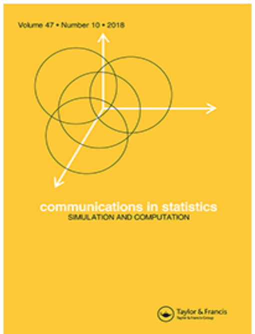

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Influence of the three different types of standby components on the performance of a *k*–out–of–*n*:F system in the dynamic stress–strength model

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ABSTRACT
We consider the effect of adding three different types of standby components to a *k*–out–of–*n*:F system based on the dynamic stress–strength model. For this purpose, the system reliability in the three different types of standby components, i.e., cold, warm and hot standby components, are calculated. Also, it is assumed that the stress and the strength components follow the Weibull distribution and the Weibull process, respectively. The maximum likelihood estimators of the redundancy system reliability are obtained. In addition, a simulation study is performed based on the Monte Carlo method. Finally, the results are illustrated via a practical example that is the time of successive failures of the air conditioning system of every member of a fleet of jet airplanes.

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Cold standby; Hot standby; k–out–of–n:F system; Reliability; Stress–strength model; Warm standby; Weibull process

1. Introduction

The stress–strength model is one of the most widely used and essential reliability models. This concept was introduced by Birnbaum and McCarty (1958) and further developed by Kotz, Lumelskii, and Pensky (2003). In this model, a system fails if its stress exceeds its strength. Therefore, the reliability of the system corresponds to the probability that the system's strength exceeds its stress ($\mathcal{R} = P(Y > X)$). There are many applications of stress–strength models in engineering and medicine. Examples of the application of this model can be found in studies by Johnson (1988) and Ghanbari, Rezaei Roknabadi, and Salehi (2022). Following are some practical examples of this model. Suppose that a successful rocket engine is designed with X representing the highest stress on the piston while Y represents the piston strength of the rocket against imposed stress. In this case, the stress–strength model's parameter ($\mathcal{R}$) denotes the probability of successfully turning on the rocket engine. In another example, consider a dam in which Y is the dam's strength against the water flow, and X is the amount of water stress caused by a vigorous flood. Then $\mathcal{R}$ represents the probability of the dam being successful in its design.

In many studies, the stress–strength model's parameter was estimated based on the assumption that X and Y are parametrically distributed. Kundu and Raqab (2009) used a three–parameter Weibull distribution for random variables X and Y and estimated the parameter of this model. Eryilmaz (2010) considered the reliability of stress–strength model in the general coherent system by assuming exponential distribution for stress and strength variables. The reliability of this model was also investigated by Salehi and Ahmadi (2015) using upper–record ranked set

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sampling by assuming exponential distribution for random variables. Other studies have estimated the $\mathcal{R}$ parameter for random variables X and Y using record–breaking data (Sadeghpour, Salehi, and Nezakati 2020a, 2020b). A recent study by Ghanbari et al. (2022) looked at the $\mathcal{R}$ parameter of stress–strength model based on progressively Type–II censored samples and the Marshall-Olkin distribution for the random variables X and Y. A generalized survival signature, progressively Type–II censored samples, and the copula function have been applied to assess the reliability of multistate systems (Liu et al. 2018b), Maurya and Tripathi (2020), Kohansal (2019) and Bai et al. (2018).

The stress and strength values of the system or component change over time in the industry. Therefore, dynamic modeling is more appropriate than static modeling. In this condition, the system reliability is time–dependent. Assume $Y_i(t)$ is the strength of the ith component at time t and $X_i(t)$ is the stress imposed on the ith component over time. Then, the lifetime of the ith component can be considered as the following random variable:

$$T_i = \inf\{t \geq 0 : X_i(t) > Y_i(t)\}. \tag{1}$$

At the time s, the reliability function of the ith component, $\mathcal{R}_i(s)$, is the probability of still being active at the time s, that is, $P\{T_i > s\}$. As a result of (1), the following equation would be the reliability

$$\mathcal{R}_i(s) = P(T_i > s) = P\left(\inf_{0 \leq t \leq s}\{Y_i(t) - X_i(t)\} > 0\right). \tag{2}$$

For more details in this field, we refer the reader to the following works. Eryilmaz (2013a) studied the stress–strength reliability when the random variable Y is time–dependent, and the random variable X is constant. In addition, he calculated the probabilistic relationships between variables for this model. Siju and Kumar (2016) investigated time–dependent variables with a constant cycle in the stress–strength model. Also, see Yadav (1973), Gopalan and Venkateswarlu (1982), and Finkelstein (2007).

In this article, we assume the following hypotheses:

i. $Y_i(t)$ is decreasing in time, that is $Y_i(t_2) < Y_i(t_1)$ for all $t_1 < t_2$ and $i = 1, 2, ..., n$. (our reason for expressing this assumption is that most of the time, in reality, the strength of the components decreases over time.)
ii. $X_i(t) = X$, that is, the stress entered into the components has been fixed over time for $i = 1, 2, ..., n$.

One of the concepts used in this article is the k–out–of–n:F system. We have briefly defined this system, and stated the research done in this field. As stated in the reliability literature, A system consisting of n components is called a k–out–of–n:F system, whenever the system fails if and only if at least k components fail. This system is widely used in engineering. For more details, see Bhattacharyya and Johnson (1974), Rao (2014), Dey et al. (2017).

The redundancy of a system is an essential aspect of increasing its reliability. It is possible to create redundancy in the system by using standby components. Standby components can be classified as cold, warm, and hot.

Hot standby redundancy starts working when the system is active. In other words, hot standby redundancy is functional in the standby case. So it may fail in the standby case. Cold standby redundancy will start when the system fails, so in the standby case, it does not fail. On the other hand, warm standby redundancy works in a milder environment in the standby case. Thus, this redundancy component fails in the standby case with less probability than other system components. In short, warm standby redundancy is a case between hot and cold redundancy cases. For more details in this field, we refer the interested readers to Cha, Mi, and Yun (2007) who examined the system reliability of three types of standby components using accelerated life testing, Li, Zhang, and Wu (2009) studied the optimal allotment of a general standby component in a series system with two independent components, and Liu et al. (2018a) investigated the system reliability with N subsystems so that each subsystem contains M components, only one subsystem works under stress, and other subsystems are on standby case.Also, see Eryilmaz (2012, 2014), Li et al. (2013, 2015), Hazra and Nanda (2017), Li and Li (2013), and Zhao et al. (2017).

In this article, we want to investigate the effect of adding standby components to a k–out–of–n:F system in the dynamic stress–strength model and obtain the maximum likelihood estimator (MLE) of the stress-strength model parameter. The rest of this article is organized as follows. Preliminaries are presented in Sec. 2. The system reliability is assessed in Sec. 3 by assuming that the $Y(t)$ and X follow the Weibull process and the Weibull distribution, respectively. In addition, a sensitivity analysis of the reliability function is done to investigate how varying parameters of the Weibull process affect it. The maximum likelihood estimator (MLE) for the parameters is presented in Sec. 4, and a simulation study is conducted in Sec. 5. As a real application of this model, Sec. 6 presents a real data set and Sec. 7 provides the conclusions.

2. Preliminaries

The stochastic process $\{\psi(t), \quad t \geq 0\}$ is the Weibull process (WP) with $\alpha(t)$ and β parameters in the case of the one–dimensional distribution if $\psi(t)$ has the following density function (see, Eryilmaz (2013a))

$$g_{WP}(x;\alpha(t),\beta) = \frac{\beta}{\alpha(t)^{\beta}} \quad x^{\beta-1} \ e^{-\left(\frac{x}{\alpha(t)}\right)^{\beta}}, \ x > 0 \ , \beta > 0, \tag{3}$$

where the shape parameter β is assumed to be time–independent, and the intensity function $\alpha(t) = \frac{\alpha}{t}$ decreases over time. Assume that $Y_i(t) \sim WP(\alpha_1(t), \beta)$ and $Z(t) \sim WP(\alpha_2(t), \lambda)$ represent the ith component's strength and the standby component's strength at time t, respectively. Also, the random variable X represents the stress imposed by the system on the components following the Weibull distribution with the density function given by

$$f_W(x;\theta,\tau) = \frac{\tau}{\theta^{\tau}} x^{\tau-1} e^{-\left(\frac{x}{\theta}\right)^{\tau}}, \qquad x > 0, \ \ \theta > 0, \tau > 0. \tag{4}$$

Now, we are interested in calculating the reliability of a k–out–of–n:F system by using the above assumptions in two cases without and with the presence of a standby component.

