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On the three-component mixture of exponential distributions: A Bayesian framework to model data with multiple lower and upper outliers

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Abstract

The presence of lower and upper outliers in the dataset may cause misleading inferential conclusions in the applied statistical problems. This paper introduces the three-component mixture of exponential (3-CME) distributions as an alternative platform for analyzing positive datasets in the presence of multiple lower and upper outliers. We obtain the parameter estimates with a focus on the Bayesian methodology. In order to investigate the performance of the presented approach, five simulation studies are conducted. We show that the proposed outlier model can be selected as an appropriate alternative model in dealing with the data with and without lower and upper outliers. The performance of the Bayes estimators under different loss functions with various sample sizes and the number of outliers are also investigated. Finally, two examples of real data are studied to illustrate the superiority of the 3-CME distributions in analyzing dataset and detecting lower and upper outliers.

Keywords: Lower and upper outliers; Exponential distribution; Mixture model; Bayesian analysis; Gibbs sampler

1. Introduction

Statistical methods are important in any statistical dataset analysis because, if there exist "unrepresentative" or "rogue" points in the data, the statistical method applied may exert amiss influence on the conclusions of statistical data analysis. These observations are commonly called as "outliers" or "bad points" [1]. The term outlier commonly refers to extreme observations that are far from the remainder of observed data points. However, one can find more information and general definition of outlier in [6,16,39]. A main reason of arising outliers are the contaminations along with the human error and ignorance. Outliers can affect on the results of statistical data analysis, so they should be detected. Some commonly applied methods for outlier detection can be found in McCulloch [22], Hodge and Austin [13], Jabbari Nooghabi et al. [15], Xue et al. [40], Jabbari Nooghabi [14] and Shadrokh and Pazira [36].

Because of the memoryless property and constant failure rate of the exponential model, it is well known that it is widely employed in many areas of life testing experiments and reliability engineering. One can find applications of the exponential model in [18]. Also, the mixture of exponential distributions plays an important role in various

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research areas. For instance, [38] introduced a new infinite mixture model for analyzing the mortality rate of demography based on the assumption that the populations' members differ in their endowment for longevity. The interested reader can also find more applications of the mixture of the exponential distributions in [3,8,26,37,41]. Referring to wide applicability in the actual sciences, the presence of outliers in an exponential sample has been discussed extensively in the literature. Pettit [32] studied a Bayesian method for the modeling of outliers in an exponential sample. One may acknowledge [19,27,31,33,34,42] to find more recently developed models. As can be seen, all of the aforementioned works presented a statistical approach and test in the presence of upper (or lower) outliers in the sample. In these cases, the researcher must rely on the assumption that the rogue points are only presented in the one side of the sample. In other words, the researcher should consider that the outliers are coming either from the exponential distribution with a smaller mean (lower outliers) or exponential distribution with a larger mean (upper outliers). However, [20] stated that this is often not the case in practice, hence the problem of identifying lower and upper outliers simultaneously in dataset may be of considerable importance. Although Kumar and Lin [20] proposed a test statistic for testing upper and lower outliers simultaneously in the exponentially distributed samples, to the best of our knowledge, there is no contribution for modeling and analyzing the exponentially distributed data in the presence of lower and upper outliers. In this paper, we present a robust outlier model for analyzing an exponentially distributed sample in the presence of multiple lower and upper outliers.

Enhancing the probability density function (pdf) is one of the most considered approaches for modeling and detecting outliers. In this regard, the Bayesian statistical approach can provide an alternative framework to the classical data analysis with two advantages; (1) providing additional information about the number of outliers, and (2) conducting a predictive inference on the parameter uncertainty [31]. An excellent review of the Bayesian approach is available in [10]. Also, the interested readers can find papers of the Bayesian inference for the mixture models in [2,4,5]. The analysis of outliers via the Bayesian approach was discussed by [11] who considered the contaminated location-shift normal distribution. Verdinelli and Wasserman [39] discussed the Bayesian analysis of outlier models via the contaminated normal distribution. Later, Scollnik [35] considered Pareto sample in the presence of upper outliers and developed a Bayesian analysis using the Gibbs sampler. Following [35], Okhli and Jabbari Nooghabi [31] considered the contaminated exponential distribution and extended a Bayesian approach using the Gibbs sampler for positive-valued insurance data when lower and upper outliers exist in the sample. They showed that the computation of posterior density function can be simplified by using Markov chain Monte Carlo (MCMC) such as the Gibbs sampler. Bayesian analysis via the MCMC procedures can provide feasible, low programming, and fast implementation approach. The MCMC methods such as the Gibbs sampler and Metropolis–Hastings algorithms bring considerable conceptual and calculating simplicity for calculating the posterior marginal probability of the observation to be an outlier (see [9,10,30] to an overview of the MCMC methods). To reach our aim, we will focus on the Bayesian approach using the Gibbs sampler for data modeling.

To enhance the resistance against lower and upper outliers, the main goal of this article is to suggest a new model based on the exponential distribution, named the three-component mixture of exponential (3-CME) model. Steaming form [31], we do Bayesian inference for the parameters of the 3-CME distributions. Numerical studies illustrate that the proposed model can be utilized with a small, moderate, or large sample size. Since we focus on the Bayesian approach, it is exhibited that the effect of lower and upper outliers are automatically reflected in the posterior distribution. The posterior probability of the 3-CME model is computed for lower and upper outliers identification purposes. Furthermore, we exhibit the proposed model as an alternative model in dealing with the data with and without lower and upper outliers.

The rest of the paper is outlined as follows. Section 2 discusses the formulation of 3-CME distributions and Bayesian method for computing its parameter estimates. The results and analysis of the simulation studies with different sample sizes (small, moderate, and large) are presented in Section 3. In Section 4, we analyze two examples of the real dataset to illustrate the outperformance of the 3-CME distributions. Finally, the paper is ended in Section 5 by presenting a brief conclusion and possible future directions.

2. Bayesian inference of the 3-CME model

This section begins with a definition and notation of the 3-CME distributions. For the cause of notation, let random variable X follow an 3-CME distributions, denoted by $X \sim 3-CME(\alpha, \theta, \beta, \rho, \tau)$. Thus, the pdf of X is given by

$$f_{\text{3-CME}}(x; \alpha, \theta, \beta, \rho, \tau) = \rho f_{\text{E}}(x; \alpha\theta) + \tau f_{\text{E}}(x; \alpha\beta) + (1 - \rho - \tau) f_{\text{E}}(x; \alpha), \quad x, \alpha, \theta, \beta > 0, \tag{1}$$

where the parameters $\rho \in [0, 1]$ and $\tau \in [0, 1]$ are the probability that an observation comes from the exponential distribution with means $\alpha\theta$ and $\alpha\beta$, respectively. It can obviously be seen that the pdf (1) is a mixture of three pdfs with mixing proportions ρ and τ. In fact, we suppose the sample x is not an outlier point if it is generated from the exponential distribution with parameter α, denoted by $E(\alpha)$, and pdf

$$f_{\mathrm{E}}(x;\alpha) = \alpha e^{-\alpha x}, \quad x, \alpha > 0, \tag{2}$$

and alternatively, x is an outlier observation comes from the exponential distribution with means $\alpha\theta$ if it is arisen from

$$f_{\mathrm{E}}(x;\alpha\theta) = \alpha\theta e^{-\alpha\theta x}, \quad x, \alpha, \theta > 0, \tag{3}$$

and similarly, x is an outlier observation comes from the exponential distribution with means $\alpha\beta$ if it is arisen from

$$f_{\mathrm{E}}(x;\alpha\beta) = \alpha\beta e^{-\alpha\beta x}, \quad x, \alpha, \beta > 0. \tag{4}$$

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^\top$ be a set of n realizations coming from a $3-CME(\alpha, \theta, \beta, \rho, \tau)$. For conducting Bayesian inference on the parameters of the model, it is convenient to introduce two vectors of independent Bernoulli random variables $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^\top$ and $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n)^\top$ with success probability ρ and τ, denoted by $Ber(\rho)$ and $Ber(\tau)$, respectively. We therefore have

$$X_i|(\boldsymbol{\omega}, \boldsymbol{\nu}) \sim E(\alpha\theta^{\omega_i}\beta^{\nu_i}), \quad i = 1, 2, \ldots, n. \tag{5}$$

It should be emphasised that X_is are conditionally independent of each other, and also of ρ and τ, given the other model parameters. The conditional likelihood is then given by

$$L(\boldsymbol{x}|\alpha, \theta, \beta, \boldsymbol{\omega}, \boldsymbol{\nu}, \rho, \tau) \propto \alpha^n \theta^{\sum_{i=1}^n \omega_i} \beta^{\sum_{i=1}^n \nu_i} \exp\left\{-\alpha \sum_{i=1}^n \theta^{\omega_i}\beta^{\nu_i} x_i\right\}, \tag{6}$$

where $\sum_{i=1}^n \omega_i$ and $\sum_{i=1}^n \nu_i$ are the number of outliers with outlier factors θ and β, respectively.

