally efficient if and only if their defined binary relation based on the given preference profile is acyclic. We could read this Lemma, in our context, as *a random assignment P is ordinal efficient if and only if it has no ex-ante P-feasible cycle.* Lemma 1 proves a similar statement for a robust ex-post Pareto efficient random assignment.

Lemma 1. *A random assignment P is robust ex-post Pareto efficient if and only if it has no ex-post P-feasible cycle.*

Proof. First, we show that if a random assignment P is robust ex-post Pareto efficient, then it has no ex-post P-feasible cycle. Let $\succ$ be a preference profile and suppose that P is robust ex-post Pareto efficient. Suppose for some $r \in CL^{\succ}$, the cycle r is ex-post P-feasible. Then by Definition 2, there exists a decomposition of P to deterministic assignments, i.e., $P = \lambda_1\Pi_1 + \lambda_2\Pi 2 + \ldots + \lambda_k\Pi_k$ for some k, such that for some Π_j with $\lambda_j \neq 0$, we have for all $1 \le t \le m$, $(\Pi_j)_{i_t a_t} = 1$. Note that Π_j is not Pareto efficient and thus $\lambda_1\Pi_1 + \lambda_2\Pi 2 + \ldots + \lambda_k\Pi_k$ is a decomposition of P where there exists a non-Pareto efficient deterministic assignment in its support. It contradicts with P is robust ex-post Pareto efficient. For the converse, suppose that P is not robust ex-post Pareto efficient. Then, it has a convex decomposition with a non-Pareto efficient deterministic assignment Π which has a cycle, say $r = \langle(a_1,i_1),(a_2,i_2),\ldots,(a_m,i_m)\rangle$. Since Π is in the support of P, by Definition 2, r is an ex-post P-feasible cycle, a contradiction. □

The Proposition 1 proves that any ordinal efficient random assignment is robust ex-post Pareto efficient but not necessarily vice versa.

Proposition 1. *Ordinal efficiency implies robust ex-post Pareto efficiency while the reverse does not hold.*

Proof. It is straightforward to show that if a trading cycle is ex-post P-feasible, it is ex-ante P-feasible. Moreover, Bogomolnaia and Moulin (2001) implicitly stated that a random assignment P is ordinal efficient if and only if it has no ex-ante P-feasible cycle. Therefore, by Lemma 1, it is easily seen that ordinal efficiency implies robust ex-post Pareto efficiency. The other direction holds by the following simple counterexample that shows although the support of any decomposition of a robust ex-post Pareto efficient random assignment has only Pareto efficient deterministic assignment, it could be ordinally inefficient. For the preference profile (1), the random assignment Q, in example 2, has a unique decomposition into Pareto efficient deterministic assignments, and hence it is robust ex-post Pareto efficient. But, Q is not ordinally efficient since it is stochastically dominated by

$$\begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}. \qquad \square$$

There exists a hierarchy between different notions of efficiency. Fig. 1 illustrates the relationship between ordinal efficiency (OE), robust ex-post Pareto efficiency (REPE), and ex-post Pareto efficiency (EPE).

5. Main results: impossibilities

In this Section, we prove that any strategy-proof mechanism that satisfies robust ex-post Pareto efficiency is not possible to achieve with different notions of fairness, namely, equal treatment of equals or equal division lower bound. Proposition 2 shows the first impossibility when only four agents and objects exist. Then, Theorem 1 proves it for the general case.

To prove our impossibility results when there are only four agents and objects, we suppose by a contradictory argument that a strategy-proof and fair mechanism satisfies robust ex-post Pareto efficiency. Since, Mennle and Seuken (2021) proved

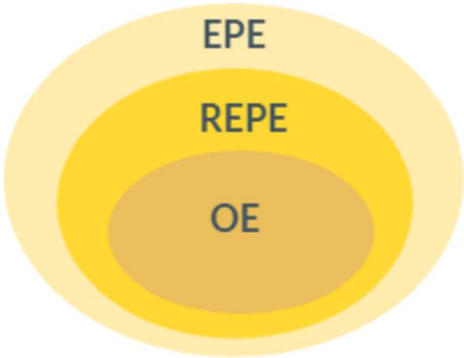

Fig. 1. Relationship between different notions of efficiency.

that a mechanism is strategy-proof if and only if it is *swap monotonic*,[3] *upper invariant*,[4] and *lower invariant*,[5] our mechanism should also satisfy these three properties. We first determine the mechanism's output for a profile where all agents have the same preference. Then, we change an agent's preference marginally in any subsequent profiles until we get to a profile for which the mechanism cannot output any bistochastic random assignment. This contradiction shows that it is impossible to have such a mechanism in the first place.

Proposition 2. *For $n = 4$, there is no mechanism satisfying robust ex-post Pareto efficiency, equal treatment of equals, and strategy-proofness.*

Proof. See the Appendix. □

Now, we are equipped to prove our first impossibility result in the general case with at least four agents and objects.

Theorem 1. *For $n \geq 4$, there is no strategy-proof mechanism satisfying robust ex-post Pareto efficiency and equal treatment of equals.*

Proof. Let $N_1 = \{1, 2, 3, 4\}$, $N_2 = \{5, 6, ..., n\}$, $A_1 = \{a, b, c, d\}$, and $A_2 = \{o_5, o_6, ..., o_n\}$. We extend each profile Qk $(1 \leq k \leq 14)$ in proof of Proposition 2 to Qk' in the way that

- for $i \in N_1$, the preference of i in profile Qk' over objects in A_1 is the same as her preference in profile Qk,
- for $i \in N_1$, for all $x \in A_1$ and all $y \in A_2$, we have $x \succ_i' y$,
- for all $i, j \in N_1$, for all $x, y \in A_2$, $x \succ_i' y$ if and only if $x \succ_j' y$,
- for $i \in N_2$, the first-best of i is o_i, and
- for $i \in N_2$, for all $x \in A_2$ and all $y \in A_1$, we have $x \succ_i' y$.

Because of robust ex-post Pareto efficiency, using Lemma 4 (in the Appendix), we have $\phi_{io_i}(Qk') = 1$ for all $i \in N_2$. Therefore, the same argument of Proposition 2 works for profiles Qk' $(1 \leq k \leq 14)$ as well. □

Martini (2016) proved that if agents have outside options, no mechanism is strategy-proof, non-wasteful, and satisfies equal treatment of equals. However, if we allow the mechanism to be wasteful, the impossibility does not hold.[6] The impossibility result of Bogomolnaia and Moulin (2001) is a conclusion of our Theorem 1. Chun and Yun (2020) introduced a

[3] For each pair $\succ_i, \succ_i' \in \mathcal{F}$, $\succ_i'$ is **adjacent** to $\succ_i$ if $\succ_i'$ is attained from $\succ_i$ by swapping two sequentially ranked objects without changing the rank of any other objects. A mechanism satisfies **swap monotonicity** if one agent alters her preference to another adjacent one, then either the agent gets the same allocation, or she is more likely to receive the object with a higher rank in the revised preference. More formally, for each $\succ_i \in \mathcal{F}^N$, each $i \in N$, each $\succ_i' \in \mathcal{F}$, and each $a, b \in A$, if $\succ_i'$ is adjacent to $\succ_i$ *i.e.*, $a \succ_i b$, and $b \succ_i' a$, then either $\mu_i(\succ_i', \succ_{-i}) = \mu_i(\succ_i, \succ_{-i})$ or $\mu_{ib}(\succ_i', \succ_{-i}) > \mu_{ib}(\succ_i, \succ_{-i})$.