3. System reliability

In a k–out–of–n:F system, suppose X is the random variable that denotes the stress imposed by the system on the components with cumulative distribution function (CDF) $F_X(x)$, and it is independent of the strength random variables. As well, the random variable $Y_i(t)$ represents the ith component's strength at time t with CDF $G_t(y)$, and $Y_i(t)$'s are iid. Then, the lifetime of such a system without a standby component is equal to the kth order statistics, $T_{k:n}$, that is,

$$T_{k:n} = \inf\{s > 0, \quad X > Y_{j_1}(s), X > Y_{j_2}(s), ..., X > Y_{j_k}(s)\}, \tag{5}$$

where $\{j_1, ..., j_k\}$ is a permutation of $\{1, 2, ..., n\}$. Therefore, the system reliability function at time t is obtained from (2) and (5) as follows

$$\begin{aligned} \mathcal{R}_{k:n}(t) &= P(T_{k:n} > t) = P(Y_{k:n}(t) > X) \\ &= \sum_{i=0}^{k-1} \binom{n}{i} \int_0^{+\infty} (G_t(x))^i \ (\bar{G}_t(x))^{n-i} dF_X(x), \end{aligned} \tag{6}$$

where $Y_{k:n}(t)$ is equal to the kth order statistics of the strength components at time t and $\bar{G}_t(x) = 1 - G_t(x)$. Suppose $Z(t)$ represents the standby component's strength at time t, with CDF $H_t(z)$ in the functional case and CDF $H_t(\delta(t))$ in the standby case, where $\delta(t)$ is an increasing function that follows $\delta(0) = 0$ and $\delta(t) \leq t$. In the standby case, the standby component that operates under the standard working environment until time $t > 0$ has a virtual age $\upsilon(t)$, compared to the calendar age t, where $\upsilon(t)$ is an increasing function satisfying $\upsilon(0) = 0$ and $\upsilon(t) \leq t$. For more details, see Li and Li (2013), Li, Zhang, and Wu (2009), and Nezakati and Razmkhah (2018). It is possible to have both cold and hot standby components if $\delta(t) = \upsilon(t) = 0$ and $\delta(t) = \upsilon(t) = t$, respectively.

Based on the assumption that Z is independent of $Y_1, ..., Y_n$ and $\mathcal{T}^*_{n-k+1}$ is the system's strength equipped with a standby component, Eryilmaz (2013b) calculated the reliability function for a k–out–of–n:G system with the standby component as follows

$$\begin{aligned} \bar{F}_{\mathcal{T}^*_{n-k+1}}(t) &= P(\mathcal{T}^*_{n-k+1} > t) \\ &= \frac{n!\bar{G}^{k-1}(t)}{(n-k)!(k-1)!} \int_0^t \frac{\bar{H}(\upsilon(y) + t - y)}{\bar{H}(\upsilon(y))} \\ &\quad \times \bar{H}(\delta(y)) G^{n-k}(y) dG(y) + P(Y_{n-k+1:n} > t), \end{aligned} \tag{7}$$

where $\bar{H}(\cdot) = 1 - H(\cdot)$ and $\bar{G}(\cdot) = 1 - G(\cdot)$. Suppose $\mathcal{T}^*_k$ is the system's strength equipped with a standby component. According to (7) and doing some changes, the stress–strength reliability function for a k–out–of–n:F system with the standby component, denoted by $\mathcal{R}^*(t)$, is as follows:

$$\begin{aligned} \mathcal{R}^*(t) = P(\mathcal{T}^*_k > X) = n\binom{n-1}{k-1} \int_0^{+\infty} \int_0^x \bar{G}_t^{n-k}(x) \frac{\bar{H}_t(\upsilon(y) + x - y)}{\bar{H}(\upsilon(y))} \\ \times \bar{H}(\delta(y)) G_t^{k-1}(y) \; dG_t(y) \; dF_X(x) + \mathcal{R}_{k:n}(t). \end{aligned} \tag{8}$$

From (8), it is clear that the system reliability increases when a standby component is used. Note that the standby component is subject to the common stress X as the other components. In the following, we assume that $\mathcal{R}^*_1(t)$, $\mathcal{R}^*_2(t)$, and $\mathcal{R}^*_3(t)$ represent system reliability with cold, warm, and hot standby components, respectively. By assuming, $\delta(y) = \upsilon(y) = 0$, $\delta(y) = \upsilon(y) = \frac{y}{2}$ and $\delta(y) = \upsilon(y) = y$ for the cold, warm, and hot standby components, respectively, and $Y(t) \sim WP(\frac{\alpha_1}{t}, \beta)$, $Z(t) \sim WP(\frac{\alpha_2}{t}, \lambda)$, $X \sim W(\theta, \tau)$, from (6) and (8), we have

$$\begin{aligned} \mathcal{R}_{k:n}(t) = \sum_{i=0}^{k-1}\sum_{j=0}^{i} \binom{n}{i}\binom{i}{j} \frac{\tau}{\theta^\tau} (-1)^j \int_0^{+\infty} x^{\tau-1} e^{-(\frac{x}{\theta})^\tau} \\ \times \quad e^{-x^\beta (n-i+j)(\frac{t}{\alpha_1})^\beta} \; dx \end{aligned} \tag{9}$$

and

$$\begin{aligned} \mathcal{R}^*_u(t) = \mathcal{R}_{k:n}(t) + n\binom{n-1}{k-1} \frac{\beta t^\beta \tau}{\alpha_1^\beta \theta^\tau} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \\ \times \int_0^\infty \int_0^x e^{-\left(\frac{(x-(\frac{3-u}{2})y)t}{\alpha_2}\right)^\lambda} x^{\tau-1} y^{\beta-1} e^{-(\frac{x}{\theta})^\tau} \\ \times \; e^{-x^\beta (n-k)(\frac{t}{\alpha_1})^\beta} e^{-y^\beta (\frac{(j+1)t}{\alpha_1})^\beta} \; dy \; dx. \end{aligned} \tag{10}$$

Figure 1 shows the behavior of $\mathcal{R}^*_u(t)$, $u = 1, 2, 3$ over time when parameters are $\alpha_1 = 1.5$, $\alpha_2 = 0.5$, $\beta = 1.6$, $\theta = 1.2$, $\lambda = 2.7$, $\tau = 2.4$, in 2–out–of–10:F, 5–out–of–10:F, and 8–out–of–10:F systems. From this figure, observed $\mathcal{R}^*_u(t)$ is a decreasing function of time. Also, increasing the

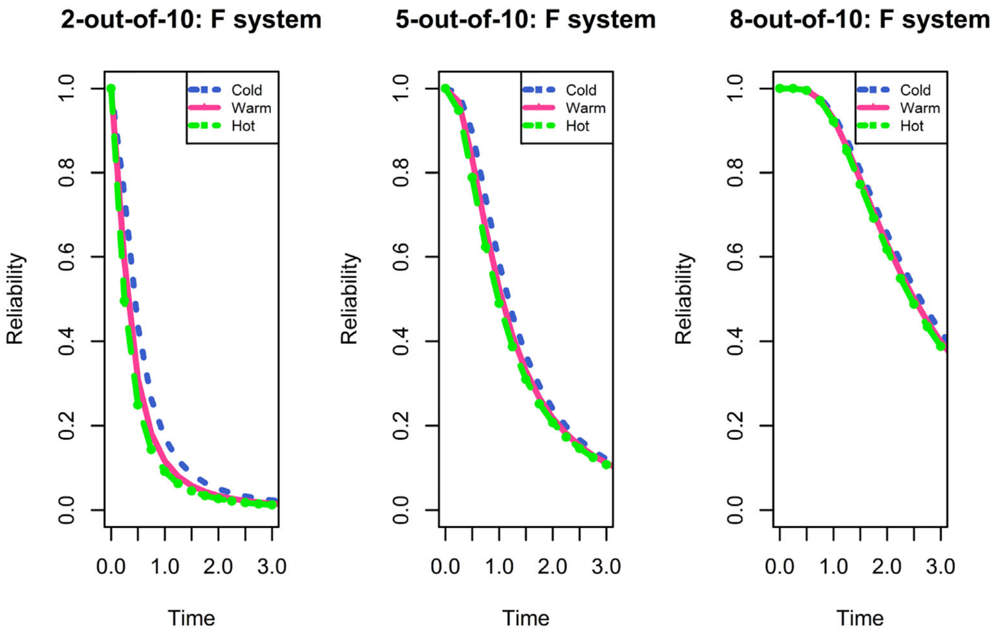

Figure 1. Reliability of the three different systems versus time.