Note 1. *In model (1), we have*

(1) If $\beta < 1$ and $\theta > 1$, then $\sum_{i=1}^n \nu_i$ and $\sum_{i=1}^n \omega_i$ are the number of upper outliers and the number of lower outliers, respectively.
(2) If $\beta = 1$ (equivalently $\sum_{i=1}^n \nu_i = 0$), then $\sum_{i=1}^n \omega_i$ is the number of lower or upper outliers (similarly for $\theta = 1$ or $\sum_{i=1}^n \omega_i = 0$).
(3) If $\beta = \theta = 1$ (equivalently $\sum_{i=1}^n \nu_i = \sum_{i=1}^n \omega_i = 0$), then the 3-CME model is the same as the basic (no outlier) model (2).

2.1. Prior describing

As a natural way of the Bayesian parameter estimation, it is necessary to consider the joint prior as

$$\pi(\alpha, \theta, \beta, \boldsymbol{\omega}, \boldsymbol{\nu}, \rho, \tau) = \pi(\alpha)\pi(\theta)\pi(\beta)\pi(\boldsymbol{\omega}|\rho)\pi(\rho)\pi(\boldsymbol{\nu}|\tau)\pi(\tau),$$

for the parameters of the 3-CME model. Following [28,31] and [32], it can easily be seen from the conditional likelihood (6) that the gamma distribution is a conjugate prior choose for the parameters α, θ, and β, i.e. we suppose that $\alpha \sim gamma(a_1, a_2)$, $\beta \sim gamma(b_1, b_2)$, and $\theta \sim gamma(d_1, d_2)$ where a_1, a_2, b_1, b_2, d_1, and d_2 are the hyperparameter values, and $gamma(\vartheta, \lambda)$ represent the gamma distribution with pdf

$$\frac{\lambda^\vartheta}{\Gamma(\vartheta)}\alpha^{\vartheta-1}e^{-\lambda\alpha}, \quad \vartheta > 0\,, \lambda > 0.$$

From the re-expression model (5), we can also observe that the prior probability mass functions of independent variables ω_i and ν_i for $i = 1, 2, \ldots, n$ are, respectively of

$$\pi(\boldsymbol{\omega}|\rho) = \prod_{i=1}^n \rho^{\omega_i}(1-\rho)^{1-\omega_i} \qquad \text{and} \qquad \pi(\boldsymbol{\nu}|\tau) = \prod_{i=1}^n \tau^{\nu_i}(1-\tau)^{1-\nu_i}. \tag{7}$$

Moreover, two conjugate priors for ρ and τ, recommended by [31,35] and [39], can be assumed as

$$\rho \sim Beta(q_1, q_2), \quad q_1, q_2 > 0, \qquad \text{and} \qquad \tau \sim Beta(t_1, t_2), \qquad t_1, t_2 > 0, \tag{8}$$

where $Beta(.,.)$ denotes the beta distribution and q_1, q_2, t_1, and t_2 are the hyperparameter values.

Note that the Bayesian parameter estimates and its corresponding inference depend on the hyperparameter values. In the numerical parts, we follow [31] and [32] to obtain initial hyperparameter values a_1, a_2, d_1, d_2, b_1, and b_2. In this regard, points that we guess are not outliers (observations greater than $\bar{x} - 3\sigma(x)$ and smaller than $\bar{x} + 3\sigma(x)$) are taken for calculating the maximum likelihood estimate (MLE) of α, say $\hat{\alpha}_{MLE}$, based on them. Now by setting proportion $a_1/a_2 \simeq \hat{\alpha}_{MLE}$, one can fix a_1 to compute the hyperparameter a_2. Based on the points that we guess are lower outliers (the observations smaller than $\bar{x} - 3\sigma(x)$ or the smallest observation), we compute the MLE of $\gamma = \alpha\theta$, say $\hat{\gamma}_{MLE}$. Consequently, by setting proportion $d_1/d_2 \simeq \hat{\gamma}_{MLE}/\hat{\alpha}_{MLE}$, one can fix d_1 to calculate the hyperparameter d_2. Similarly, based on the observations that we guess are upper outliers (the observations greater than $\bar{x} + 3\sigma(x)$ or the largest observation), we compute the MLE of $\varphi = \alpha\beta$, say $\hat{\varphi}_{MLE}$. Consequently, by setting proportion $b_1/b_2 \simeq \hat{\varphi}_{MLE}/\hat{\alpha}_{MLE}$, one can fix b_1 to calculate the hyperparameter b_2. Following [31,35] and [39], we consider hyperparameter values q_1, q_2, t_1, and t_2 that allocate any sample "less than half a chance of being an outlier with high probability", that is, $Pr(\rho < 0.5) = 0.99$ and $Pr(\tau < 0.5) = 0.99$. Thus, we can set hyperparameter values $q_1 = t_1 = 0.1842$ and $q_2 = t_2 = 3.5$.

2.2. Posterior analysis

Exploiting Eqs. (6) to (8), one can obtain the joint posterior distribution of the parameters involved in (6) as the joint posterior distribution for all of the proposed model parameters is given by

$$\begin{aligned} f(\alpha, \theta, \beta, \boldsymbol{\omega}, \boldsymbol{\nu}, \rho, \tau|\boldsymbol{x}) &\propto L(\boldsymbol{x}|\alpha, \theta, \beta, \boldsymbol{\omega}, \boldsymbol{\nu}, \rho, \tau)\pi(\alpha)\pi(\theta)\pi(\beta)\pi(\boldsymbol{\omega}|\rho)\pi(\rho)\pi(\boldsymbol{\nu}|\tau)\pi(\tau) \\ &\propto \alpha^{n+a_1-1}\theta^{\sum_{i=1}^n \omega_i + d_1 - 1}\beta^{\sum_{i=1}^n \nu_i + b_1 - 1}\rho^{\sum_{i=1}^n \omega_i + q_1 - 1}\tau^{\sum_{i=1}^n \nu_i + t_1 - 1} \\ &\quad \times (1-\rho)^{n+q_2-\sum_{i=1}^n \omega_i - 1}(1-\tau)^{n+t_2-\sum_{i=1}^n \nu_i - 1} \\ &\quad \times \exp\{-d_2\theta - b_2\beta\}\exp\{-\alpha(a_2 + \sum_{i=1}^n \theta^{\omega_i}\beta^{\nu_i}x_i)\}. \end{aligned} \tag{9}$$

In order to implement a Gibbs sampler, we should first compute the full conditional posterior distributions of each unknown parameter. Now, the joint posterior distribution (9) leads to the full conditional posterior distributions of ρ and τ, respectively as

$$\rho|(\boldsymbol{x}, \boldsymbol{\omega}) \sim Beta\Big(\sum_{i=1}^n \omega_i + q_1, n + q_2 - \sum_{i=1}^n \omega_i\Big) \quad \text{and} \quad \tau|(\boldsymbol{x}, \boldsymbol{\nu}) \sim Beta\Big(\sum_{i=1}^n \nu_i + t_1, n + t_2 - \sum_{i=1}^n \nu_i\Big).$$

It is easily seen that the full conditional distributions for ρ and τ depend only on $\boldsymbol{\omega}$ and $\boldsymbol{\nu}$, respectively. Also, the full conditional posterior distribution of α is

$$\alpha|(\boldsymbol{x}, \theta, \beta, \boldsymbol{\omega}, \boldsymbol{\nu}) \sim gamma\Big(n + a_1, a_2 + \sum_{i=1}^n \theta^{\omega_i}\beta^{\nu_i}x_i\Big).$$