[4] Let $U(\succ_i, a) = \{b \in A | b \succ_i a\}$ be the (strict) **upper contour** set of a in $\succ_i$. A mechanism satisfies **upper invariance**, introduced by Hashimoto et al. (2014), if an agent replaces her preference with another adjacent one, the probabilities of getting any object in the strict upper-contour set of the two swapping objects should not be changed. More formally, for $\succ_i \in \mathcal{F}^N$, each $i \in N$, each $\succ_i' \in \mathcal{F}$, and each $a, b \in A$, if $\succ_i'$ is adjacent to $\succ_i$ *i.e.*, $a \succ_i b$, and $b \succ_i' a$, then $\mu_{ic}(\succ_i', \succ_{-i}) > \mu_{ic}(\succ_i, \succ_{-i})$ for each $c \in U(\succ_i, a)$.

[5] Let $L(\succ_i, a) = \{b \in A | a \succ_i b\}$ be the (strict) **lower contour** set of a in $\succ_i$. A mechanism meets **lower invariance** if an agent switches her preference to another adjacent one, the probabilities of receiving any object in the strict lower-contour set of the two swapping objects should not be altered. More formally, for $\succ_i \in \mathcal{F}^N$, each $i \in N$, each $\succ_i' \in \mathcal{F}$, and each $a, b \in A$, if $\succ_i'$ is adjacent to $\succ_i$ *i.e.*, $a \succ_i b$, and $b \succ_i' a$, then $\mu_{ic}(\succ_i', \succ_{-i}) > \mu_{ic}(\succ_i, \succ_{-i})$ for each $c \in L(\succ_i, b)$.

[6] Given a preference profile with n agents and m objects (excluding the null object), the market designer gives each agent only $\frac{1}{\max\{n,m\}}$ of her first best object. This mechanism is clearly strategy-proof since no agent could benefit by misreporting her preferences. It also satisfies equal treatment of equals as

weakening of strategy-proofness, called *upper-contour strategy-proofness*, which showed to be equivalent to the combination of upper invariance and lower invariance. They proved that the impossibility of Bogomolnaia and Moulin (2001) still holds even with the introduced notion of strategy-proofness. As we also used only upper invariance and lower invariance in our proof of Proposition 2, one could read Chun and Yun (2020) impossibility as a conclusion of our Theorem 1.

Proposition 3 shows the second impossibility when only four agents and objects exist. Then, Theorem 2 proves it for the general case.

Proposition 3. *For $n = 4$, there is no strategy proof mechanism satisfying robust ex-post Pareto efficiency and equal division lower bound.*

Proof. See the Appendix. □

Our Proposition 3 strengthens the impossibility of Nesterov (2017) by weakening ordinal efficiency to robust ex-post Pareto efficiency.

Theorem 2. *For $n \geq 4$, there is no strategy-proof mechanism satisfying robust ex-post Pareto efficiency and equal division lower bound.*

Proof. The proof is straightforward. It is easily derived from Proposition 3 similar to the proof of Theorem 1. □

6. Conclusion

An assignment is robust ex-post Pareto efficient whenever for any of its lottery decomposition, each deterministic assignment in its support is Pareto efficient. We showed that robust ex-post Pareto efficiency is an intermediate notion of efficiency weaker than ordinal efficiency but stronger than ex-post Pareto efficiency. In other words, ordinal efficiency implies robust ex-post Pareto efficiency while the reverse does not hold. We also introduced ex-ante/ex-post feasible trading cycles to differentiate robust ex-post Pareto efficiency from ordinal efficiency. Given a random assignment, we say that a trading cycle is ex-ante feasible whenever each agent/object pair in the cycle happens with a positive probability, and ex-post feasible whenever there exists a decomposition of the random assignment such that the cycle happens in one of the deterministic assignments in its support. We proved that a random assignment is robust ex-post Pareto efficient if and only if it has no ex-post feasible cycle. Finally, we proved two impossibility results. We knew that strategy-proof and ordinal efficient mechanisms satisfy neither equal treatment of equals nor equal division lower bound. We proved that these impossibilities prevail, for at least four agents and objects, even when we replace ordinal efficiency with a weaker notion of robust ex-post Pareto efficiency.

Appendix A

A.1. Auxiliary lemmas

For a trading cycle $r = \langle (a_1, i_1), (a_2, i_2), ..., (a_k, i_k) \rangle$, we collect all agents within the cycle in $N_r = \{i_1, i_2, ..., i_k\}$, and all objects within the cycle in $A_r = \{a_1, a_2, ..., a_k\}$.

Lemma 2. *Let $P = [p_{ia}]_{i \in N, a \in A}$ be a random assignment, and $r = \langle (a_1, i_1), (a_2, i_2), ..., (a_k, i_k) \rangle$ be a trading cycle. If r is **ex-ante P-feasible** but not **ex-post P-feasible**, then for every permutation $\sigma : (N - N_r) \to (A - A_r)$, we have $p_{i_{k+1}\sigma(i_{k+1})} \times p_{i_{k+2}\sigma(i_{k+2})} \times ... \times p_{i_n\sigma(i_n)} = 0$.*

Proof. We prove by contradiction. Suppose for some permutation σ, we have

$$p_{i_{k+1}\sigma(i_{k+1})} \times p_{i_{k+2}\sigma(i_{k+2})} \times ... \times p_{i_n\sigma(i_n)} \neq 0. \tag{2}$$

Then, consider the deterministic assignment Π where $\Pi_{i_t a_t} = 1$ for $1 \leq t \leq k$ and $\Pi_{i_t\sigma(i_t)} = 1$ for $t > k$. Let $\varepsilon = \min\{p_{ia} | a \in A, i \in N, \Pi_{ia} = 1\}$. Note that given (2), and the assumption that r is ex-ante P-feasible, $\varepsilon > 0$. For $P' = P - \varepsilon\Pi$, the matrix $\frac{1}{1-\varepsilon}P'$ is bistochastic, and by the Birkhoff-von Neumann theorem, has a convex decomposition into deterministic assignments, i.e., $\frac{1}{1-\varepsilon}P' = \lambda_1\Pi_1 + ... + \lambda_m\Pi_m$ where $\Pi_1, ..., \Pi_m$ are deterministic assignments. Then, $P = \varepsilon\Pi + P' = \varepsilon\Pi + (1-\varepsilon)\lambda_1\Pi_1 + ... + (1-\varepsilon)\lambda_m\Pi_m$. As r is a cycle for Π, by Definition 2, it is ex-post P-feasible, that is a contradiction. □

two agents with the same first best object get the same allocation. Finally, it is robust ex-post Pareto efficient since, in all deterministic assignments of its support, agents get an object only if it is their first best, and no agent exchanges her first best object with any object of another agent.