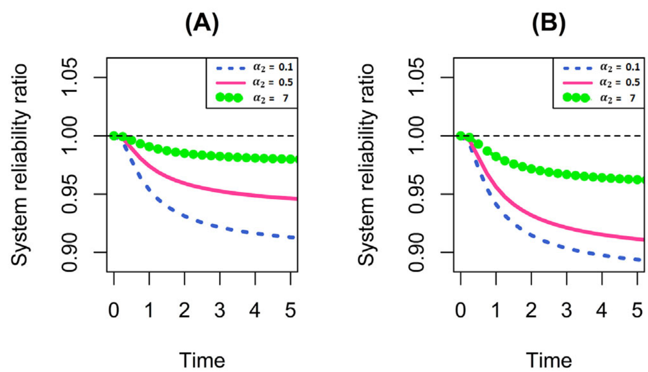

Figure 2. Reliability ratio of the 7–out–of–10:F system with (A) the warm–to–cold standby component (B) the hot–to–cold standby component versus time, respectively, within left, and right charts, when $\alpha_1 = 1.5,\ \lambda = 1,\ \beta = 1,\ \theta = 1.2,\ \tau = 1$.

amount of k increases the value of $\mathcal{R}_u^*(t)$. In all cases, the inequality of $\mathcal{R}_3^*(t) \leq \mathcal{R}_2^*(t) \leq \mathcal{R}_1^*(t)$ holds.

We now examine the effect of the λ and α_2 parameters related to the standby component's strength on the system reliability.

Lemma 3.1. *Suppose that $Z_1(t)$ and $Z_2(t)$ are the standby components' strengths of two different k–out–of–n:F systems by the reliability functions $\mathcal{R}_u^{*1}(t)$ and $\mathcal{R}_u^{*2}(t),\ u = 1, 2, 3$. Also, $Z_i(t)$s follow $WP\left(\frac{\alpha_2^i}{t}, 1\right),\ i = 1, 2$. By assuming $\beta = 1$ and $\tau = 1$, if $\alpha_2^1 < \alpha_2^2$, then $\mathcal{R}_u^{*1}(t) < \mathcal{R}_u^{*2}(t)$.*

Proof. See the Appendix. □

The reliability ratio of a 7–out–of–10:F system equipped with a warm (hot)–to–cold standby component is less than one, as shown in Figure 2A and B. In other words, the system reliability with a cold component is higher than system reliability with a warm (hot) component.

Additionally, the system reliability increases as the α_2 parameter increases. As it is displayed, Figure 2 confirms Lemma 3.1.

In the following, we want to consider the effect of the λ parameter on the system reliability. We used plots of the system reliability ratio with a warm (hot)-to-cold component versus time and λ because we could not directly prove the sign of the derivative of $\mathcal{R}_u^*(t)$ relative to the λ. In addition, we have used the system reliability ratio function instead of the reliability function that compares the reliability of a system with different types of standby components and expresses the effect of the λ parameter on the system reliability function. Figures 3–6 show the results for some selected values of the parameters. From these figures, we observed that the reliability ratio of the system with a warm (hot)–to–cold standby component is less than one. Thus, the system reliability with a cold component is higher than system reliability with a warm (hot) component. Also, the system reliability ratio is a decreasing function of the λ. As a result, the system reliability is a decreasing function of the λ.

4. Point estimation of $\mathcal{R}$

In this section, we estimate the system reliability before and after entering a cold, a warm, or a hot standby component. For this purpose, suppose that we put M systems with n components into one experiment. Then m systems with n components are observed at times $t_1, t_2, ..., t_m$. The strength and stress samples observed are respectively as

$$\mathbf{Y}(\mathbf{t}) = \begin{pmatrix} Y_{11}(t_1) & \cdots & Y_{1m}(t_m) \\ \vdots & \ddots & \vdots \\ Y_{n1}(t_1) & \cdots & Y_{nm}(t_m) \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix},$$

where $\underline{\mathbf{t}} = (t_1, t_2, ..., t_m)$ and $Y_{ij}(t_j)$ is ith strength component in the jth system at the time t_j. We now consider the following cases:

Case A: System without the standby component

The likelihood function of the parameters α_1, β, θ, and τ given $\boldsymbol{x}$ and $\boldsymbol{y}(\boldsymbol{t})$ is obtained as (see, e.g., Balakrishnan and Aggarwala (2000), 117–138)

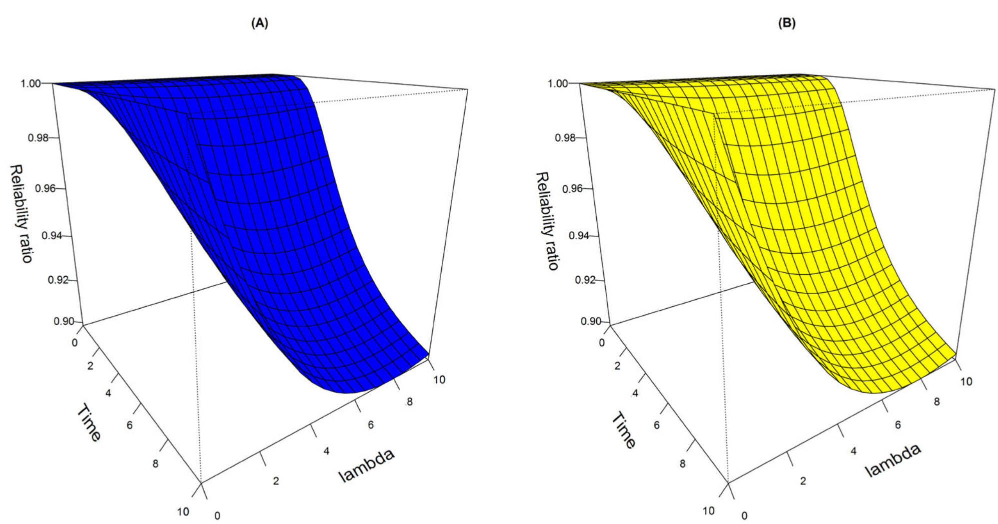

Figure 3. Reliability ratio of the 7–out–of–10: F system with (A) the warm–to–cold standby component, (B) the hot–to–cold standby component versus time, respectively, within left, and right charts, when $\alpha_1 = 0.5$, $\alpha_2 = 1.5$, $\beta = 1.6$, $\theta = 1.2$, $\tau = 2.4$.

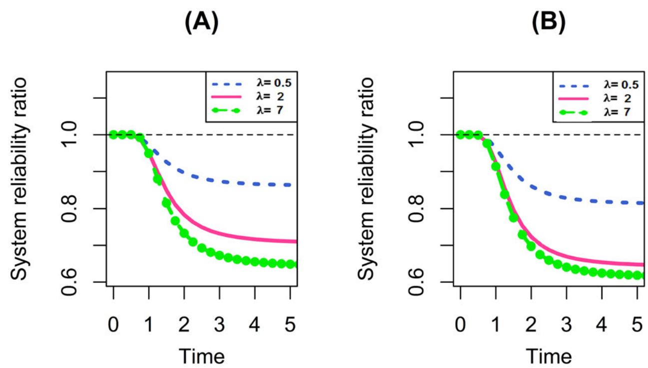

Figure 4. Reliability ratio of the 7–out–of–10: F system with (A) the warm–to–cold standby component, (B) the hot–to–cold standby component versus time, respectively, within left, and right charts, when $\alpha_1 = 1.5$, $\alpha_2 = 1.5$, $\beta = 1.6$, $\theta = 1.2$, $\tau = 2.4$.

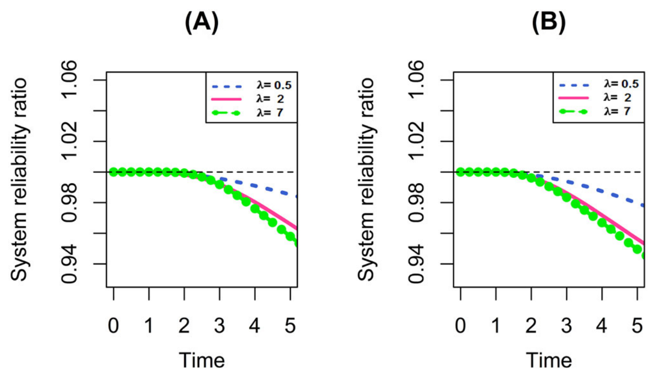

Figure 5. Reliability ratio of the 7–out–of–10: F system with (A) the warm–to–cold standby component, (B) the hot–to–cold standby component versus time, respectively, within left, and right charts, when $\alpha_1 = 1.5$, $\alpha_2 = 0.5$, $\beta = 0.6$, $\theta = 0.2$, $\tau = 2.4$.