Accordingly, the full conditional posterior distributions of θ and β are

$$\theta|(\boldsymbol{x}, \alpha, \beta, \boldsymbol{\omega}, \boldsymbol{\nu}) \sim gamma\Big(\sum_{i=1}^n \omega_i + d_1, d_2 + \alpha\sum_{i=1}^n \beta^{\nu_i}x_i\boldsymbol{I}(\omega_i = 1)\Big),$$

$$\beta|(\boldsymbol{x}, \alpha, \theta, \boldsymbol{\omega}, \boldsymbol{\nu}) \sim gamma\Big(\sum_{i=1}^n \nu_i + b_1, b_2 + \alpha\sum_{i=1}^n \theta^{\omega_i}x_i\boldsymbol{I}(\nu_i = 1)\Big),$$

where $\boldsymbol{I}(.)$ denotes the indicator function. It is observed that the full conditional posterior distributions of α, θ, and β are independent of ρ and τ. Moreover, the posterior probability mass functions of ω_i and ν_i, $i = 1, \ldots, n$, which are computed from (7) and (9), take the following forms

$$Pr(\omega_i|\boldsymbol{x}, \alpha, \theta, \beta, \boldsymbol{\nu}, \rho) \propto \rho^{\omega_i}(1-\rho)^{\omega_i - 1}\theta^{\omega_i}\exp\{-\alpha\theta^{\omega_i}\beta^{\nu_i}x_i\}, \quad i = 1, \ldots, n, \tag{10}$$

$$Pr(\nu_i|\boldsymbol{x},\alpha,\theta,\beta,\boldsymbol{\omega},\tau) \propto \tau^{\nu_i}(1-\tau)^{\nu_i-1}\beta^{\nu_i}\exp\{-\alpha\theta^{\omega_i}\beta^{\nu_i}x_i\}, \quad i=1,\ldots,n, \tag{11}$$

respectively, where ω_i and ν_i are equal to either 0 or 1. Eqs. (10) and (11) show that given the observed data and the other model parameters, $\{\omega_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$ are conditionally independent. By assuming $\theta > 1$, an interesting outcome which is obtained from (10) is that the full conditional posterior probability that x_i $(i = 1,\ldots,n)$ to be an lower outlier is

$$l_i(\alpha,\theta,\beta,\nu_i,\rho) = Pr(\omega_i = 1|\boldsymbol{x},\alpha,\theta,\beta,\boldsymbol{\nu},\rho) = \frac{\rho\theta e^{-\alpha\theta x_i}}{\rho\theta e^{-\alpha\theta x_i} + (1-\rho)\beta^{\nu_i}e^{-\alpha\beta^{\nu_i}x_i}}.$$

Also by assuming $\beta < 1$, the full conditional posterior probability that x_i $(i = 1,\ldots,n)$ to be an upper outlier is obtained from (11) as

$$u_i(\alpha,\theta,\beta,\omega_i,\tau) = Pr(\nu_i = 1|\boldsymbol{x},\alpha,\theta,\beta,\boldsymbol{\omega},\tau) = \frac{\tau\beta e^{-\alpha\beta x_i}}{\tau\beta e^{-\alpha\beta x_i} + (1-\tau)\theta^{\omega_i}e^{-\alpha\theta^{\omega_i}x_i}}.$$

These two probabilities enable us to identify lower and upper outliers automatically.

In order to estimate the posterior marginals and Bayesian inference, samples are drawn from the full conditional posterior distribution by adopting Gibbs sampler. We firstly put the number of iteration to $r = 0$. Then, by considering an appropriate initial values of the parameters the following steps are repeated (N times).

(i) Generate samples for ρ_{r+1}, τ_{r+1}, and α_{r+1} from $Beta(\sum_{i=1}^n \omega_{r,i} + q_1, n + q_2 - \sum_{i=1}^n \omega_{r,i})$, $Beta(\sum_{i=1}^n \nu_{r,i} + t_1, n + t_2 - \sum_{i=1}^n \tau_{r,i})$, and $gamma\big(n + a_1, a_2 + \sum_{i=1}^n \theta_r^{\omega_{r,i}}\beta_r^{\nu_{r,i}}x_i\big)$, respectively.

(ii) Draw a sample for θ_{r+1} from $gamma\big(\sum_{i=1}^n \omega_{r,i} + d_1, d_2 + \alpha_{r+1}\sum_{i=1}^n \beta_r^{\nu_{r,i}}x_i\boldsymbol{I}(\omega_{r,i} = 1)\big)$.

(iii) Draw a sample for scβ_{r+1} from $gamma\big(\sum_{i=1}^n \nu_{r,i} + b_1, b_2 + \alpha_{r+1}\sum_{i=1}^n \theta_{r+1}^{\omega_{r,i}}x_i\boldsymbol{I}(\nu_{r,i} = 1)\big)$.

(iv) Draw samples for $\boldsymbol{\omega}_{r+1,i}$, $i = 1,2,\ldots,n$ (conditional posterior probability that x_i is an outlier with mean $\alpha\theta$) from $Ber\big(l_i(\alpha_{r+1},\theta_{r+1},\beta_{r+1},\nu_{r,i},\rho_{r+1})\big)$.

(v) Finally, generate $\boldsymbol{\nu}_{r+1,i}$, $i = 1,2,\ldots,n$ (conditional posterior probability that x_i is an outlier with mean $\alpha\beta$) from $Ber\big(u_i(\alpha_{r+1},\theta_{r+1},\beta_{r+1},\omega_{r+1,i},\tau_{r+1})\big)$.

3. Simulation study

In this section, five simulation studies are conducted to check the performance of the 3-CME model under various (small, moderate, and large) sample sizes and the number of lower and upper outliers. The statistical software R is used for implementing simulations. In order to generate lower and upper outliers, we consider presumed parameters $\theta > 1$ and $\beta < 1$. For the sample size n, the number of lower outliers $\sum_{i=1}^n \omega_i = l$ and the number of upper outliers $\sum_{i=1}^n \omega_i = u$ the below steps are followed.

(i) By assuming $\omega_i = \nu_i = 0$, generate main sample of size $n - l - u$ (no lower and upper outliers) from $E(\alpha\theta^{\omega_i}\beta^{\nu_i})$.

(ii) By assuming $\omega_i = 1$ and $\nu_i = 0$, generate lower outlier of size l from $E(\alpha\theta^{\omega_i}\beta^{\nu_i})$.

(iii) By assuming $\omega_i = 0$ and $\nu_i = 1$, generate upper outlier of size u from $E(\alpha\theta^{\omega_i}\beta^{\nu_i})$.

In simulations, it is assumed that $N = 10\,000$ with a burn-in period of 2000 in each replication of 1000 trials. Hence, our results are based on a total of 8000 kept chains (following burn-in period of 2000 for reducing correlation). In order to monitor convergence of MCMC simulations, we utilize the scale reduction factor estimate $\sqrt{\hat{R}} = \sqrt{Var(\Upsilon)/Var(WMCMC)}$ as suggested by [10], where for the between- and within-sequence variances, respectively denoted by $Var(BMCMC)$ and $Var(WMCMC)$, the variance of estimated parameter of interest is calculated as $Var(\Upsilon) = (N-1)Var(WMCMC)/N + Var(BMCMC)/N$. In simulations for all MCMC chains, the scale factors for the sequences of α,θ,β,ρ, and τ are chosen within interval (0.99995, 1.0000) which shows their convergence.

3.1. Estimation and performance in dealing with multiple lower and upper outliers

An experiment in this subsection is conducted to verify the performance of our model and computational method under different number of lower and upper outliers simultaneously. We simulate from the 3-CME distributions by

Table 1
Bayesian parameter estimates of the 3-CME distributions under setting 1.

n	$\sum_{i=1}^{n}\omega_i$	$\sum_{i=1}^{n}\nu_i$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{\tau}$	$AL\alpha$	$AL\theta$	$AL\beta$	$AL\rho$	$AL\tau$	GV
20	1	1	0.010674	38.673	0.012210	0.081652	0.065569	0.011364	162.244	0.037417	0.335817	0.196944	2.350695E−08
20	1	2	0.010701	40.281	0.012591	0.075863	0.106957	0.011501	165.911	0.031797	0.298712	0.249831	3.631717E−08
20	1	3	0.010896	41.157	0.012303	0.069405	0.150491	0.012108	167.648	0.027297	0.273462	0.291503	3.461081E−08
20	2	1	0.011127	45.827	0.012619	0.104890	0.066634	0.012077	166.217	0.038302	0.343215	0.199279	7.474726E−08
20	2	2	0.011276	47.007	0.012217	0.096914	0.107592	0.012365	169.342	0.030858	0.317573	0.250104	8.571885E−08
20	2	3	0.011212	48.149	0.011986	0.090956	0.150097	0.012713	172.710	0.026721	0.298583	0.290611	7.836126E−08
20	3	1	0.011320	48.800	0.012517	0.138784	0.067625	0.013064	157.581	0.037919	0.375183	0.202004	1.095126E−07
20	3	2	0.011455	50.789	0.012199	0.127218	0.109155	0.013059	162.750	0.030653	0.348286	0.252528	1.526186E−07
20	3	3	0.011714	52.819	0.012019	0.118218	0.150280	0.013779	168.994	0.026904	0.330821	0.291268	1.451630E−07
25	1	1	0.010440	38.328	0.012732	0.076520	0.054719	0.009904	162.437	0.038751	0.317699	0.165689	2.061855E−08
25	1	2	0.010532	38.654	0.012678	0.068598	0.088878	0.010176	162.903	0.031792	0.284954	0.210653	3.127227E−08
25	1	3	0.010797	39.543	0.012088	0.062557	0.124879	0.010435	164.837	0.026360	0.253490	0.245020	2.615842E−08
25	2	1	0.010812	42.967	0.012570	0.092426	0.054492	0.010244	161.982	0.038286	0.322424	0.164643	3.885106E−08
25	2	2	0.010839	44.506	0.012362	0.087302	0.090263	0.010537	165.136	0.030705	0.294843	0.211638	6.152878E−08
25	2	3	0.010950	45.840	0.011956	0.081053	0.124175	0.010759	168.508	0.026143	0.273041	0.244124	6.167876E−08
25	3	1	0.010976	47.453	0.012630	0.113019	0.054738	0.010724	160.849	0.038513	0.331674	0.165819	7.016266E−08
25	3	2	0.011067	49.120	0.012555	0.110229	0.090524	0.011045	162.739	0.031261	0.314394	0.212905	8.791658E−08
25	3	3	0.011236	49.880	0.011769	0.103963	0.124604	0.011371	164.309	0.025857	0.298301	0.245083	7.982358E−08
50	1	1	0.010169	34.984	0.012695	0.059636	0.029575	0.006678	153.487	0.038285	0.275117	0.090721	9.190797E−09
50	1	2	0.010117	36.296	0.013129	0.051795	0.048084	0.006537	157.244	0.032222	0.232802	0.116084	1.034353E−08
50	1	3	0.010240	36.760	0.012593	0.048378	0.066624	0.006702	158.954	0.026927	0.212892	0.136558	1.198663E−08
50	2	1	0.010193	38.551	0.012761	0.064825	0.029278	0.006674	156.482	0.038750	0.264928	0.090241	1.548786E−08
50	2	2	0.010294	40.089	0.012713	0.060182	0.048016	0.006767	159.666	0.031257	0.234825	0.116168	2.730384E−08
50	2	3	0.010257	41.369	0.012496	0.056483	0.066428	0.006693	161.161	0.026610	0.210582	0.135822	1.999472E−08
50	3	1	0.010258	42.081	0.012894	0.077247	0.029483	0.006848	154.703	0.038966	0.269312	0.090646	2.430747E−08
50	3	2	0.010392	42.764	0.012711	0.071563	0.048345	0.006891	156.179	0.031075	0.238693	0.116393	3.174892E−08
50	3	3	0.010376	44.446	0.012600	0.067552	0.066782	0.006907	160.487	0.026830	0.221910	0.136729	3.051395E−08