Lemma 3. *Let $N = \{i, j, h, k\}$ be a set of agents and $\succ$ be a profile of preferences over objects $A = \{a, b, c, d\}$. Suppose for two arbitrary objects, without loss of generality say a and b, and two arbitrary agents, without loss of generality say i and j, we have $a \succ_i b$ and $b \succ_j a$. For every robust ex-post Pareto efficient random assignment P if $p_{ib} > 0$ and $p_{ja} > 0$, then either for object c, $p_{hc} = p_{kc} = 0$, or for object d, $p_{hd} = p_{kd} = 0$.*

Proof. Suppose P is a robust ex-post Pareto efficient random assignment. Then, $[(b, i), (a, j)]$ is not an ex-post P-feasible cycle. By Lemma 2-(ii), since $p_{ib} > 0$ and $p_{ja} > 0$, we have $p_{hc}p_{kd} = 0$ and $p_{hd}p_{kc} = 0$, which in turn imply that at least one of the following happens:

(i) $p_{hc} = p_{kc} = 0$
(ii) $p_{hd} = p_{kd} = 0$
(iii) $p_{hc} = p_{hd} = 0$
(iv) $p_{kc} = p_{kd} = 0$

If (i) or (ii) happens, we are done. Suppose (iii) happens, i.e.,

$$p_{hc} = p_{hd} = 0. \tag{3}$$

Then,

$$p_{ha} + p_{hb} = 1. \tag{4}$$

Since $p_{ib} > 0$ and $p_{ja} > 0$, we have $p_{ha} > 0$ and $p_{hb} > 0$. We also have either $a \succ_h b$ or $b \succ_h a$. In the former case, $[(b, h), (a, j)]$ is a trading cycle and in the latter case, $[(a, h), (b, i)]$ is a trading cycle. Assume the former case happens.[7] Since P is robust ex-post Pareto efficient, the trading cycle $[(b, h), (a, j)]$ is not ex-post P-feasible. Thus, by Lemma 2-(ii), we have

$$p_{ic}p_{kd} = p_{id}p_{kc} = 0. \tag{5}$$

Because of (4) and $p_{ja} > 0$, we have $p_{ia} + p_{ib} < 1$ and thus either $p_{ic} \neq 0$ or $p_{id} \neq 0$. Therefore, by (5) either $p_{kd} = 0$ or $p_{kc} = 0$. So, using (3), we have either $p_{kd} = p_{hd} = 0$ or $p_{kc} = p_{hc} = 0$, and we are done. The argument for (iv) is similar. □

Lemma 4. *Let $N = N_1 \cup N_2$ (where $N_1 \cap N_2 = \varnothing$), and $A = A_1 \cup A_2$ (where $A_1 \cap A_2 = \varnothing$) with $|N_1| = |A_1|$ and $|N_2| = |A_2|$, and $\succ$ be a profile of preferences where*

(1) for all $i \in N_2$, the first-best of agent i, denoted by FB_i, is not the first-best of any other agent in N,
(2) for all $i \in N_2$, for all $a \in A_2$, and all $b \in A_1$, $a \succ_i b$,
(3) for all $i \in N_1$, for all $b \in A_1$, and all $a \in A_2$, $b \succ_i a$.

Then, for every Pareto efficient deterministic assignment Π regarding profile $\succ$, $\Pi(i) = FB_i$, for all $i \in N_2$.

Proof. It is straightforward. □

A.2. Omitted proofs

Proof of Proposition 2. *Let $N = \{1, 2, 3, 4\}$ be a set of agents, and $A = \{a, b, c, d\}$ be a set of objects. Suppose by a contradictory argument that ϕ is a mechanism satisfying robust ex-post Pareto efficiency, equal treatment of equals, and strategy-proofness.* □

Profile 1. $Q1$: for all agents i, $\succ_i$: $abcd$. By equal treatment of equals, we have

$$\phi(Q1) = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}.$$

Profile 2. $Q2$: for $i = 1, 2, 3$, $\succ_i$: $abcd$, and $\succ_4$: $bacd$. By lower invariance, which is a consequence of strategy-proofness, $\phi_{4c}(Q2) = \phi_{4c}(Q1) = 1/4$, and $\phi_{4d}(Q2) = \phi_{4d}(Q1) = 1/4$. We apply Lemma 3 to objects a and b to show that $\phi_{4a}(Q2) = 0$ (suppose it is not the case, and $\phi_{4a}(Q2) > 0$. Then, for some agent $j \neq 4$, $\phi_{jb}(Q2) > 0$. By Lemma 3, either for all $i \in N \backslash \{4, j\}$, $\phi_{ic}(Q2) = 0$ or for all $i \in N \backslash \{4, j\}$, $\phi_{id}(Q2) = 0$. Suppose the former case happens. Since ϕ satisfies equal treatment of equals, we have $\phi_{1c}(Q2) = \phi_{2c}(Q2) = \phi_{3c}(Q2) = 0$, which contradicts with $\phi_{4c}(Q2) = 1/4$ and the fact that $\phi(Q2)$ is a bistochastic matrix. The argument

[7] The argument for the latter case is similar.

for the latter case is similar. Thus, $\phi_{4a}(Q2) = 0$). Therefore, $\phi_{4b}(Q2) = 1/2$ and by equal treatment of equals, we can construct the allocations of agents 1, 2, and 3.

$$\phi(Q2) = \begin{pmatrix} 1/3 & 1/6 & 1/4 & 1/4 \\ 1/3 & 1/6 & 1/4 & 1/4 \\ 1/3 & 1/6 & 1/4 & 1/4 \\ 0 & 1/2 & 1/4 & 1/4 \end{pmatrix}.$$

Profile 3. $Q3$*: for* $i = 1, 2$, $\succ_i = abcd$, *for* $i = 3, 4$, $\succ_i = bacd$. *By lower invariance,* $\phi_{3c}(Q3) = \phi_{3c}(Q2) = 1/4$ *and* $\phi_{3d}(Q3) = \phi_{3d}(Q2) = 1/4$. *By equal treatment of equals,* $\phi_{4c}(Q3) = \phi_{4d}(Q3) = 1/4$. *Again, by equal treatment of equals,* $\phi_{1c}(Q3) = \phi_{2c}(Q3) = 1/4$, *and* $\phi_{1d}(Q3) = \phi_{2d}(Q3) = 1/4$.

$$\phi(Q3) = \begin{pmatrix} - & - & 1/4 & 1/4 \\ - & - & 1/4 & 1/4 \\ - & - & 1/4 & 1/4 \\ - & - & 1/4 & 1/4 \end{pmatrix}.$$

We apply Lemma 3 to objects a and b. Since for all $i \in N$, $\phi_{ic}(Q3) = \phi_{id}(Q3) = 1/4$, *either* $\phi_{1b}(Q3) = 0$ *or* $\phi_{3a}(Q3) = 0$. *If* $\phi_{1b}(Q3) = 0$, *then* $\phi_{1a}(Q3) = 1/2$ *and by equal treatment of equals,* $\phi_{2b}(Q3) = 0$, *and* $\phi_{2a}(Q3) = 1/2$. *Then,* $\phi_{3a}(Q3) = 0$, $\phi_{4a}(Q3) = 0$, $\phi_{3b}(Q3) = 1/2$, $\phi_{4a}(Q3) = 0$ *and* $\phi_{4b}(Q3) = 1/2$. *If* $\phi_{3a}(Q3) = 0$, *with a similar argument, we derive the same results. Finally, we have*

$$\phi(Q3) = \begin{pmatrix} 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 0 & 1/2 & 1/4 & 1/4 \\ 0 & 1/2 & 1/4 & 1/4 \end{pmatrix}.$$