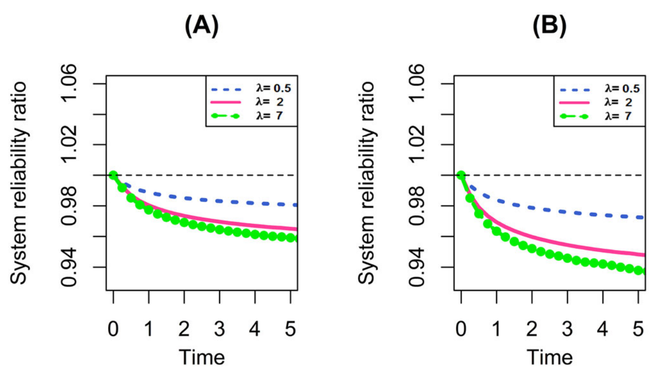

Figure 6. Reliability ratio of the 7–out–of–10: F system with (A) the warm–to–cold standby component, (B) the hot–to–cold standby component versus time, respectively, within left, and right charts, when $\alpha_1 = 0.5$, $\alpha_2 = 0.5$, $\beta = 0.6$, $\theta = 0.2$, $\tau = 0.4$.

$$L_1 = \prod_{j=1}^{m} \left(\prod_{i=1}^{n} g_{WP}(y_{ij}, \alpha_1(t_j), \beta) \right) \times f_W(x_j; \theta, \tau). \tag{11}$$

By placing (3) and (4) into (11), the log–likelihood function follows as

$$\begin{aligned}\ell_1 = \ln L_1 &= nm(\ln\beta - \beta\ln(\alpha_1)) + m\ln\tau - m\tau\ln\theta \\ &+ n\beta\sum_{j=1}^{m}\ln t_j - \sum_{j=1}^{m}\left(\frac{x_j}{\theta}\right)^{\tau} - \frac{1}{\alpha_1^{\beta}}\sum_{j=1}^{m}\sum_{i=1}^{n}(y_{ij}t_j)^{\beta} \\ &+ (\tau-1)\sum_{j=1}^{m}\ln x_j + (\beta-1)\sum_{j=1}^{m}\sum_{i=1}^{n}\ln y_{ij},\end{aligned}$$

where $x = (x_1, ..., x_m)$ and $y = (y_{11}, ..., y_{nm})$ are the observations of $\boldsymbol{X}$ and $\boldsymbol{Y}(\boldsymbol{t})$, respectively. The MLEs of parameters θ and α_1 denoted by $\hat{\theta}$ and $\hat{\alpha}_1$, respectively, are calculated as follows

$$\begin{aligned}\hat{\theta} &= \left(\frac{\sum_{j=1}^{m} x_j^{\hat{\tau}}}{m}\right)^{\frac{1}{\hat{\tau}}}, \\ \hat{\alpha}_1 &= \left(\frac{\sum_{j=1}^{m}\sum_{i=1}^{n}(y_{ij}t_j)^{\hat{\beta}}}{mn}\right)^{\frac{1}{\hat{\beta}}}.\end{aligned}$$

Also, the maximum likelihood estimators of the parameters τ and β, shown by $\hat{\tau}$ and $\hat{\beta}$, respectively, are obtained by solving the following nonlinear equations

$$\begin{aligned}\frac{\partial\ell_1}{\partial\beta} &= nm\left(\frac{1}{\beta} - \ln(\alpha_1)\right) + n\sum_{j=1}^{m}\ln t_j + \sum_{j=1}^{m}\sum_{i=1}^{n}\ln y_{ij} \\ &- \sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{y_{ij}t_j}{\alpha_1}\right)^{\beta}\ln\left(\frac{y_{ij}t_j}{\alpha_1}\right), \\ \frac{\partial\ell_1}{\partial\tau} &= \frac{m}{\tau} - m\ln\theta + \sum_{j=1}^{m}\ln x_j - \sum_{j=1}^{m}\left(\frac{x_j}{\theta}\right)^{\tau}\ln\left(\frac{x_j}{\theta}\right).\end{aligned}$$

Thus, $\hat{\mathcal{R}}_{k:n}(t)$ is derived based on the invariance property of MLE by substituting $\hat{\alpha}_1$, $\hat{\beta}$, $\hat{\theta}$, and $\hat{\tau}$ in (9).

Case B: System with a standby component

The strength and stress samples observed after entering the standby component, respectively, are equal to

$$\begin{pmatrix}\mathbf{Y}(\mathbf{t}) \\ \mathbf{Z}(\mathbf{t})\end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix}X_1 \\ \vdots \\ X_m\end{pmatrix}.$$

where $\mathbf{Z}(\mathbf{t}) = (Z_1(t_1), Z_2(t_2), ..., Z_m(t_m))$, and $\mathbf{Y}(\mathbf{t})$ is defined in Part A. The likelihood function of the parameters α_1, α_2, β, θ, τ, and λ, given $\boldsymbol{x}$, $\boldsymbol{y}(\boldsymbol{t})$, and $\boldsymbol{z}(\boldsymbol{t})$, is as follows

$$\begin{aligned}L_2 = \prod_{j=1}^{m}&\left(\prod_{i=1}^{n} g_{WP}(y_{ij};\alpha_1(t_j),\beta)\right) \\ &\times g_{WP}(z_j;\alpha_2(t_j),\lambda) \times f_W(x_j,\theta,\tau).\end{aligned} \tag{12}$$

By placing (3) and (4) into (12), the log–likelihood function follows as

$$\begin{aligned}\ell_2 = \ln(L_2) &= nm(\ln\beta - \beta\ln(\alpha_1)) + m(\ln(\lambda) - \lambda\ln(\alpha_2) \\ &+ \ln\tau - \tau\ln\theta) + n\beta\sum_{j=1}^{m}\ln t_j + (\beta-1)\sum_{j=1}^{m}\sum_{i=1}^{n}\ln y_{ij} \\ &+ \lambda\sum_{j=1}^{m}\ln(t_j) + (\lambda-1)\sum_{j=1}^{m}\ln(z_j) - \sum_{j=1}^{m}\left(\frac{x_j}{\theta}\right)^{\tau} \\ &- \frac{1}{\alpha_1^{\beta}}\sum_{j=1}^{m}\sum_{i=1}^{n}(y_{ij}t_j)^{\beta} + (\tau-1)\sum_{j=1}^{m}\ln x_j - \sum_{j=1}^{m}\left(\frac{t_j z_j}{\alpha_2}\right)^{\lambda},\end{aligned}$$

where $x = (x_1, ..., x_m)$, $z = (z_1, ..., z_m)$, and $y = (y_{11}, ..., y_{nm})$ are the observations of $\boldsymbol{X}$, $\boldsymbol{Z}(\boldsymbol{t})$, and $\boldsymbol{Y}(\boldsymbol{t})$, respectively. Also, $\hat{\alpha}_2$ is the MLE of the parameter α_2. The $\hat{\theta}$, $\hat{\alpha}_1$, and $\hat{\alpha}_2$, are calculated as follows

$$\begin{aligned}\hat{\theta} &= \left(\frac{\sum_{j=1}^{m} x_j^{\hat{\tau}}}{m}\right)^{\frac{1}{\hat{\tau}}}, \\ \hat{\alpha_1} &= \left(\frac{\sum_{j=1}^{m}\sum_{i=1}^{n}(y_{ij}t_j)^{\hat{\beta}}}{mn}\right)^{\frac{1}{\hat{\beta}}}, \\ \hat{\alpha_2} &= \left(\frac{\sum_{j=1}^{m}(z_j t_j)^{\hat{\lambda}}}{m}\right)^{\frac{1}{\hat{\lambda}}}.\end{aligned} \tag{13}$$