(*continued on next page*)

generating x_i from

$$E(\alpha\theta^{\omega_i}\beta^{\nu_i}), \quad i = 1, \ldots, n,$$

where $\omega_i = \nu_i = 0$, $\omega_i = 1$ and $\nu_i = 0$, and $\omega_i = 0$ and $\nu_i = 1$, for the main (without outlier), lower outlier and upper outlier samples, respectively. We consider $n = 20, 25, 50, 100, 500$, targeting to have the small, moderate, and large samples, and the presumed parameters $\alpha = 0.01, 0.1$, $\theta = 40, 50$, and $\beta = 0.01, 0.05$. For each sample size, we also take various number of lower and upper outliers ranging from 1 to 10, i.e. $\sum_{i=1}^{n}\omega_i = 1, 2, 3, 5, 10$ and $\sum_{i=1}^{n}\nu_i = 1, 2, 3, 5, 10$. Moreover, the taken hyperparameter values are $a_1 = b_1 = d_1 = 2/3$ whereas a_2, b_2, and d_2 are computed via straightforward technique studied in Section 2.1. In the experiment of Section 3.4, we compare the squared error loss function (SELF) with some others alternative loss function. Based on the SELF, the Bayes estimates, average lengths (AL) of the 95% credible intervals, generalized variance (GV) for estimators of the parameters, the posterior probability that if an individual sample is a lower outlier, and the posterior probability that if an individual sample is an upper outlier, are calculated over the remaining samples, to investigate how well the proposed Bayesian approach for the 3-CME model works. The GV of the Bayes estimate of parameters $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\theta}$ is computed as

$$GV(\hat{\alpha}, \hat{\beta}, \hat{\theta}) = \begin{vmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha}, \hat{\theta}) & Cov(\hat{\alpha}, \hat{\beta}) \\ Cov(\hat{\alpha}, \hat{\theta}) & Var(\hat{\theta}) & Cov(\hat{\theta}, \hat{\beta}) \\ Cov(\hat{\alpha}, \hat{\beta}) & Cov(\hat{\theta}, \hat{\beta}) & Var(\hat{\beta}) \end{vmatrix}.$$

Tables 1 and 2 report the results of simulation study for the parameters setting 1 ($\alpha = 0.01, \theta = 40, \beta = 0.01$) and setting 2 ($\alpha = 0.1, \theta = 50, \beta = 0.05$), respectively. Fig. 1 shows a graphical visualization of the GV values for setting 1. From Tables 1 and 2 and Fig. 1, the following features can be seen. (i) Although the GV is an increasing function with respect to the number of lower ($\sum_{i=1}^{n}\omega_i = l$) or upper ($\sum_{i=1}^{n}\nu_i = u$) outliers for a fixed sample size, it is a decreasing function with respect to the sample size n. (ii) By increasing the sample size, the AL of the 95% credible intervals for α, θ, ρ, and τ decreases, meanwhile the AL of β is slowly increased, (iii) For a fixed sample sizes and the number of upper outliers, the AL of the 95% credible intervals for α, θ, ρ, and τ increases when the number of lower outliers increased, (iv) By increasing the number of upper outliers under a fixed sample sizes and

Table 1 (*continued*).

n	$\sum_{i=1}^{n}\omega_i$	$\sum_{i=1}^{n}\nu_i$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{\tau}$	$AL\alpha$	$AL\theta$	$AL\beta$	$AL\rho$	$AL\tau$	GV
100	1	1	0.009981	32.027	0.012790	0.047186	0.015270	0.004626	149.096	0.038537	0.251162	0.047334	3.185397E−09
100	1	2	0.010067	33.100	0.012695	0.042574	0.024820	0.004582	150.243	0.031068	0.212111	0.060849	5.081967E−09
100	1	3	0.010012	33.994	0.012572	0.037335	0.034816	0.004507	153.020	0.026464	0.181753	0.072497	4.948821E−09
100	2	1	0.009972	34.737	0.012969	0.050045	0.015198	0.004630	151.075	0.039281	0.243601	0.047240	5.150168E−09
100	2	2	0.010089	35.266	0.012786	0.045144	0.024883	0.004630	150.970	0.031256	0.204422	0.061107	6.662490E−09
100	2	3	0.010029	36.682	0.012485	0.041398	0.034553	0.004560	154.808	0.026306	0.181348	0.072146	6.449340E−09
100	3	1	0.010131	37.437	0.012779	0.055683	0.015095	0.004732	149.692	0.038807	0.235235	0.046981	7.694112E−09
100	3	2	0.010182	37.890	0.012886	0.050271	0.025084	0.004700	151.856	0.031371	0.205518	0.061437	1.153771E−08
100	3	3	0.010150	38.904	0.012187	0.045352	0.034714	0.004623	154.526	0.025626	0.180985	0.072242	8.628423E−09
100	5	5	0.010239	46.265	0.011570	0.053202	0.054004	0.004699	153.779	0.019870	0.156839	0.089629	1.282853E−08
100	10	10	0.010389	56.121	0.010784	0.086777	0.102880	0.005094	132.616	0.013886	0.163240	0.121007	1.082762E−08
500	1	1	0.009933	28.444	0.012728	0.032614	0.003141	0.002177	139.531	0.038764	0.215149	0.009898	6.813554E−10
500	1	2	0.009933	28.960	0.013334	0.028379	0.005197	0.002130	141.738	0.032679	0.180803	0.013008	8.124975E−10
500	1	3	0.009956	29.344	0.012792	0.024743	0.007151	0.002059	142.967	0.027065	0.153051	0.015307	9.686164E−10
500	2	1	0.009930	29.335	0.012584	0.031102	0.003139	0.002157	141.090	0.038283	0.201940	0.009865	7.853822E−10
500	2	2	0.009954	30.000	0.012990	0.027499	0.005145	0.002105	142.279	0.032012	0.170809	0.012892	7.886522E−10
500	2	3	0.009947	30.322	0.012382	0.024550	0.007167	0.002049	143.243	0.026145	0.147788	0.015305	8.934207E−10
500	3	1	0.009955	30.089	0.012906	0.033449	0.003176	0.002208	140.936	0.039148	0.210097	0.010011	9.061176E−10
500	3	2	0.009952	31.170	0.013017	0.028761	0.005108	0.002096	142.525	0.032169	0.168380	0.012870	1.312680E−09
500	3	3	0.009954	32.361	0.012693	0.025569	0.007169	0.002067	146.671	0.026777	0.147073	0.015329	1.494359E−09
500	5	5	0.009981	34.711	0.011921	0.023141	0.011115	0.002010	146.217	0.020375	0.113557	0.019099	1.298711E−09
500	10	10	0.010013	43.574	0.011077	0.023925	0.021146	0.001980	149.927	0.013808	0.076901	0.026221	1.152362E−09