Profile 4. $Q4$*: for* $i = 1, 2, 3$, $\succ_i$: *abcd, and* $\succ_4$: *bcad. By lower invariance,* $\phi_{4d}(Q4) = \phi_{4d}(Q2) = 1/4$, *and by upper invariance,* $\phi_{4b}(Q4) = \phi_{4b}(Q2) = 1/2$. *We apply Lemma 3 to objects a and b to show that* $\phi_{4a}(Q4) = 0$ *(by a contradictory argument, suppose* $\phi_{4a}(Q4) > 0$. *Then, for some agent* $i = 1, 2$, *or 3, we have* $\phi_{ib}(Q4) > 0$. *Without loss of generality, assume* $i = 1$. *Now, by Lemma 3, either* $\phi_{2c}(Q4) = \phi_{3c}(Q4) = 0$ *or* $\phi_{2d}(Q4) = \phi_{3d}(Q4) = 0$. *Suppose the former case happens. Then, by equal treatment of equals,* $\phi_{1c}(Q4) = 0$, *which implies* $\phi_{4c}(Q4) = 1$, *a contradiction. The argument for the latter case is similar). Finally, we have* $\phi_{4c}(Q4) = 1/4$. *By equal treatment of equals, we can construct the allocations of agents 1, 2, and 3. Finally, we have* $\phi(Q4) = \phi(Q2)$.

Profile 5. $Q5$*: for* $i = 1, 2$, $\succ_i$: *abcd,* $\succ_3$: *bacd, and* $\succ_4$: *bcad. By lower invariance,* $\phi_{3c}(Q5) = \phi_{3c}(Q4) = 1/4$, *and* $\phi_{3d}(Q5) = \phi_{3d}(Q4) = 1/4$. *By upper invariance,* $\phi_{4b}(Q5) = \phi_{4b}(Q3) = 1/2$, *and by lower invariance* $\phi_{4d}(Q5) = \phi_{4d}(Q3) = 1/4$. *By equal treatment of equals, we get* $\phi_{1d}(Q5) = \phi_{2d}(Q5) = 1/4$. *Now, if* $\phi_{1b}(Q5) > 0$ *and* $\phi_{4a}(Q5) > 0$, *then by Lemma 3, either* $\phi_{2c}(Q5) = \phi_{3c}(Q5) = 0$ *or* $\phi_{2d}(Q5) = \phi_{3d}(Q5) = 0$. *However, both cases are impossible as we already showed* $\phi_{3c}(Q5) = 1/4 \neq 0$ *and* $\phi_{3d}(Q5) = 1/4 \neq 0$. *Therefore, Lemma 3 implies* $\phi_{1b}(Q5) = 0$ *or* $\phi_{4a}(Q5) = 0$, *where in both cases* $\phi_{3a}(Q5) = 0$ *holds.*

In the former case where $\phi_{1b}(Q5) = 0$, *by equal treatment of equals,* $\phi_{2b}(Q5) = 0$, *and we derive* $\phi_{3b}(Q5) = 1/2$ *which implies* $\phi_{3a}(Q5) = 0$. *In the latter case where* $\phi_{4a}(Q5) = 0$, *we have* $\phi_{4c}(Q5) = 1/4$. *Since* $\phi_{4c}(Q5) \neq 0$, *and* $\phi_{2d}(Q5) \neq 0$, *with a similar argument, Lemma 3 implies* $\phi_{1b}(Q5) = 0$ *or* $\phi_{3a}(Q5) = 0$. *Note that we already showed that the former case* $\phi_{1b}(Q5) = 0$ *leads to* $\phi_{3a}(Q5) = 0$.

Since $\phi(Q5)$ *is bistochastic, we have* $\phi_{3b}(Q5) = 1/2$ *and by equal treatment of equals, we derive* $\phi_{1b}(Q5) = \phi_{2b}(Q5) = 0$. *Moreover,* $\phi_{1c}(Q5) > 0$ *since otherwise* $\phi_{4c}(Q5) = 3/4$ *and* $\phi_{4c}(Q5) + \phi_{4b}(Q5) > 1$, *which is a contradiction. Now, we apply Lemma 3 to objects a and c to show* $\phi_{4a}(Q5) = 0$. *By a contradictory argument, suppose* $\phi_{4a}(Q5) > 0$. *Since* $\phi_{1c}(Q5) > 0$, *applying Lemma 3 to objects a and c where* $c \succ_4 a$ *and* $a \succ_1 c$, *implies either* $\phi_{2b}(Q5) = \phi_{3b}(Q5) = 0$ *or* $\phi_{2d}(Q5) = \phi_{3d}(Q5) = 0$. *But, we already derived that* $\phi_{3b}(Q5) = 1/2 \neq 0$ *and* $\phi_{3d}(Q5) = 1/4 \neq 0$, *a contradiction. Hence,* $\phi_{4c}(Q5) = 1/4$, *and by equal treatment of equals, we get* $\phi_{1c}(Q5) = \phi_{2c}(Q5) = 1/4$, *and* $\phi_{1a}(Q5) = \phi_{2a}(Q5) = 1/2$. *Therefore,* $\phi(Q5) = \phi(Q3)$.

Profile 6. $Q6$*: for* $i = 1, 2$, $\succ_i$: *abcd, and for* $i = 3, 4$, $\succ_i$: *bcad. By upper invariance,* $\phi_{3b}(Q6) = \phi_{3b}(Q5) = 1/2$, *and by lower invariance,* $\phi_{3d}(Q6) = \phi_{3d}(Q5) = 1/4$. *By equal treatment of equals,* $\phi_{4b}(Q6) = 1/2$ *and* $\phi_{4d}(Q6) = 1/4$. *Note that* $\phi_{1c}(Q6) > 0$. *Since otherwise, by equal treatment of equals,* $\phi_{1c}(Q6) = \phi_{2c}(Q6) = 0$ *and* $\phi_{3c}(Q6) = \phi_{4c}(Q6) = 1/2$ *which is a contradiction as* $\phi_{4c}(Q6) + \phi_{4b}(Q6) + \phi_{4d}(Q6) > 1$.

Now, we apply Lemma 3 to objects a and c to show $\phi_{3a}(Q6) = 0$. *By contradiction, assume that* $\phi_{3a}(Q6) > 0$. *As we also have* $\phi_{1c}(Q6) > 0$, $c \succ_3 a$, *and* $a \succ_1 c$, *by Lemma 3, either* $\phi_{2b}(Q6) = \phi_{4b}(Q6) = 0$ *or* $\phi_{2d}(Q6) = \phi_{4d}(Q6) = 0$. *Thus, by equal treatment of equals, either for all* $i \in N$, $\phi_{ib}(Q6) = 0$ *or for all* $i \in N$, $\phi_{id}(Q6) = 0$, *a contradiction. Since* ϕ *satisfies equal treatment of equals,* $\phi_{4a}(Q6) = 0$ *and* $\phi_{1a}(Q6) = \phi_{2a}(Q6) = 1/2$. *Since* $\phi(Q6)$ *is a bistochastic matrix and satisfies equal treatment of equals, we simply derive* $\phi_{3c}(Q6) = \phi_{4c}(Q6) = 1/4$. *Altogether, we have* $\phi(Q6) = \phi(Q5)$.