The $\hat{\lambda}$ is the MLE of the parameter λ. Also, we obtained $\hat{\tau}$, $\hat{\beta}$, and $\hat{\lambda}$ by solving the following nonlinear equations

$$\begin{aligned}\frac{\partial\ell_2}{\partial\beta} &= mn\left(\frac{1}{\beta} - \ln(\alpha_1)\right) + n\sum_{j=1}^{m}\ln(t_j) + \sum_{j=1}^{m}\sum_{i=1}^{n}\ln(y_{ij}) \\ &- \sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{y_{ij}t_j}{\alpha_1}\right)^{\beta}\ln\left(\frac{y_{ij}t_j}{\alpha_1}\right), \\ \frac{\partial\ell_2}{\partial\tau} &= \frac{m}{\tau} - m\ln(\theta) + \sum_{j=1}^{m}\ln(x_j) - \sum_{j=1}^{m}\ln\left(\frac{x_j}{\theta}\right)\left(\frac{x_j}{\theta}\right)^{\tau}, \\ \frac{\partial\ell_2}{\partial\lambda} &= m\left(\frac{1}{\lambda} - \ln(\alpha_2)\right) + \sum_{j=1}^{m}\ln(t_j) + \sum_{j=1}^{m}\ln(z_j) \\ &- \sum_{j=1}^{m}\ln\left(\frac{z_j t_j}{\alpha_2}\right)\left(\frac{z_j t_j}{\alpha_2}\right)^{\lambda}.\end{aligned} \tag{14}$$

Therefore, based on the invariance property of MLE, $\widehat{\mathcal{R}_u^*}(t)$ is derived by substituting $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}$, $\hat{\theta}$, $\hat{\tau}$ and $\hat{\lambda}$ in (10).

5. Simulation study

In this section, we examined the reliability of a 7–out–of–10: F system without and with cold, warm, or hot standby components. Also, we used the Monte Carlo simulation method to observe the effect of the standby component and determine which type of standby component has the most impact on the system reliability. For this purpose, we used the following algorithm to calculate biases and the mean square errors (*MSEs*) criteria of the system reliability.

Algorithm 6.1 Step1: Based on the independent observed samples $\mathbf{X} \sim W(\tau, \theta)$, $\mathbf{Y}(\mathbf{t}) \sim WP\left(\beta, \frac{\alpha_1}{t}\right)$, and $\mathbf{Z}(\mathbf{t}) \sim WP\left(\lambda, \frac{\alpha_2}{t}\right)$, calculate $\mathcal{R}_{k:n}(t)$ and $\mathcal{R}^*_u(t)$, $u = 1, 2, 3$, using (9) and (10), respectively.

Step2: Given the independent observed samples in Step 1, calculate $\hat{\theta}$, $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}$, $\hat{\tau}$, $\hat{\lambda}$, $\hat{\mathcal{R}}_{k:n}(t)$, and $\hat{\mathcal{R}}^*_u(t)$ using the formulas given in sec. 4.

Step3: For $d = 1, \ldots, B$, repeat Step 2 to derive $\hat{\mathcal{R}}^{[d]}_{k:n}(t)$ and $\hat{\mathcal{R}}^{*[d]}_u(t)$, $d = 1, \ldots, B$.

Step4: By the following definitions, calculate the Bias and MSE for $\mathcal{R}_{k:n}(t)$ and $\mathcal{R}^*(t)$, respectively.

$$MSE(\hat{\mathcal{R}}_{k:n}(t), \mathcal{R}_{k:n}(t)) = \frac{1}{B}\sum_{d=1}^{B}\left(\hat{\mathcal{R}}^{[d]}_{k:n}(t) - \mathcal{R}_{k:n}(t)\right)^2,$$
$$Bias(\hat{\mathcal{R}}_{k:n}(t), \mathcal{R}_{k:n}(t)) = \frac{1}{B}\sum_{d=1}^{B}\left(\hat{\mathcal{R}}^{[d]}_{k:n}(t) - \mathcal{R}_{k:n}(t)\right),$$

and

$$MSE(\hat{\mathcal{R}}^*_u(t), \mathcal{R}^*_u(t)) = \frac{1}{B}\sum_{d=1}^{B}\left(\hat{\mathcal{R}}^{*[d]}_u(t) - \mathcal{R}^*_u(t)\right)^2,$$
$$Bias(\hat{\mathcal{R}}^*_u(t), \mathcal{R}^*_u(t)) = \frac{1}{B}\sum_{d=1}^{B}\left(\hat{\mathcal{R}}^{*[d]}_u(t) - \mathcal{R}^*_u(t)\right),$$

where $\mathcal{R}_{k:n}(t)$ and $\mathcal{R}^*_u(t)$ are already given by Step 1. We presented the results in Figures 7–12 for $B = 1000$. Also, all combinations used in this section are $n = 10$, $k = 7$, $m = 15$, $\beta = 0.8$, 0.9, 1.9, $\lambda = 0.5$, 0.8, 1.3, $\alpha_1 = 0.5$, 1.5, $\alpha_2 = 0.8$, 0.9, 7, $\tau = 0.5$, 2.2, and $\theta = 0.8$, 1.2, 1.5.

According to Figures 7, 9, and 11, panel A shows the system reliability and its estimation over time for different parameters. As expected, the system reliability decreases over time. Moreover, we have $\mathcal{R}^*_3(t) < \mathcal{R}^*_2(t) < \mathcal{R}^*_1(t)$ and $\hat{\mathcal{R}}^*_3(t) < \hat{\mathcal{R}}^*_2(t) < \hat{\mathcal{R}}^*_1(t)$. Also, panel B in these figures shows the bias value of the MLE estimator of $\mathcal{R}$ over time. As can be seen from these figures, the bias value of the MLE estimator of $\mathcal{R}$ with and without the standby component is positive, and we have

$$Bias(\hat{\mathcal{R}}^*_1(t), \mathcal{R}^*_1(t)) < Bias(\hat{\mathcal{R}}^*_2(t), \mathcal{R}^*_2(t)) < Bias(\hat{\mathcal{R}}^*_3(t), \mathcal{R}^*_3(t)) < Bias(\hat{\mathcal{R}}_{k:n}(t), \mathcal{R}_{k:n}(t))$$

According to Figures 8, 10, and 12, panel A shows the MSE of the MLE estimator of $\mathcal{R}^*$ over time. As expected, MSE $(\hat{\mathcal{R}}^*_1(t), \mathcal{R}^*_1(t))$ reduced after the impact of the standby component. Also

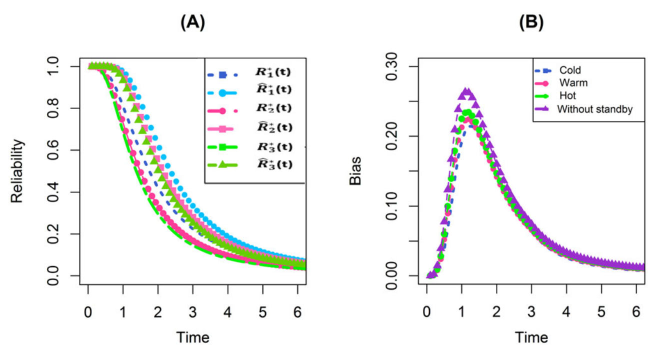

Figure 7. The behavior of the reliability, bias, MSE functions of $\mathcal{R}^*$ for parameters $\beta = 0.9$, $\alpha_1 = 1.5$, $\alpha_2 = 0.9$, $\lambda = 0.8$, $\theta = 1.5$, $\tau = 2.2$.

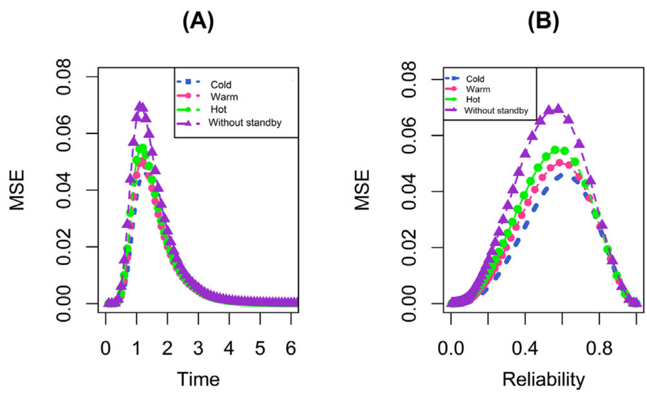

Figure 8. The behavior of the MSE functions of $\mathcal{R}^*$ for parameters $\beta = 0.9$, $\alpha_1 = 1.5$, $\alpha_2 = 0.9$, $\lambda = 0.8$, $\theta = 1.5$, $\tau = 2.2$.