Table 2
Bayesian parameter estimates of the 3-CME distributions under setting 2.

n	$\sum_{i=1}^{n}\omega_i$	$\sum_{i=1}^{n}\nu_i$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{\tau}$	$AL\alpha$	$AL\theta$	$AL\beta$	$AL\rho$	$AL\tau$	GV
20	1	1	0.110068	51.323	0.060003	0.077592	0.076281	0.140362	209.452	0.197991	0.313779	0.277535	8.927139E−05
20	1	2	0.111178	52.037	0.057297	0.075182	0.114501	0.147338	210.667	0.160370	0.298983	0.321023	1.127442E−04
20	1	3	0.109689	52.672	0.057233	0.070868	0.151923	0.150575	213.490	0.142116	0.285897	0.358350	1.038577E−04
20	2	1	0.116441	60.318	0.060081	0.100108	0.080505	0.158018	214.353	0.193985	0.326747	0.291714	1.999302E−04
20	2	2	0.117224	60.803	0.056532	0.096509	0.118871	0.161869	215.381	0.156003	0.315020	0.327666	2.687429E−04
20	2	3	0.117892	62.502	0.055208	0.091986	0.154405	0.166366	219.851	0.137329	0.300446	0.357803	3.135785E−04
20	3	1	0.121548	65.413	0.060677	0.130395	0.082588	0.177916	205.647	0.195635	0.351991	0.299427	3.366987E−04
20	3	2	0.123500	67.002	0.055843	0.125338	0.121205	0.181490	210.683	0.153646	0.341758	0.330956	5.469139E−04
20	3	3	0.122143	70.366	0.054255	0.120343	0.154817	0.178378	218.825	0.135053	0.328864	0.356368	4.898849E−04
25	1	1	0.109729	48.935	0.061580	0.069992	0.066263	0.123031	203.936	0.200710	0.295195	0.244795	7.067024E−05
25	1	2	0.109798	50.322	0.059388	0.065424	0.098053	0.125506	207.918	0.162997	0.271057	0.281608	9.430402E−05
25	1	3	0.108642	51.288	0.056984	0.063746	0.127863	0.128620	209.997	0.139387	0.260329	0.308350	9.376848E−05
25	2	1	0.113208	56.009	0.062363	0.089010	0.068864	0.138916	205.826	0.202405	0.301393	0.259450	1.262224E−04
25	2	2	0.113602	57.508	0.058266	0.085112	0.099414	0.138053	209.336	0.159616	0.286321	0.286240	2.197857E−04
25	2	3	0.112171	60.045	0.057261	0.079226	0.129881	0.137090	217.055	0.140023	0.269916	0.312497	1.999923E−04
25	3	1	0.117415	64.056	0.063401	0.112570	0.072634	0.153282	205.362	0.202073	0.316392	0.273057	2.687866E−04
25	3	2	0.116049	64.500	0.059813	0.107317	0.102479	0.149229	208.037	0.162694	0.303340	0.297088	3.157449E−04
25	3	3	0.116232	67.006	0.055943	0.103192	0.134744	0.148763	214.271	0.134699	0.292008	0.317861	3.847634E−04
50	1	1	0.105395	45.467	0.067072	0.053004	0.041256	0.083703	198.201	0.217101	0.249345	0.169212	4.430059E−05
50	1	2	0.103875	46.553	0.063523	0.048332	0.057494	0.080230	200.168	0.173037	0.221804	0.181186	5.122266E−05
50	1	3	0.103186	47.232	0.061353	0.045297	0.074711	0.078741	201.574	0.144654	0.202075	0.192000	5.614666E−05
50	2	1	0.105779	51.054	0.067063	0.059989	0.041729	0.084987	200.112	0.216637	0.239999	0.172986	7.034698E−05
50	2	2	0.105836	52.538	0.064874	0.056438	0.059495	0.083419	202.956	0.173310	0.216043	0.186122	9.642409E−05
50	2	3	0.105176	52.408	0.061917	0.053148	0.076658	0.084615	203.131	0.144353	0.202136	0.199441	1.018898E−04
50	3	1	0.108038	55.596	0.067167	0.072091	0.043684	0.093322	196.587	0.214101	0.236523	0.182774	1.043168E−04
50	3	2	0.106330	58.364	0.063826	0.066904	0.060643	0.086258	203.566	0.169366	0.217743	0.188347	1.514228E−04
50	3	3	0.107960	59.378	0.060162	0.063901	0.077158	0.086709	207.091	0.138800	0.205283	0.195631	1.474496E−04
100	1	1	0.101279	42.417	0.068904	0.040943	0.023145	0.054204	191.140	0.228153	0.217183	0.104921	1.701619E−05
100	1	2	0.101163	43.208	0.066487	0.036900	0.031812	0.052222	191.939	0.182257	0.187118	0.107317	2.202607E−05
100	1	3	0.101229	43.918	0.065569	0.034244	0.042240	0.051956	193.979	0.153725	0.165775	0.116464	2.201239E−05

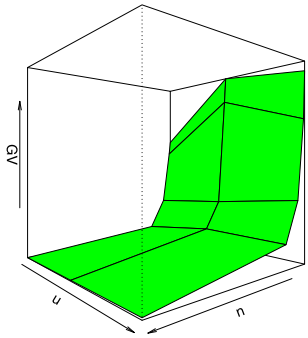
(*continued on next page*)

the number of lower outliers, the AL of the 95% credible intervals for α, θ, ρ, and τ increases, meanwhile the AL of β is decreased, (v) By increasing the value of all parameters, the AL of the 95% credible intervals for α, θ, β, τ, and GV increases, meanwhile the AL of ρ is decreased.

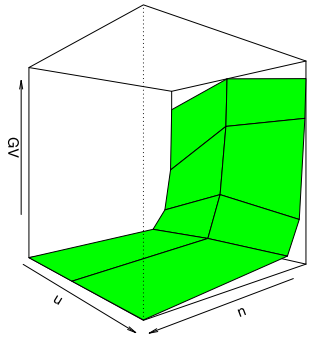
Moreover, for $\sum_{i=1}^{n}\omega_i = l$ and $\sum_{i=1}^{n}\nu_i = u$ where u and l vary in $1, 2, 3$ and under parameters setting 1, the posterior probability that the sample is a lower outlier and the posterior probability that the sample is an upper

Table 2 (*continued*).

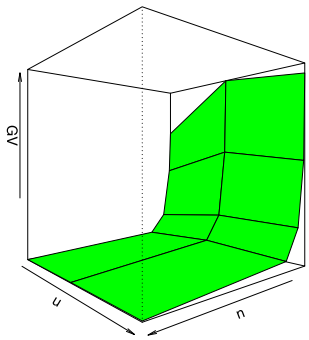
n	$\sum_{i=1}^{n}\omega_i$	$\sum_{i=1}^{n}\nu_i$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{\tau}$	$AL\alpha$	$AL\theta$	$AL\beta$	$AL\rho$	$AL\tau$	GV
100	2	1	0.102333	45.655	0.069564	0.043361	0.024483	0.057933	193.209	0.228764	0.205436	0.114813	2.878004E−05
100	2	2	0.102279	48.003	0.065933	0.039812	0.032777	0.053563	197.211	0.178866	0.179954	0.111761	3.812927E−05
100	2	3	0.101520	48.914	0.064336	0.037444	0.040775	0.052446	199.752	0.154206	0.162688	0.115227	4.176943E−05
100	3	1	0.103935	50.151	0.070724	0.046827	0.025975	0.059683	194.899	0.229212	0.193046	0.117679	5.436582E−05
100	3	2	0.102853	49.997	0.066244	0.043177	0.033216	0.054813	195.284	0.178987	0.172046	0.114600	3.851675E−05
100	3	3	0.103305	52.111	0.064274	0.041150	0.042386	0.055342	199.183	0.151678	0.159219	0.119910	5.969678E−05
100	5	5	0.103735	62.141	0.061140	0.050215	0.061357	0.054291	198.853	0.114631	0.143783	0.128682	7.416053E−05
100	10	10	0.104668	76.765	0.054516	0.085429	0.107310	0.058491	172.239	0.075395	0.153096	0.152729	7.026723E−05
500	1	1	0.100563	37.816	0.075037	0.024529	0.007296	0.024088	180.976	0.255411	0.165588	0.037848	7.348630E−06
500	1	2	0.099911	38.276	0.072501	0.022408	0.008770	0.022950	182.353	0.208144	0.146985	0.038103	1.229483E−05
500	1	3	0.099788	38.398	0.068504	0.020167	0.009993	0.021666	182.348	0.168149	0.127631	0.033582	4.325178E−06
500	2	1	0.100199	39.396	0.073160	0.024595	0.007208	0.023718	183.410	0.246164	0.160999	0.038785	7.465588E−06
500	2	2	0.100265	39.605	0.072533	0.022244	0.008500	0.022151	182.727	0.204084	0.141875	0.036237	7.058397E−06
500	2	3	0.100102	40.987	0.069563	0.019924	0.010148	0.021386	184.726	0.168144	0.118240	0.033418	5.866644E−06
500	3	1	0.100416	40.279	0.076249	0.025015	0.007312	0.024235	181.079	0.260037	0.155348	0.041474	6.792086E−06
500	3	2	0.100346	40.938	0.071340	0.022859	0.008331	0.022173	182.144	0.202592	0.138476	0.034022	8.120469E−06
500	3	3	0.100099	41.662	0.067903	0.020641	0.009912	0.021308	183.939	0.162987	0.115794	0.031651	7.380731E−06
500	5	5	0.100407	47.006	0.063169	0.019140	0.013758	0.021071	189.723	0.121773	0.088466	0.033156	7.835592E−06
500	10	10	0.100741	58.854	0.056624	0.020986	0.022970	0.020630	189.946	0.079577	0.063490	0.035895	8.374848E−06