Profile 7. $Q7$*: for* $i = 1, 2, 3$, $\succ_i$: *abcd, and* $\succ_4$: *bcda. By upper invariance,* $\phi_{4b}(Q7) = \phi_{4b}(Q4) = 1/2$ *and* $\phi_{4c}(Q7) = \phi_{4c}(Q4) = 1/4$. *By equal treatment of equals,* $\phi_{1b}(Q7) = \phi_{2b}(Q7) = \phi_{3b}(Q7) = 1/6$ *and* $\phi_{1c}(Q7) = \phi_{2c}(Q7) = \phi_{3c}(Q7) = 1/4$. *We apply*

Lemma 3 to objects a and b to show $\phi_{4a}(Q7)=0$. Suppose $\phi_{4a}(Q7)>0$. As we also have $\phi_{1b}(Q7)>0$, Lemma 3 implies either $\phi_{2c}(Q7)=\phi_{3c}(Q7)=0$ or $\phi_{2d}(Q7)=\phi_{3d}(Q7)=0$ which by equal treatment of equals implies either for all $i\in N$, $\phi_{ic}(Q7)=0$ or for all $i\in N$, $\phi_{id}(Q7)=0$. We derive either $\phi_{4c}(Q7)=1$ or $\phi_{4d}(Q7)=1$ which is a contradiction as ϕ is a bistochastic matrix. Together, we have $\phi(Q7)=\phi(Q2)$.

Profile 8. $Q8$: for $i=1,2$, $\succ_i$: abcd, $\succ_3$: bacd, and $\succ_4$: bcda. By lower invariance, $\phi_{3c}(Q8)=\phi_{3c}(Q7)=1/4$ and $\phi_{3d}(Q8)=\phi_{3d}(Q7)=1/4$, and by upper invariance, $\phi_{4b}(Q8)=\phi_{4b}(Q5)=1/2$ and $\phi_{4c}(Q8)=\phi_{4c}(Q5)=1/4$. Note that $\phi_{1d}(Q8)>0$ since otherwise $\phi_{1d}(Q8)=\phi_{2d}(Q8)=0$ that implies $\phi_{4d}(Q8)=3/4$, which is a contradiction as $\phi_{4b}(Q8)+\phi_{4d}(Q8)>1$. We apply Lemma 3 to objects a and d to show $\phi_{4a}(Q8)=0$. Suppose it is not the case, then $\phi_{4a}(Q8)>0$ and $\phi_{1d}(Q8)>0$. Lemma 3 implies either $\phi_{2b}(Q8)=\phi_{3b}(Q8)=0$ or $\phi_{2c}(Q8)=\phi_{3c}(Q8)=0$. Since ϕ satisfies equal treatment of equals, we have either $\phi_{1b}(Q8)=\phi_{2b}(Q8)=\phi_{3b}(Q8)=0$ or $\phi_{1c}(Q8)=\phi_{2c}(Q8)=\phi_{3c}(Q8)=0$. The former case implies $\phi_{4b}(Q8)=1$ which contradicts with $\phi_{4b}(Q8)=1/2$. The latter case is also impossible as we already showed $\phi_{3c}(Q8)=1/4$. Therefore, since $\phi(Q8)$ is a bistochastic matrix, $\phi_{4d}(Q8)=1/4$ and since it is a bistochastic matrix and satisfies equal treatment of equals, $\phi_{1c}(Q8)=\phi_{2c}(Q8)=\phi_{1d}(Q8)=\phi_{2d}(Q8)=1/4$.

Also, note that $\phi_{1b}(Q8)=0$ or $\phi_{3a}(Q8)=0$. Since otherwise if $\phi_{1b}(Q8)>0$ and $\phi_{3a}(Q8)>0$, then by Lemma 3 we have either $\phi_{2c}(Q8)=\phi_{4c}(Q8)=0$ or $\phi_{2d}(Q8)=\phi_{4d}(Q8)=0$. However, both cases are impossible as we already showed $\phi_{2c}(Q8)=1/4\neq 0$ and $\phi_{2d}(Q8)=1/4\neq 0$. If $\phi_{1b}(Q8)=0$ then by equal treatment of equals, $\phi_{2b}(Q8)=0$ and $\phi_{1a}(Q8)=\phi_{2a}(Q8)=1/2$ and since $\phi(Q8)$ is a bistochastic matrix, $\phi_{3a}(Q8)=0$ and $\phi_{3b}(Q8)=1/2$. If $\phi_{3a}(Q8)=0$ then $\phi_{3b}(Q8)=1/2$. By equal treatment of equals, $\phi_{2b}(Q8)=\phi_{2b}(Q8)=0$ and $\phi_{1a}(Q8)=\phi_{2a}(Q8)=1/2$. Altogether, we have $\phi(Q8)=\phi(Q3)$.

Profile $8'$. $Q8'$: $\succ_1$: bacd, for $i=2,3$, $\succ_i$: abcd, and $\succ_4$: bcda. Profile $Q8'$ is a permutation of profile $Q8$ and we have

$$\phi(Q8')=\begin{pmatrix}0 & 1/2 & 1/4 & 1/4\\ 1/2 & 0 & 1/4 & 1/4\\ 1/2 & 0 & 1/4 & 1/4\\ 0 & 1/2 & 1/4 & 1/4\end{pmatrix}.$$

Profile $8''$. $Q8''$: for $i=1,3$, $\succ_i$: abcd, $\succ_2$: bacd, and $\succ_4$: bcda. Profile $Q8''$ is a permutation of profile $Q8$ and we have

$$\phi(Q8'')=\begin{pmatrix}1/2 & 0 & 1/4 & 1/4\\ 0 & 1/2 & 1/4 & 1/4\\ 1/2 & 0 & 1/4 & 1/4\\ 0 & 1/2 & 1/4 & 1/4\end{pmatrix}.$$

Profile 9. $Q9$: for $i=1,2,3,4$, $\succ_i$: bacd. By equal treatment of equals, we have $\phi(Q9)=\phi(Q1)$.