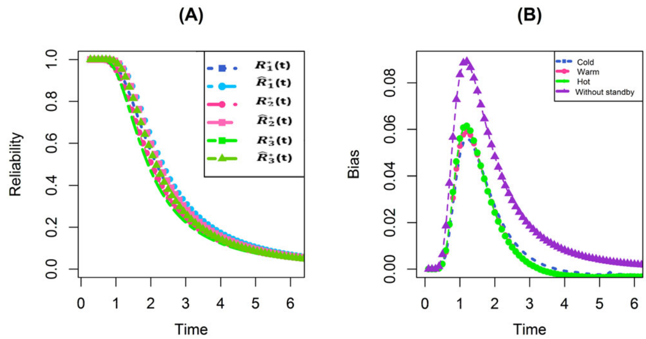

Figure 9. The behavior of the reliability, bias, MSE functions of $\mathcal{R}^*$ for parameters $\beta = 1.9$, $\alpha_1 = 1.5$, $\alpha_2 = 7$, $\lambda = 1.3$, $\theta = 1.2$, $\tau = 2.2$.

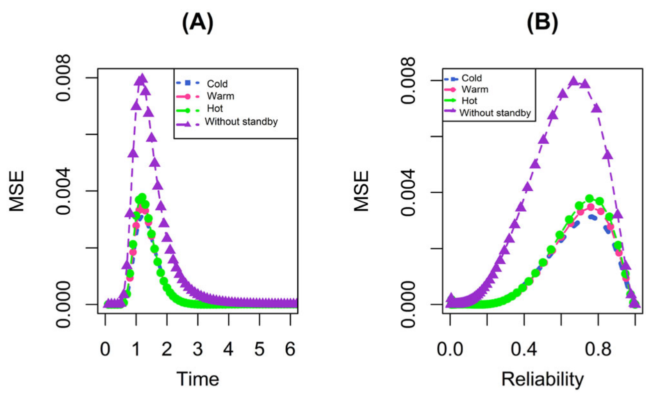

Figure 10. The behavior of the MSE functions of $\mathcal{R}^*$ for parameters $\beta = 1.9$, $\alpha_1 = 1.5$, $\alpha_2 = 7$, $\lambda = 1.3$, $\theta = 1.2$, $\tau = 2.2$.

$$MSE(\hat{\mathcal{R}}_1^*(t), \mathcal{R}_1^*(t)) < MSE(\hat{\mathcal{R}}_2^*(t), \mathcal{R}_2^*(t)) < MSE(\hat{\mathcal{R}}_3^*(t), \mathcal{R}_3^*(t)) < MSE(\hat{\mathcal{R}}_{k:n}(t), \mathcal{R}_{k:n}(t))$$

Also, panel B in these figures shows the MSE of the MLE estimator of $\mathcal{R}^*$ relative to the system reliability. Based on these figures, we conclude that MSE decreases after the impact of standby. All the results obtained from panel A of these Figures are also correct for panel B.

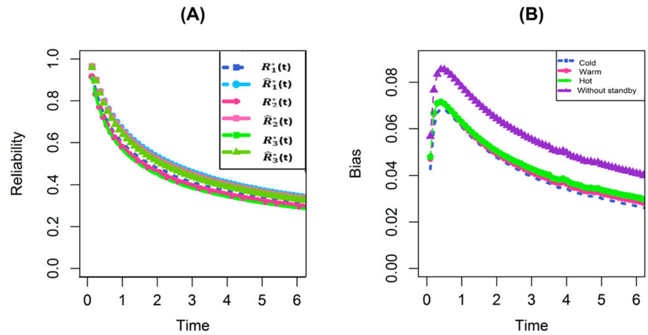

Figure 11. The behavior of the reliability, bias, MSE functions of $\mathcal{R}^*$ for parameters $\beta = 0.8$, α_1=0.5, α_2=0.8, λ =0.5, θ =0.9, $\tau = 0.5$.

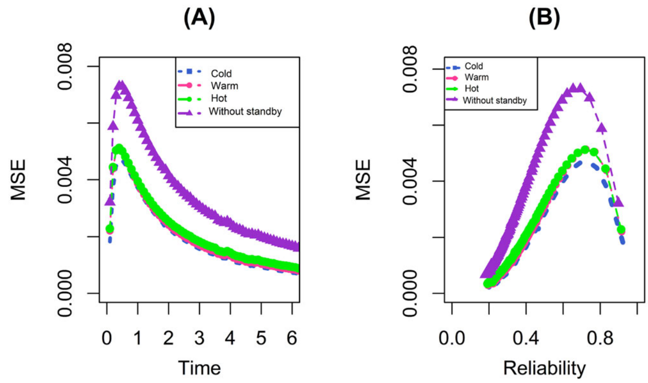

Figure 12. The behavior of the MSE functions of $\mathcal{R}^*$ for parameters β =1.9, α_1=1.5, α_2=7, $\lambda = 1.3$, θ =1.2, $\tau = 2.2$.

6. Real data analysis

Here, we use a real dataset to illustrate the theoretical results obtained. The dataset is related to the time of successive failures of the air conditioning system of each member of a fleet of jet airplanes. Proschan (1963) studied this dataset in other different works. Suppose, $Y_j(t_1),\ j = 1, \ldots, 4$ indicate the time interval of the first failures of the air conditioning systems of 7909, 7910, 7913, and 7914 airplanes, respectively. Also, X_1 shows the average time interval of the first failures of air conditioning systems of 7908, 7911, 7915, 8044, and 8045 airplanes. $Y_j(t_2),\ j = 1, \ldots, 4$ indicate the time interval of the second failures of the air conditioning systems of 7909, 7910, 7913, 7914 airplanes, respectively. The X_2 shows the average time interval of the second failures of air conditioning systems of 7908, 7911, 7915, 8044, and 8045 airplanes. We continue an equivalent process up to the time interval of the tenth failure of the air conditioning systems, so $m = 10$. Also, different air conditioning systems are set for X_j and $Y_i(t_j)$ data to reduce and eliminate the dependence between these variables. Now we can suggest the following scenario to check the quality of the air conditioning system of each member of a fleet of jet airplanes. If at least 3 of 4 air conditioning systems, the failure time interval is less than the average failure time interval on them. Then, we can conclude that the quality of the air conditioning systems is lower than the average quality. In other words, air conditioning systems need to be repaired in a short period. Here, We have a 3–out–of–4:F system at times $t_i,\ i = 1, \ldots, 10$. The observed data sets X and $Y(t)$ are shown in (15) and Table 1, respectively.

$$\mathbf{x}' = \begin{pmatrix} 283.2 & 114 & 48 & 95 & 215 & 68.8 & 62.4 & 113.4 & 247.6 & 173 \end{pmatrix}. \tag{15}$$

and, First, we want to check whether the Weibull distribution is adequate to fit data set X or not.

Table 1. Real data display for $y(t)$.

y(t) \ Time	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
$y_1(t)$	90	10	60	186	61	49	14	24	56	20
$y_2(t)$	74	57	48	29	502	12	70	21	29	386
$y_3(t)$	97	51	11	4	141	18	142	68	77	80
$y_4(t)$	50	44	102	72	22	39	3	15	197	188

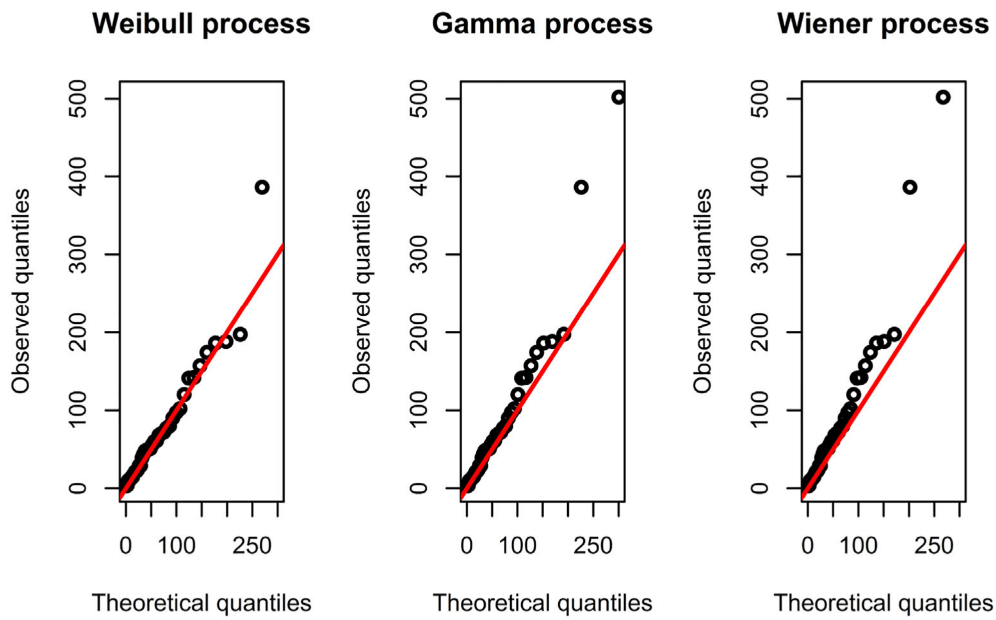

Figure 13. The $Q–Q$ plots of the $Y(t)$ data set under three different processes.