(a) $\sum_{i=1}^{n}\omega_i = 1$

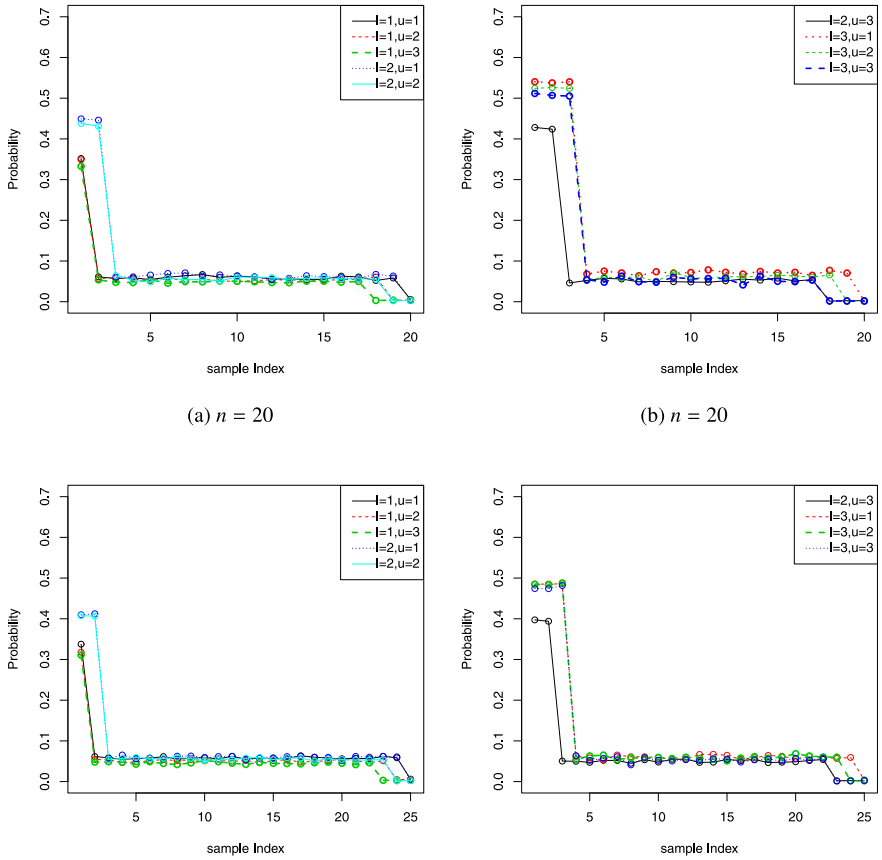


(b) $\sum_{i=1}^{n}\omega_i = 2$



(c) $\sum_{i=1}^{n}\omega_i = 3$

Fig. 1. The GV plot as a function of n and u under setting 1 for $\sum_{i=1}^{n}\omega_i = 1, 2,$ and 3.



(a) $n = 20$ (b) $n = 20$

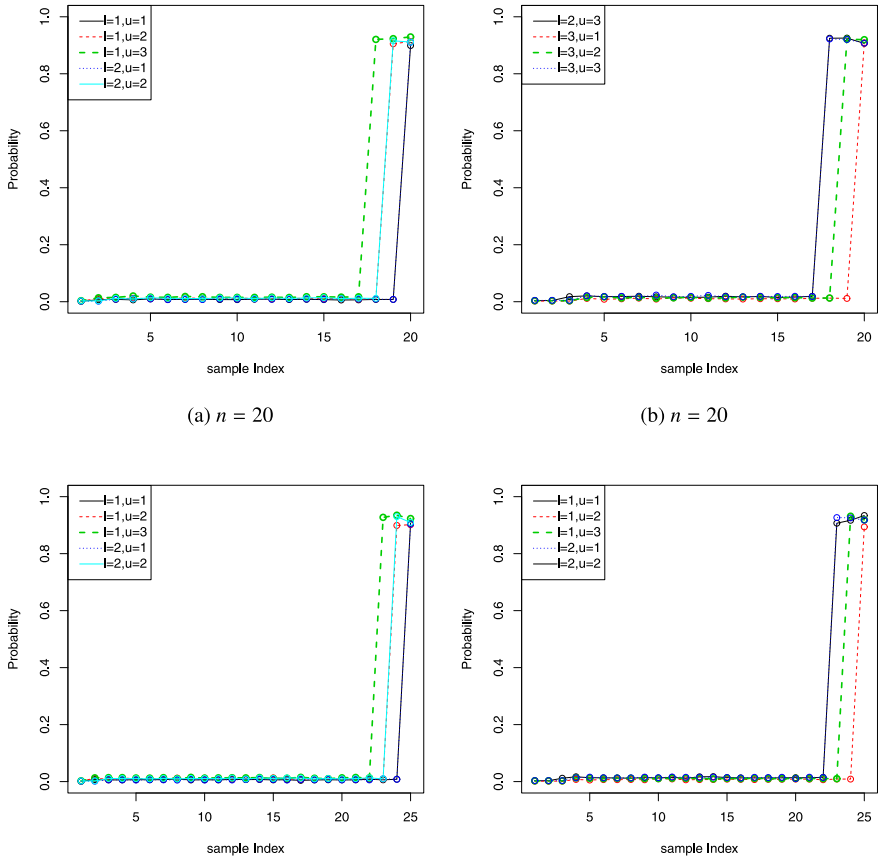
(c) $n = 25$ (d) $n = 25$

Fig. 2. Posterior probability that each generated sample is a lower outlier under setting 1.

outlier are displayed in Figs. 2 and 3, respectively. Note that for a fixed number of lower outliers l, the posterior probability that a small sample is a lower outlier is plotted at the first lth sample index in Fig. 2. Similarly, for a fixed number of upper outliers u, the posterior probability that a big sample is an upper outlier is plotted at the last uth sample index of Fig. 3. The posterior probability that an sample is a lower outlier, increases by increasing the number of lower outliers. Also, the posterior probability that the sample is an upper outlier, increases by increasing the number of upper outliers. From Fig. 2, it can be seen that the posterior probability of the lower outlier points in these experiments is moderately high (at least 0.204), while the no lower outlier points have a small posterior probability of being a lower outlier. From Fig. 3, it can be observed that the posterior probability of the upper outlier points is high (at least 0.912) in these experiments, while the no upper outlier points have a small posterior probability of being an upper outlier. It is obvious that the lower and upper outlier points are detected with large posterior probabilities of being lower and upper outliers, respectively, but this probability tends to zero for the remainder points. It indicates that the 3-CME model and its Bayesian method are efficient to identify lower and upper outliers. One must note that the same outcomes are observed under parameter setting 2.

3.2. Model performance when no lower and upper outlier in sample

An experiment in this subsection is carried out to verify the performance of our presented model when no lower and upper outliers are in the sample. In this regard, we generate samples from an exponential distribution with $\alpha = 0.1$. Under the SELF, the posterior probability that a homogeneous sample is a lower outlier and the posterior



(a) $n = 20$ (b) $n = 20$

(c) $n = 25$ (d) $n = 25$

Fig. 3. Posterior probability that each generated sample is an upper outlier under setting 1.