Profile 10. $Q10$: for $i=1,2,3$, $\succ_i$: bacd and $\succ_4$: bcad. By upper invariance, $\phi_{4b}(Q10)=\phi_{4b}(Q9)=1/4$, and by lower invariance $\phi_{4d}(Q10)=\phi_{4d}(Q9)=1/4$. Since $\phi_{4b}(Q10)=1/4$, by equal treatment of equals, $\phi_{1b}(Q10)=\phi_{2b}(Q10)=\phi_{3b}(Q10)=1/4$. Since $\phi_{4d}(Q10)=1/4$ by equal treatment of equals, $\phi_{1d}(Q10)=\phi_{2d}(Q10)=\phi_{3d}(Q10)=1/4$. By Lemma 3 applied to objects a and c, $\phi_{4a}(Q10)=0$. Otherwise, $\phi_{4a}(Q10)>0$ and $\phi_{1c}(Q10)>0$. By Lemma 3, either $\phi_{2b}(Q10)=\phi_{3b}(Q10)=0$ or $\phi_{2d}(Q10)=\phi_{3d}(Q10)=0$, but we already showed that $\phi_{2b}(Q10)=1/4$ (note that $\phi_{1c}(Q10)>0$, since otherwise by equal treatment of equals, $\phi_{1c}(Q10)=\phi_{2c}(Q10)=\phi_{3c}(Q10)=0$ which implies $\phi_{4c}(Q10)=1$ that contradicts with $\phi_{4b}(Q10)=1/4$) and $\phi_{2d}(Q10)=1/4$, a contradiction. Therefore, $\phi_{4a}(Q10)=0$ and we get $\phi_{4c}(Q10)=1/2$. By equal treatment of equals,

$$\phi(Q10)=\begin{pmatrix}1/3 & 1/4 & 1/6 & 1/4\\ 1/3 & 1/4 & 1/6 & 1/4\\ 1/3 & 1/4 & 1/6 & 1/4\\ 0 & 1/4 & 1/2 & 1/4\end{pmatrix}.$$

Profile 11. $Q11$: for $i=1,2,3$, $\succ_i$: bacd and $\succ_4$: bcda. By upper invariance, $\phi_{4b}(Q11)=\phi_{4b}(10)=1/4$ and $\phi_{4c}(Q11)=\phi_{4c}(10)=1/2$. Note that because of equal treatment of equals, and $\phi_{4c}(Q11)=1/2>0$,

$$\text{for all object } o\in A,\ \phi_{1o}(Q11)=\phi_{2o}(Q11)=\phi_{3o}(Q11)>0. \tag{6}$$

By Lemma 3 applied to objects a and c, we get $\phi_{4a}(Q11)=0$. Suppose it is not the case. Then, $\phi_{4a}(Q11)>0$ and $\phi_{1c}(Q11)>0$. By Lemma 3, we have either $\phi_{2b}(Q11)=\phi_{3b}(Q11)=0$ or $\phi_{2d}(Q11)=\phi_{3d}(Q11)=0$ which contradicts with (6). Hence, $\phi_{4d}(Q11)=1/4$ and by equal treatment of equals, $\phi(Q11)=\phi(Q10)$.

Profile 12. $Q12$: for $i=1,2,4$, $\succ_i$: bacd and $\succ_3$: abcd. By lower invariance, $\phi_{3c}(Q12)=\phi_{3c}(Q9)=1/4$ and $\phi_{3d}(Q12)=\phi_{3d}(Q9)=1/4$. Note that because of equal treatment of equals, and $\phi_{3c}(Q12)=1/4>0$,

$$\text{for all object } o\in A,\ \phi_{1o}(Q12)=\phi_{2o}(Q12)=\phi_{4o}(Q12)>0. \tag{7}$$

By Lemma 3 applied to objects a and b, we get $\phi_{3b}(Q\,12)=0$. Suppose it is not the case. Then, since $\phi_{3b}(Q\,12)>0$ and $\phi_{1a}(Q\,12)>0$, using Lemma 3, we have either $\phi_{2c}(Q\,12)=\phi_{4c}(Q\,12)=0$or $\phi_{2d}(Q\,12)=\phi_{4d}(Q\,12)=0$ which contradicts with (7). Hence, $\phi_{3a}(Q\,12)=1/2$. By equal treatment of equals,

$$\phi(Q\,12)=\begin{pmatrix}1/6 & 1/3 & 1/4 & 1/4\\ 1/6 & 1/3 & 1/4 & 1/4\\ 1/2 & 0 & 1/4 & 1/4\\ 1/6 & 1/3 & 1/4 & 1/4\end{pmatrix}.$$

Profile 13. $Q\,13$: for $i=1,2$, $\succ_i$: bacd, $\succ_3$: abcd and $\succ_4$: bcad. By upper invariance, $\phi_{4b}(Q\,13)=\phi_{4b}(Q\,12)=1/3$.

$$\phi(Q\,13)=\begin{pmatrix}- & - & - & -\\ - & - & - & -\\ - & - & - & -\\ - & 1/3 & - & -\end{pmatrix}.$$

Profile 14. $Q\,14$: for $i=1,2$, $\succ_i$: bacd, $\succ_3$: abcd and $\succ_4$: bcda. By lower invariance, $\phi_{2c}(Q\,14)=\phi_{2c}(Q^{\,8'})=1/4$, $\phi_{2d}(Q\,14)=\phi_{2d}(Q^{\,8'})=1/4$, $\phi_{1c}(Q\,14)=\phi_{1c}(Q\,8'')=1/4$, $\phi_{1d}(Q\,14)=\phi_{1d}(Q\,8'')=1/4$, $\phi_{3c}(Q\,14)=\phi_{3c}(Q\,11)=1/6$, and $\phi_{3d}(Q\,14)=\phi_{3d}(Q\,11)=1/4$. By Lemma 3, $\phi_{3b}(Q\,14)=0$ (suppose not, then for some $i\in N\backslash\{3\}$, we have $\phi_{ia}(Q\,14)>0$. Now, by Lemma 3, either for all $j\in N\backslash\{3,i\}$, $\phi_{jc}(Q\,14)=0$ or for all $j\in N\backslash\{3,i\}$, $\phi_{jd}(Q\,14)=0$. Either agent 1 or agent 2 belongs $N\backslash\{3,i\}$. Without loss of generality, assume that $1\in N\backslash\{3,i\}$. Since $\phi_{1c}(Q\,14)=\phi_{1d}(Q\,14)=1/4\neq 0$, we face a contradiction) which implies $\phi_{3a}(Q\,14)=7/12$. By Lemma 3 applied to objects a and d, we get $\phi_{4a}(Q\,14)=0$. Suppose it is not the case, then $\phi_{4a}(Q\,14)>0$ and we already have $\phi_{3d}(Q\,14)=1/4>0$. By Lemma 3, either $\phi_{1b}(Q\,14)=\phi_{2b}(Q\,14)=0$ or $\phi_{1c}(Q\,14)=\phi_{2c}(Q\,14)=0$. If the former case happens, we have $\phi_{1a}(Q\,14)=\phi_{2a}(Q\,14)=1/2$ while $\phi_{1a}(Q\,14)+\phi_{2a}(Q\,14)+\phi_{3a}(Q\,14)>1$. The latter case also does not happen as we already showed $\phi_{2c}(Q\,14)=1/4$, a contradiction. Therefore, $\phi_{4a}(Q\,14)=0$. Now, we have $\phi_{1a}(Q\,14)=\phi_{2a}(Q\,14)=5/24$. By upper invariance, $\phi_{4b}(Q\,14)=\phi_{4b}(Q\,13)=1/3$. By equal treatment of equals, $\phi_{1b}(Q\,14)=\phi_{2b}(Q\,14)=1/3$. But $\phi_{1a}(Q\,14)+\phi_{2a}(Q\,14)+\phi_{3a}(Q\,14)+\phi_{4a}(Q\,14)=5/24+1/3+1/4+1/4>1$, a contradiction, and finally we are done.

$$\phi(Q\,14)=\begin{pmatrix}5/24 & 1/3 & 1/4 & 1/4\\ 5/24 & 1/3 & 1/4 & 1/4\\ 7/12 & 0 & 1/6 & 1/4\\ 0 & 1/3 & 1/3 & 1/4\end{pmatrix}.$$

QED.