For this purpose, we use the Kolmogorov-Smirnov (K–S) test. The K–S test statistic and corresponding p–value for X are equal to 0.2307 and 0.5846, respectively. Thus, we can say that the Weibull distribution is a good distribution for fitting on X data because the p–value of the K–S test is more than 0.05. Also, we used three stochastic processes to consider the appropriate process for $Y(t)$ data. Stochastic processes are Weibull, Gamma, and Wiener. We use $Q–Q$ plots as a graphical method to check the goodness of fit of the three stochastic processes. Figure 13 shows the $Q–Q$ plots.

As shown in Figure 13, the Weibull process outperforms the other models. In addition to the $Q–Q$ plots, the Cramer–von Mises (CVM) goodness of fit test and Akaike's information criterion (AIC) and bayesian information criterion (BIC) of the processes are shown in Table 2.

Table 2 also confirms the results of Figure 13. The results of the MLE of parameter $\mathcal{R}(t)$ are shown in Table 3.

Assume $z(t_i), \quad i = 1, ..., 10$ is the time interval between the ith failure of airplane 7912s air conditioning system. Moreover, we assume that the standby component is subjected to the same stress as the other components. Table 4 shows the data set with cold or hot standby components.

Table 2. The CVM statistic, p–value, AIC, and BIC for the $Y(t)$ data set in Table 1.

	CVM statistic	p–value	AIC	BIC
Weibull process	0.0826	0.6795	439.9765	438.3542
Gamma process	0.1239	0.5345	470.1357	482.1287
Wiener process	0.1510	0.4363	521.8702	511.1257

Table 3. The ML estimates of parameters α, β, τ, θ and $\mathcal{R}_{k:n}$.

MLE's		$\hat{\mathcal{R}}_{k:n}(t_1)$	$\hat{\mathcal{R}}_{k:n}(t_2)$	$\hat{\mathcal{R}}_{k:n}(t_3)$	$\hat{\mathcal{R}}_{k:n}(t_4)$	$\hat{\mathcal{R}}_{k:n}(t_5)$
$\hat{\alpha} = 1.501$	$\hat{\beta} = 0.154$	0.5957	0.5581	0.5361	0.5128	0.5085
		$\hat{\mathcal{R}}_{k:n}(t_6)$	$\hat{\mathcal{R}}_{k:n}(t_7)$	$\hat{\mathcal{R}}_{k:n}(t_8)$	$\hat{\mathcal{R}}_{k:n}(t_9)$	$\hat{\mathcal{R}}_{k:n}(t_{10})$
$\hat{\tau} = 0.181$	$\hat{\theta} = 1.512$	0.4987	0.4905	0.4835	0.4772	0.4717

Table 4. The data set with a standby component.

Time	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
$y_1(t)$	90	10	60	186	61	49	14	24	56	20
$y_2(t)$	74	57	48	29	502	12	70	21	29	386
$y_3(t)$	97	51	11	4	141	18	142	68	77	80
$y_4(t)$	50	44	102	72	22	39	3	15	197	188
$z(t)$	28	261	87	7	120	14	62	47	225	71

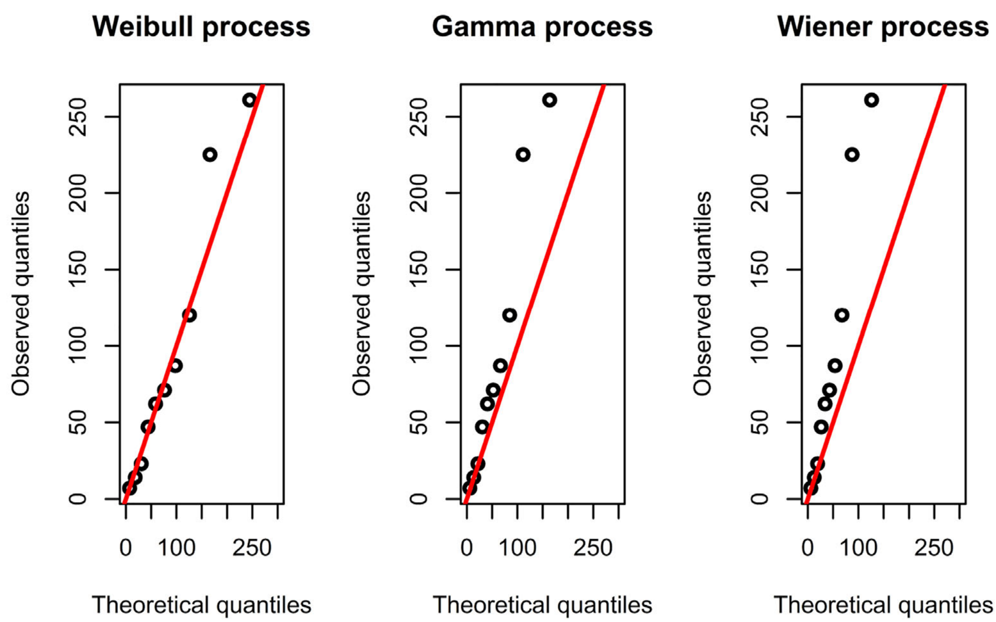

Figure 14. The Q–Q plots of the $Z(t)$ data set under three different processes.

Table 5. The CVM statistic, p–value, AIC, and BIC for the $z(t)$ data set.

	CVM statistic	p–value	AIC	BIC
Weibull process	0.0209	0.9978	114.3476	114.9528
Gamma process	0.0812	0.7813	135.7814	129.2514
Wiener process	0.1871	0.4369	178.9318	182.1819

Table 6. The ML estimates of the parameters α_1, α_2, β, λ, θ and $\mathcal{R}^*{}_1(t)$.

MLE's		$\hat{\mathcal{R}}^*{}_1(t_1)$	$\hat{\mathcal{R}}^*{}_1(t_2)$	$\hat{\mathcal{R}}^*{}_1(t_3)$	$\hat{\mathcal{R}}^*{}_1(t_4)$	$\hat{\mathcal{R}}^*{}_1(t_5)$
$\hat{\alpha}_1 = 1.501$	$\hat{\beta} = 0.121$	0.6572	0.6307	0.6100	0.5918	0.5869
$\hat{\alpha}_2 = 1.501$	$\hat{\tau} = 0.174$	$\hat{\mathcal{R}}^*{}_1(t_6)$	$\hat{\mathcal{R}}_{k:n}(t_7)$	$\hat{\mathcal{R}}^*{}_1(t_8)$	$\hat{\mathcal{R}}_{k:n}(t_9)$	$\hat{\mathcal{R}}^*{}_1(t_{10})$
$\hat{\lambda} = 0.118$	$\hat{\theta} = 1.251$	0.5785	0.5714	0.5653	0.5598	0.5549

The previous three stochastic processes are considered to determine the appropriate stochastic process for $Z(t)$ data. For this purpose, the Q–Q plots of the three stochastic processes are present in Figure 14. As shown in Figure 14, the Weibull process outperforms the other models. Also,

Table 7. The ML estimates of the parameters α_1, α_2, β, λ, θ and $\mathcal{R}^*{}_3(t)$.

MLE's		$\hat{\mathcal{R}}^*{}_3(t_1)$	$\hat{\mathcal{R}}^*{}_3(t_2)$	$\hat{\mathcal{R}}^*{}_3(t_3)$	$\hat{\mathcal{R}}^*{}_3(t_4)$	$\hat{\mathcal{R}}^*{}_3(t_5)$
$\hat{\alpha}_1 = 1.501$	$\hat{\beta} = 0.121$	0.6519	0.6235	0.6061	0.5907	0.5835
$\hat{\alpha}_2 = 1.501$	$\hat{\tau} = 0.174$	$\hat{\mathcal{R}}^*{}_3(t_6)$	$\hat{\mathcal{R}}^*{}_3(t_7)$	$\hat{\mathcal{R}}^*{}_3(t_8)$	$\hat{\mathcal{R}}^*{}_3(t_9)$	$\hat{\mathcal{R}}^*{}_3(t_{10})$
$\hat{\lambda} = 0.118$	$\hat{\theta} = 1.251$	0.5753	0.5683	0.5623	0.5569	0.5521

Table 5 shows the values of the AIC, BIC indices, and CVM statistics. This table confirms the results of Figure 14. The results of the MLE of the parameter $\mathcal{R}^*$ are presented in Table 6.