Table 3
The posterior probability that each homogeneous sample be a lower outlier.

n	The posterior probability
20	0.0559, 0.0510, 0.0518, 0.0562, 0.0567, 0.0572, 0.0548, 0.0578, 0.0541, 0.0545, 0.0554, 0.0521, 0.0548, 0.0585, 0.0557, 0.0571, 0.0518 0.0521, 0.0537, 0.0494
25	0.0552, 0.0500, 0.0532, 0.0574, 0.0475, 0.0525, 0.0555, 0.0516, 0.0492, 0.0521, 0.0545, 0.0536, 0.0558, 0.0502, 0.0522, 0.0529, 0.0517 0.0506, 0.0522, 0.0489, 0.0526, 0.0482, 0.0530, 0.0558, 0.0504
50	0.0473, 0.0477, 0.0459, 0.0444, 0.0447, 0.0461, 0.0465, 0.0454, 0.0433, 0.0479, 0.0464, 0.0446, 0.0461, 0.0460, 0.0433, 0.0422, 0.0443 0.0462, 0.0472, 0.0462, 0.0432, 0.0420, 0.0431, 0.0436, 0.0477, 0.0469, 0.0496, 0.0448, 0.0425, 0.0405, 0.0408, 0.0485, 0.0464, 0.0428 0.0450, 0.0415, 0.0449, 0.0443, 0.0455, 0.0452, 0.0450, 0.0473, 0.0463, 0.0473, 0.0457, 0.0428, 0.0448, 0.0440, 0.0433, 0.0425

probability that a homogeneous sample is an upper outlier are computed, to investigate the performance of 3-CME model when no lower and upper outliers are in the generated sample.

Tables 3 and 4 present the results of the simulation study for three taken sample sizes $n = 20, 25, 50$. It is clear that the posterior probability of each sample to be a lower outlier or an upper outlier is very small, indicating that the presented methodology does not classify some observations as outliers. Hence, the 3-CME model can be selected as an alternative model in dealing with the data without lower and upper outliers. One must note that the same outcomes are observed for a large sample sizes.

Table 4
The posterior probability that each homogeneous sample be an upper outlier.

n	The posterior probability
20	0.0196, 0.0213, 0.0195, 0.0190, 0.0194, 0.0222, 0.0215, 0.0194, 0.0199, 0.0224, 0.0181, 0.0213, 0.0233, 0.0207, 0.0199, 0.0201, 0.0191 0.0187, 0.0201, 0.0213
25	0.0177, 0.0194, 0.0197, 0.0182, 0.0176, 0.0213, 0.0184, 0.0197, 0.0184, 0.0202, 0.0192, 0.0199, 0.0177, 0.0218, 0.0213, 0.0188, 0.0202 0.0184, 0.0184, 0.0212, 0.0205, 0.0196, 0.0197, 0.0173, 0.0187
50	0.0137, 0.0142, 0.0136, 0.0126, 0.0125, 0.0124, 0.0140, 0.0133, 0.0136, 0.0135, 0.0124, 0.0155, 0.0128, 0.0139, 0.0139, 0.0134, 0.0141 0.0125, 0.0136, 0.0134, 0.0148, 0.0131, 0.0130, 0.0130, 0.0135, 0.0123, 0.0135, 0.0148, 0.0148, 0.0151, 0.0153, 0.0127, 0.0139, 0.0149 0.0141, 0.0133, 0.0132, 0.0135, 0.0131, 0.0131, 0.0128, 0.0124, 0.0130, 0.0133, 0.0132, 0.0128, 0.0145, 0.0130, 0.0124, 0.0149

Table 5
Posterior probability that each generated sample be an upper outlier for presumed parameters $\alpha = 0.1, \beta = 0.05$.

n	$\sum_{i=1}^{n} \nu_i$	Upper outlier sample	No outlier sample
20	1	0.6683	0.0297, 0.0310, 0.0291, 0.0297, 0.0313, 0.0299, 0.0317, 0.0311, 0.0319, 0.0332, 0.0283, 0.0280, 0.0296 0.0315, 0.0293, 0.0280, 0.0304, 0.0294, 0.0328
20	2	0.7173, 0.7067	0.0416, 0.0408, 0.0435, 0.0423, 0.0421, 0.0402, 0.0394, 0.0414, 0.0397, 0.0411, 0.0407, 0.0420, 0.0403 0.0431, 0.0418, 0.0402, 0.0388, 0.0430
20	3	0.7076, 0.7324 0.7321	0.0514, 0.0484, 0.0513, 0.0514, 0.0487, 0.0486, 0.0530, 0.0514, 0.0502, 0.0498, 0.0511, 0.0477, 0.0498 0.0493, 0.0460, 0.0479, 0.0468
25	1	0.6774	0.0300, 0.0277, 0.0282, 0.0298, 0.0274, 0.0300, 0.0279, 0.0342, 0.0293, 0.0314, 0.0269, 0.0286, 0.0273 0.0298, 0.0302, 0.0278, 0.0276, 0.0300, 0.0289, 0.0291, 0.0268, 0.0315, 0.0291, 0.0275
25	2	0.6955, 0.6814	0.0332, 0.0354, 0.0376, 0.0320, 0.0327, 0.0348, 0.0318, 0.0347, 0.0360, 0.0367, 0.0342, 0.0314, 0.0355 0.0329, 0.0334, 0.0324, 0.0339, 0.0368, 0.0344, 0.0368, 0.0356, 0.0313, 0.0334
25	3	0.7376, 0.7433 0.7348	0.0450, 0.0440, 0.0463, 0.0447, 0.0444, 0.0460, 0.0484, 0.0404, 0.0450, 0.0421, 0.0440, 0.0457, 0.0440 0.0447, 0.0474, 0.0392, 0.0465, 0.0406, 0.0414, 0.0471, 0.0461, 0.0441

3.3. Model performance when either lower or upper outliers are in sample

An experiment in this subsection is carried out to verify the performance of our presented model when only upper outliers are in the generated sample. In this regard, we generate x_i from

$$E(\alpha\theta^{\omega_i}\beta^{\nu_i}), \quad i = 1, \ldots, n$$

where $\omega_i = \nu_i = 0$ and $\nu_i = 1$, $\omega_i = 0$, for the main (without outlier) and upper outlier sample, respectively. We consider the presumed parameters $\alpha = 0.1$ and $\beta = 0.05$. The selected hyperparameter values are $a_1 = b_1 = d_1 = 2/3$ whereas a_2, b_2 and d_2 are computed with the same strategy which is studied in Section 2.1. Under SELF, the posterior probability that a generated sample is an upper outlier and the posterior probability that a generated sample is a lower outlier are computed.

Tables 5 and 6 show the results of the experiment for two taken sample sizes $n = 20, 25$ and the number of upper outliers ranging from 1 to 3, i.e. $\sum_{i=1}^{n} \nu_i = 1, 2, 3$. It can be observed from Table 5 that the posterior probability of the upper outlier samples in these simulations is high (at least 0.67), while the no upper outlier samples have a small posterior probability of being an upper outlier. It is obvious that the upper outliers are detected with large posterior probabilities of being upper outliers, but this probability tends to zero for the other (no upper outlier) points. It shows that the presented model and its Bayesian method are efficient to identify upper outliers when only upper outliers are in the sample. Table 6 shows that the posterior probability of each sample to be a lower outlier is very small. Thus, the 3-CME model can be selected as an alternative model in dealing with the data in the presence of upper outliers. The same outcomes are obtained for a large sample sizes, too.

Table 6
Posterior probability that each generated sample be an lower outlier for presumed parameters $\alpha = 0.1, \beta = 0.05$.

n	$\sum_{i=1}^{n} \nu_i$	Upper outlier sample	No outlier sample
20	1	0.0148	0.0547, 0.0468, 0.0513, 0.0513, 0.0545, 0.0494, 0.0542, 0.0506, 0.0519, 0.0526, 0.0562, 0.0573, 0.0496 0.0532, 0.0527, 0.0518, 0.0518, 0.0535, 0.0493
20	2	0.0130, 0.0135	0.0516, 0.0481, 0.0476, 0.0495, 0.0526, 0.0500, 0.0476, 0.0471, 0.0497, 0.0488, 0.0447, 0.0456, 0.0433 0.0491, 0.0489, 0.0466, 0.0527, 0.0484
20	3	0.0118, 0.0115 0.0123	0.0462, 0.0478, 0.0454, 0.0502, 0.0487, 0.0481, 0.0472, 0.0473, 0.0420, 0.0473, 0.0452, 0.0445, 0.0487 0.0487, 0.0458, 0.0522, 0.0458
25	1	0.0150	0.0510, 0.0475, 0.0533, 0.0510, 0.0513, 0.0456, 0.0483, 0.0485, 0.0465, 0.0507, 0.0483, 0.0493, 0.0497 0.0503, 0.0513, 0.0482, 0.0463, 0.0478, 0.0442, 0.0498, 0.0507, 0.0539, 0.0512, 0.0523
25	2	0.0126, 0.0146	0.0434, 0.0435, 0.0423, 0.0445, 0.0449, 0.0495, 0.0435, 0.0458, 0.0445, 0.0437, 0.0429, 0.0420, 0.0433 0.0427, 0.0469, 0.0409, 0.0454, 0.0435, 0.0429, 0.0494, 0.0452, 0.0471, 0.0460
25	3	0.0102, 0.0110 0.0105	0.0420, 0.0460, 0.0404, 0.0453, 0.0425, 0.0402, 0.0421, 0.0429, 0.0429, 0.0446, 0.0425, 0.0429, 0.0444 0.0401, 0.0419, 0.0420, 0.0446, 0.0439, 0.0411, 0.0411, 0.0451, 0.0434

Note 2. *When only lower outliers are in the sample, the simulation results are similar to when only upper outliers are in the sample. Hence, the 3-CME model and its Bayesian method are efficient to identify lower outliers when only lower outliers are in the sample. Also, the 3-CME model can be selected as an alternative model in dealing with the data in the presence of lower outliers.*

3.4. Comparing the Bayes estimators under different loss functions

In this simulation study, we compare the Bayes estimators obtained by utilizing different loss functions. We therefore generate samples from the 3-CME distributions under settings 1 and 2 in Section 3.1. Also, we select different sample sizes and the number of lower and upper outliers similar to Section 3.1. The selected hyperparameter values are $a_1 = b_1 = d_1 = 2/3$ whereas a_2, b_2 and d_2 are computed with same strategy studied in Section 2.1. We assume SELF, precautionary loss function (PLF), and DeGroot loss function (DLF) recommended by [21,29], and [7], respectively. The posterior risks of parameters α, θ, β, ρ, and τ for SELF, PLF, and DLF are calculated, to investigate performance of the Bayes estimators under different loss functions.