Proof of Proposition 3. *Let $N=\{1,2,3,4\}$ be a set of agents, and $A=\{a,b,c,d\}$ be a set of objects. Suppose by a contradictory argument that ϕ is a mechanism satisfying robust ex-post Pareto efficiency, strategy-proofness, and equal division lower bound. We use the following profiles to complete our proof.* □

Profile 1. $Q\,1$: for $i=1,2$, $\succ_i$: abcd, and for $i=3,4$, $\succ_i$: acbd. By equal division lower bound, it is easily seen that

$$\phi_{1a}(Q\,1)=\phi_{2a}(Q\,1)=\phi_{3a}(Q\,1)=\phi_{4a}(Q\,1)=1/4, \tag{8}$$

and

$$\phi_{1d}(Q\,1)=\phi_{2d}(Q\,1)=\phi_{3d}(Q\,1)=\phi_{4d}(Q\,1)=1/4. \tag{9}$$

Now, by (8), (9), and applying Lemma 3 to objects b and c, we have $\phi_{1c}(Q\,1)\phi_{3b}(Q\,1)=0$, $\phi_{1c}(Q\,1)\phi_{4b}(Q\,1)=0$, $\phi_{2c}(Q\,1)\phi_{3b}(Q\,1)=0$, and $\phi_{2c}(Q\,1)\phi_{4b}(Q\,1)=0$.

We show that $\phi_{1c}(Q\,1)=0$. By contradiction, suppose $\phi_{1c}(Q\,1)\neq 0$. Then, since we already have $\phi_{1c}(Q\,1)\phi_{3b}(Q\,1)=0$ and $\phi_{1c}(Q\,1)\phi_{4b}(Q\,1)=0$, it is derived that $\phi_{3b}(Q\,1)=0$, and $\phi_{4b}(Q\,1)=0$. Because $\phi_{1c}(Q\,1)$ is a bistochastic matrix, we conclude that $\phi_{3c}(Q\,1)=\phi_{4c}(Q\,1)=1/2$. Hence, $\phi_{3c}(Q\,1)+\phi_{4c}(Q\,1)+\phi_{1c}(Q\,1)>1$, which is a contradiction. Thus, $\phi_{1c}(Q\,1)=0$. With similar arguments, it is proved that $\phi_{2c}(Q\,1)=0$, $\phi_{3b}(Q\,1)=0$, and $\phi_{4b}(Q\,1)=0$. Therefore,

$$\phi(Q\,1)=\begin{pmatrix}1/4 & 1/2 & 0 & 1/4\\ 1/4 & 1/2 & 0 & 1/4\\ 1/4 & 0 & 1/2 & 1/4\\ 1/4 & 0 & 1/2 & 1/4\end{pmatrix}.$$

Profile 2. $Q\,2$: $\succ_1$: abdc, $\succ_2$: abcd, and for $i=3,4$, $\succ_i$: acbd. By equal division lower bound, $\phi_{1a}(Q\,2)=\phi_{2a}(Q\,2)=\phi_{3a}(Q\,2)=\phi_{4a}(Q\,2)=1/4$. By upper invariance, $\phi_{1b}(Q\,2)=\phi_{1b}(Q\,1)=1/2$. By swap monotonicity, we derive either $\phi_{1d}(Q\,2)=\phi_{1d}(Q\,1)=$

$1/4$ or $\phi_{1d}(Q2) > \phi_{1d}(Q1) = 1/4$. The latter case is impossible as $\phi_{1a}(Q2) + \phi_{1b}(Q2) + \phi_{1d}(Q2) = 1/4 + 1/2 + \phi_{1d}(Q2) > 1$. Thus, we have $\phi_{1d}(Q2) = 1/4$ and $\phi_{1c}(Q2) = 0$. Now, by equal division lower bound, $\phi_{2d}(Q2) = \phi_{3d}(Q2) = \phi_{4d}(Q2) = 1/4$. Since for all i, $\phi_{ia}(Q2) > 0$ and $\phi_{id}(Q2) > 0$, by applying Lemma 3 to objects b and c, $\phi_{1c}(Q2)\phi_{3b}(Q2) = 0$, $\phi_{1c}(Q2)\phi_{4b}(Q2) = 0$, $\phi_{2c}(Q2)\phi_{3b}(Q2) = 0$, and $\phi_{2c}(Q2)\phi_{4b}(Q2) = 0$. With a same argument, discussed already for profile $Q1$, $\phi_{2c}(Q2) = 0$, $\phi_{3b}(Q2) = 0$, and $\phi_{4b}(Q2) = 0$. Therefore,

$$\phi(Q2) = \begin{pmatrix} 1/4 & 1/2 & 0 & 1/4 \\ 1/4 & 1/2 & 0 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \end{pmatrix}.$$

Profile $2'$. $Q2'$: Profile $Q2'$ is a permutation of profile $Q2$ as follows: $\succ_1$: $abcd$, $\succ_2$: $abdc$, and for $i = 3, 4$, $\succ_i$: $acbd$. Therefore,

$$\phi(Q2') = \begin{pmatrix} 1/4 & 1/2 & 0 & 1/4 \\ 1/4 & 1/2 & 0 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \end{pmatrix}.$$

Profile 3. $Q3$: for $i = 1, 2$, $\succ_i$: $abdc$, and for $i = 3, 4$, $\succ_i$: $acbd$. First, by equal division lower bound, $\phi_{1a}(Q3) = \phi_{2a}(Q3) = \phi_{3a}(Q3) = \phi_{4a}(Q3) = 1/4$. By upper invariance, $\phi_{2b}(Q3) = \phi_{2b}(Q2) = 1/2$ and $\phi_{1b}(Q3) = \phi_{1b}(Q'2) = 1/2$. By swap monotonicity, we derive either $\phi_{2d}(Q3) = \phi_{2d}(Q2) = 1/4$ or $\phi_{2d}(Q3) > \phi_{2d}(Q2) = 1/4$. The latter case is impossible as $\phi_{2a}(Q3) + \phi_{2b}(Q3) + \phi_{2d}(Q3) = 1/4 + 1/2 + \phi_{2d}(Q3) > 1$. Thus, $\phi_{2d}(Q3) = 1/4$ and $\phi_{2c}(Q3) = 0$. Again, regarding profile $Q3$ and $Q2'$ for agent 1, by swap monotonicity, with a similar argument, $\phi_{1d}(Q3) = 1/4$ and $\phi_{1c}(Q3) = 0$. Now, by equal division lower bound,

$$\phi(Q3) = \begin{pmatrix} 1/4 & 1/2 & 0 & 1/4 \\ 1/4 & 1/2 & 0 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \end{pmatrix}.$$