Tables 3, 6, and 7 show that the system reliability increases after the impact of the cold or hot standby components. The inequality $\mathcal{R}^*{}_1(t) > \mathcal{R}^*{}_3(t)$ holds in all cases. These results precisely confirmed the results obtained in Sections 3 and 4.

7. Conclusions

In this article, we considered the effect of adding three different types of standby components to a k–out–of–n:F system in the dynamic stress–strength model. Furthermore, we assumed that the component's strength and stress are determined by the Weibull process and the Weibull distribution, respectively. The system reliability is calculated without (and with) a standby component. Also, we performed a sensitivity analysis on the parameters of the standby component's strength (λ and α_2) to examine the impact of these parameters on the system reliability. Moreover, an actual data set is applied to illustrate the procedure. Based on the obtained results, a standby component increases system reliability. We also concluded that the cold standby component has more impact on system reliability than other standby components, and the warm standby component has more impact than the hot standby component. We suggest the following different items to generalize the current paper.

- According to our assumptions in the article, the strength components follow the Weibull process. Other stochastic processes, such as Gaussian, Gamma, and Pareto, can be studied.
- One can consider a dependency structure between X and $Y(t)$ based on different copulas, e.g., FGM, Gumble, Frank, and then investigate the influence of such models on the estimation of $\mathcal{R}$.

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Appendix

Proof. First we assume that the standby component is a hot component. It is enough to prove that $\frac{\partial \mathcal{R}_3^*(t)}{\partial \alpha_2} > 0$. For this purpose, we have

$$\begin{aligned}\frac{\partial \mathcal{R}_3^*(t)}{\partial \alpha_2} &= \frac{nt\binom{n-1}{k-1}}{\alpha_1\theta}\int_0^\infty\int_0^x\left(\frac{\partial}{\partial\alpha_2}\left(e^{-\frac{xt}{\alpha_2}}\right)\right)e^{-x(\frac{(n-k)t}{\alpha_1})}\\ &\quad\times e^{-\frac{yt}{\alpha_1}}\left(1-e^{-\frac{yt}{\alpha_1}}\right)^{k-1}e^{-(\frac{x}{\theta})}\ dy\ dx\\ &= \frac{nt\binom{n-1}{k-1}}{\alpha_1\alpha_2^2\theta}\int_0^\infty xte^{-x(\frac{t}{\alpha_2}+\frac{(n-k)t}{\alpha_1}+\frac{1}{\theta})}\\ &\quad\times\int_0^x e^{-\frac{yt}{\alpha_1}}\left(1-e^{-\frac{yt}{\alpha_1}}\right)^{k-1}\ dy\ dx\\ &= \frac{nt\binom{n-1}{k-1}}{k\alpha_1\alpha_2^2\theta}\int_0^\infty xe^{-x(\frac{t}{\alpha_2}+\frac{(n-k)t}{\alpha_1}+\frac{1}{\theta})}\left(1-e^{-\frac{xt}{\alpha_1}}\right)^k dx\end{aligned}$$

we know $\left(1-e^{-\frac{xt}{\alpha_1}}\right)^k = \sum_{j=0}^k\binom{k}{j}(-1)^j\left(e^{-\frac{xt}{\alpha_1}}\right)^j$.

$$\begin{aligned}\frac{\partial \mathcal{R}_3^*(t)}{\partial \alpha_2} &= \frac{nt\binom{n-1}{k-1}}{k\alpha_1\alpha_2^2\theta}\sum_{j=0}^k\binom{k}{j}(-1)^j\int_0^\infty xe^{-x(\frac{t}{\alpha_2}+\frac{(n-k)t}{\alpha_1}+\frac{1}{\theta}+\frac{jt}{\alpha_1})}dx\\ &= \frac{nt\binom{n-1}{k-1}}{k\alpha_1\alpha_2^2\theta}\sum_{j=0}^k\binom{k}{j}(-1)^j\frac{1}{(\frac{t}{\alpha_2}+(n-k)(\frac{t}{\alpha_1})+\frac{1}{\theta}+\frac{jt}{\alpha_1})^2}\\ &= \frac{nt\binom{n-1}{k-1}}{k\alpha_1\alpha_2^2\theta}\sum_{j=0}^k\frac{\binom{k}{j}(-1)^j}{A_j^2}\end{aligned} \tag{16}$$

The symbol $A_j = (\frac{t}{\alpha_2}+(n-k)\left(\frac{t}{\alpha_1}\right)+\frac{1}{\theta}+\frac{jt}{\alpha_1}),\quad j=0,\ldots,k$ is used for simplicity. Also, it is clear that

$$\frac{1}{A_0} > \frac{1}{A_1} > \cdots > \frac{1}{A_k} \tag{17}$$

Now it is enough to prove the Eq. (16) is positive. For this purpose, the inductive method is used.

Step1: if $k=1$, we have

$$\frac{\partial \mathcal{R}_3^*(t)}{\partial \alpha_2}=\frac{nt}{\alpha_1\alpha_2^2\theta}\left(\frac{1}{A_0^2}-\frac{1}{A_1^2}\right) \tag{18}$$

Since $\frac{1}{A_0}>\frac{1}{A_1}$, (18) is positive.

Step2: if $k=2$, we have

$$\begin{aligned}\frac{\partial \mathcal{R}_3^*(t)}{\partial \alpha_2}&=\frac{n(n-1)t}{2\alpha_1\alpha_2^2\theta}\left(\frac{1}{A_0^2}-\frac{2}{A_1^2}+\frac{1}{A_2^2}\right)\\&=\frac{n(n-1)t}{2\alpha_1\alpha_2^2\theta}\left[\left(\frac{1}{A_0^2}-\frac{1}{A_1^2}\right)+\left(\frac{1}{A_2^2}-\frac{1}{A_1^2}\right)\right]\end{aligned}$$

Given (17), we know

$$\begin{cases}\frac{1}{A_0^2}-\frac{1}{A_1^2}>0\\ \frac{1}{A_2^2}-\frac{1}{A_1^2}<0\\ \frac{1}{A_0^2}-\frac{1}{A_1^2}>\frac{1}{A_2^2}-\frac{1}{A_1^2}\end{cases}\quad\Rightarrow\quad\frac{\partial \mathcal{R}_3^*(t)}{\partial \alpha_2}>0$$

Stepk^*: Now suppose that for the Step k^*-1, the Eq. (16) is positive. It is enough to prove that (16) is also positive for $k=k^*$. For this purpose, we have

$$\begin{aligned}\frac{nt\binom{n-1}{k^*-1}}{k^*\alpha_1\alpha_2^2\theta}\sum_{j=0}^{k^*}\binom{k^*}{j}\frac{(-1)^j}{A_j^2}=&\frac{nt\binom{n-1}{k^*-1}}{k^*\alpha_1\alpha_2^2\theta}\\&\times\left[\sum_{j=0}^{k^*-1}\binom{k^*-1}{j}\frac{(-1)^j}{A_j^2}+\sum_{j=0}^{k^*-1}\binom{k^*-1}{j}\frac{(-1)^{j+1}}{A_{j+1}^2}\right]\end{aligned} \tag{19}$$

There are two cases. The first case, the sum of the sentences added is positive $\left(\sum_{j=0}^{k^*-1}\binom{k^*-1}{j}\frac{(-1)^{j+1}}{A_{j+1}^2}>0\right)$ by using induction assumption, the Eq. (19) is positive. In the second case, the sum of the sentences added is negative, i.e., $\sum_{j=0}^{k^*-1}\binom{k^*-1}{j}\frac{(-1)^j}{A_{j+1}^2}>0$. Given that $\sum_{j=0}^{k^*-1}\binom{k^*-1}{j}\frac{(-1)^j}{A_j^2}>\sum_{j=0}^{k^*-1}\binom{k^*-1}{j}\frac{(-1)^j}{A_{j+1}^2}$, the Eq. (19) is also positive. The proof is complete. The similar prove can be used for cold and warm component standby. □