Profile 4. $Q4$: for $i = 1, 2$, $\succ_i$: $abdc$, and for $i = 3, 4$, $\succ_i$: $abcd$. By equal division lower bound, we have

$$\phi_{1a}(Q4) = \phi_{2a}(Q4) = \phi_{3a}(Q4) = \phi_{4a}(Q4) = 1/4, \tag{10}$$

and

$$\phi_{1b}(Q4) = \phi_{2b}(Q4) = \phi_{3b}(Q4) = \phi_{4b}(Q4) = 1/4. \tag{11}$$

Now, by (10), (11), and applying Lemma 3 to objects c and d, $\phi_{1c}(Q4)\phi_{3d}(Q4) = 0$, $\phi_{1c}(Q4)\phi_{4d}(Q4) = 0$, $\phi_{2c}(Q4)\phi_{3d}(Q4) = 0$, $\phi_{2c}(Q4)\phi_{4d}(Q4) = 0$. We show $\phi_{1c}(Q4) = 0$. Otherwise, we have $\phi_{3d}(Q4) = 0$ and $\phi_{4d}(Q4) = 0$ which causes $\phi_{3c}(Q4) = 1/2$ and $\phi_{4c}(Q4) = 1/2$. Then, we have $\phi_{3c}(Q4) + \phi_{4c}(Q4) + \phi_{1c}(Q4) = 1/2 + 1/2 + \phi_{1c}(Q4) > 1$, contradiction. With a similar argument, we derive $\phi_{2c}(Q4) = 0$, $\phi_{3d}(Q4) = 0$, and $\phi_{4d}(Q4) = 0$. Therefore,

$$\phi(Q4) = \begin{pmatrix} 1/4 & 1/4 & 0 & 1/2 \\ 1/4 & 1/4 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \end{pmatrix}.$$

Profile 5. $Q5$: for $i = 1, 2$, $\succ_i$: $abdc$, $\succ_3$: $abcd$ and $\succ_4$: $acbd$. By equal division lower bound, we have

$$\phi_{1a}(Q5) = \phi_{2a}(Q5) = \phi_{3a}(Q5) = \phi_{4a}(Q5) = 1/4. \tag{12}$$

By lower invariance, $\phi_{4d}(Q5) = \phi_{4d}(Q4) = 0$, and $\phi_{3d}(Q5) = \phi_{3d}(Q3) = 1/4$. By equal division lower bound, we have

$$\phi_{1b}(Q5) \geq 1/4,\ \phi_{2b}(Q5) \geq 1/4,\ \phi_{3b}(Q5) \geq 1/4. \tag{13}$$

Now, we apply Lemma 3 to objects d and c to show $\phi_{1c}(Q5) = \phi_{2c}(Q5) = 0$. If $\phi_{1c}(Q5) > 0$ then $\phi_{1c}(Q5)\phi_{3d}(Q5) > 0$ and thus by Lemma 3, either $\phi_{2a}(Q5) = \phi_{3a}(Q5) = 0$ or $\phi_{2b}(Q5) = \phi_{3b}(Q5) = 0$. By (12) and (13), both cases do not happen and thus $\phi_{1c}(Q5) = 0$. With a similar argument, $\phi_{2c}(Q5) = 0$. Note that $\phi_{3c}(Q5) > 0$, since otherwise we must have $\phi_{4c}(Q5) = 1$, which is a contradiction.

We show that $\phi_{4b}(Q5) = 0$. Otherwise, since $\phi(Q5)$ is a bistochastic matrix, we either have $\phi_{1d}(Q5) > 0$ or $\phi_{2d}(Q5) > 0$. If the former case happens, then since $\phi_{4b}(Q5) > 0$, $\phi_{3c}(Q5) > 0$, $\phi_{1d}(Q5) > 0$, and $\phi_{2a}(Q5) > 0$, the probability of occurrence of the cycle $([b, 4), (3, c)]$ is greater than 0 which contradicts with the assumption that $\phi(Q5)$ is robust ex-post Pareto efficient. If the latter case happens, then since $\phi_{4b}(Q5) > 0$, $\phi_{3c}(Q5) > 0$, $\phi_{2d}(Q5) > 0$, and $\phi_{1a}(Q5) > 0$, the probability of occurrence of *the cycle* $([b,4),(3,c)]$ *is greater than* 0 *which contradicts with the assumption that* $\phi(Q5)$ *is robust ex-post Pareto efficient. Thus,* $\phi_{4b}(Q5)=0$. *As* $\phi(Q5)$*is a bistochastic matrix,* $\phi_{4c}(Q5)=3/4$, *and then* $\phi_{3c}(Q5)=1/4$ *and* $\phi_{3b}(Q5)=1/4$. *Therefore,*

$$\phi(Q5)=\begin{pmatrix} 1/4 & - & 0 & - \\ 1/4 & - & 0 & - \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 0 & 3/4 & 0 \end{pmatrix}.$$

Profile $5'$. $Q5'$: *for* $i=1,3$, $\succ_i$: *abdc*, $\succ_2$: *abcd and* $\succ_4$: *acbd*. *We have*

$$\phi(Q5')=\begin{pmatrix} 1/4 & - & 0 & - \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & - & 0 & - \\ 1/4 & 0 & 3/4 & 0 \end{pmatrix}.$$

Profile $Q5''$: *for* $i=2,3$, $\succ_i$: *abdc*, $\succ_1$: *abcd and* $\succ_4$: *acbd*. *We have*

$$\phi(Q5'')=\begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & - & 0 & - \\ 1/4 & - & 0 & - \\ 1/4 & 0 & 3/4 & 0 \end{pmatrix}.$$

Profile 6. $Q6$: *for* $i=1,2,3$, $\succ_i$: *abdc and* $\succ_4$: *acbd*. *By equal division lower bound,* $\phi_{1a}(Q6)=\phi_{2a}(Q6)=\phi_{3a}(Q6)=\phi_{4a}(Q6)=1/4$. *By upper invariance,* $\phi_{1b}(Q6)=\phi_{1b}(Q''5)=1/4$, $\phi_{2b}(Q6)=\phi_{2b}(Q'5)=1/4$, *and* $\phi_{3b}(Q6)=\phi_{3b}(Q5)=1/4$ *which together conclude* $\phi_{4b}(Q6)=1/4$. *Also, by equal division lower bound, for all* $i=1,2,3$, $\phi_{id}(Q6)\geq 1/4$. *Therefore,*

$$\phi(Q6)=\begin{pmatrix} 1/4 & 1/4 & - & \geq 1/4 \\ 1/4 & 1/4 & - & \geq 1/4 \\ 1/4 & 1/4 & - & \geq 1/4 \\ 1/4 & 1/4 & - & - \end{pmatrix}.$$

Since $\phi(Q6)$ *is a bistochastic matrix, there exists* $j\in\{1,2,3\}$ *such that* $\phi_{jc}(Q6)>0$. *Then, we have* $\phi_{4b}(Q6)\phi_{jc}(Q6)>0$. *By Lemma 3, either for all* $i\in N\backslash\{4,j\}$, we have $\phi_{ia}(Q6)=0$ *or for all* $i\in N\backslash\{4,j\}$, we have $\phi_{id}(Q6)=0$, *where both cases do not happen. Contradiction.* **QED.**

Uncited references

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