Tables 7 and 8 summarize the outputs of the simulation under settings 1 and 2, respectively. From the depicted results in Tables 7 and 8, it can be seen that the posterior risks of parameters α, β, ρ, and τ under the SELF are smaller than the PLF and DLF for various sample sizes and the number of lower and upper outliers. It indicates that the SELF is a more preferable loss function for Bayesian approach based on the 3-CME model, even though, the posterior risk of Bayes estimator of parameter θ under DLF is smaller than the SELF and PLF. Thus, in our study, SELF is a preferable loss function. The posterior risk of parameter θ under the DLF decreases when the number of lower outliers is increased. Under the SELF, the posterior risks of parameters α, θ, ρ and τ decrease when the sample size is increased. The posterior risk of parameter β under the SELF reduces by increasing the number of upper outliers.

3.5. Comparing the 3-CME model with the basic (no outlier) model (2)

An experiment in this subsection is carried out to verify which model fits better the exponential sample in the presence of lower and upper outliers. We therefore generate samples from the 3-CME distributions with various sample sizes and various numbers of lower and upper outliers for parameters settings 1 and 2 used in Section 3.1. The selected hyperparameter values are $a_1 = b_1 = d_1 = 2/3$ whereas a_2, b_2 and d_2 are computed with the strategy presented in Section 2.1. We consider discrimination measures such as Akaike information criterion

Table 7
The posterior risks for the SELF, PLF, and DLF under setting 1.

n	$\sum_{i=1}^{n}\omega_i$	$\sum_{i=1}^{n}\nu_i$	SELF					PLF					DLF				
			$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{\tau}$
20	1	1	9.61E−06	2100	1.20E−04	9.13E−03	2.87E−03	8.35E−04	41.9	7.31E−03	8.60E−02	3.79E−02	7.32E−02	0.585	0.405	0.579	0.407
20	1	2	9.73E−06	2189	8.72E−05	7.44E−03	4.41E−03	8.43E−04	42.2	5.16E−03	7.56E−02	3.78E−02	7.42E−02	0.575	0.311	0.569	0.284
20	1	3	1.09E−05	2219	6.19E−05	6.23E−03	5.88E−03	9.10E−04	42.1	3.89E−03	6.86E−02	3.68E−02	7.78E−02	0.568	0.256	0.564	0.210
20	2	1	1.13E−05	2203	1.24E−04	9.45E−03	2.94E−03	9.28E−04	38.4	7.41E−03	7.60E−02	3.82E−02	7.70E−02	0.515	0.401	0.479	0.404
20	2	2	1.13E−05	2262	8.20E−05	7.99E−03	4.41E−03	9.21E−04	38.8	5.01E−03	6.96E−02	3.77E−02	7.66E−02	0.510	0.312	0.476	0.283
20	2	3	1.20E−05	2336	5.89E−05	7.13E−03	5.82E−03	9.79E−04	39.2	3.82E−03	6.54E−02	3.66E−02	8.16E−02	0.507	0.258	0.476	0.210
20	3	1	1.46E−05	1940	1.23E−04	1.09E−02	3.05E−03	1.12E−03	32.8	7.33E−03	7.01E−02	3.87E−02	8.72E−02	0.454	0.402	0.380	0.404
20	3	2	1.29E−05	2050	8.05E−05	9.25E−03	4.50E−03	1.03E−03	33.5	4.94E−03	6.52E−02	3.79E−02	8.35E−02	0.449	0.310	0.384	0.279
20	3	3	1.45E−05	2189	6.08E−05	8.36E−03	5.85E−03	1.10E−03	34.7	3.86E−03	6.31E−02	3.67E−02	8.61E−02	0.448	0.260	0.395	0.210
25	1	1	7.26E−06	2115	1.28E−04	8.29E−03	2.03E−03	6.51E−04	42.3	7.51E−03	8.26E−02	3.20E−02	5.89E−02	0.591	0.401	0.589	0.409
25	1	2	7.60E−06	2110	8.83E−05	6.86E−03	3.16E−03	6.71E−04	42.1	5.13E−03	7.44E−02	3.24E−02	6.03E−02	0.587	0.307	0.588	0.291
25	1	3	7.84E−06	2151	5.82E−05	5.40E−03	4.14E−03	6.77E−04	42.2	3.71E−03	6.54E−02	3.12E−02	5.97E−02	0.580	0.249	0.581	0.214
25	2	1	7.65E−06	2100	1.24E−04	8.39E−03	2.00E−03	6.68E−04	38.6	7.43E−03	7.45E−02	3.18E−02	5.92E−02	0.533	0.402	0.504	0.410
25	2	2	8.10E−06	2160	8.14E−05	6.98E−03	3.17E−03	7.00E−04	38.8	4.90E−03	6.64E−02	3.21E−02	6.18E−02	0.525	0.304	0.494	0.285
25	2	3	8.40E−06	2242	5.73E−05	5.98E−03	4.12E−03	7.09E−04	39.2	3.68E−03	6.14E−02	3.12E−02	6.15E−02	0.520	0.251	0.492	0.215
25	3	1	9.15E−06	2034	1.26E−04	8.63E−03	2.06E−03	7.59E−04	34.9	7.49E−03	6.66E−02	3.23E−02	6.41E−02	0.478	0.403	0.418	0.413
25	3	2	9.39E−06	2082	8.53E−05	7.75E−03	3.23E−03	7.82E−04	34.7	5.02E−03	6.16E−02	3.25E−02	6.52E−02	0.465	0.304	0.408	0.285
25	3	3	9.49E−06	2092	5.63E−05	6.88E−03	4.14E−03	7.79E−04	34.6	3.65E−03	5.84E−02	3.13E−02	6.54E−02	0.461	0.252	0.410	0.215
50	1	1	3.15E−06	1908	1.26E−04	6.38E−03	6.12E−04	3.01E−04	41.2	7.36E−03	7.70E−02	1.78E−02	2.90E−02	0.611	0.396	0.634	0.416
50	1	2	2.93E−06	1986	9.00E−05	4.63E−03	9.53E−04	2.82E−04	41.6	5.08E−03	6.51E−02	1.80E−02	2.74E−02	0.601	0.296	0.628	0.297
50	1	3	3.07E−06	2029	6.29E−05	3.87E−03	1.29E−03	2.92E−04	42.0	3.72E−03	5.86E−02	1.81E−02	2.80E−02	0.600	0.239	0.621	0.229
50	2	1	3.15E−06	1981	1.29E−04	5.92E−03	6.06E−04	3.00E−04	39.9	7.49E−03	6.97E−02	1.77E−02	2.88E−02	0.575	0.398	0.583	0.420
50	2	2	3.20E−06	2062	8.56E−05	4.75E−03	9.58E−04	3.01E−04	39.9	4.95E−03	6.03E−02	1.81E−02	2.87E−02	0.565	0.297	0.566	0.299
50	2	3	3.08E−06	2074	6.00E−05	3.76E−03	1.27E−03	2.91E−04	39.6	3.65E−03	5.26E−02	1.79E−02	2.79E−02	0.551	0.238	0.549	0.228
50	3	1	3.30E−06	1933	1.29E−04	5.97E−03	6.09E−04	3.14E−04	36.6	7.50E−03	6.31E−02	1.77E−02	3.00E−02	0.526	0.397	0.503	0.418
50	3	2	3.30E−06	1961	8.42E−05	4.74E−03	9.60E−04	3.08E−04	36.5	4.88E−03	5.44E−02	1.80E−02	2.90E−02	0.520	0.295	0.489	0.296
50	3	3	3.30E−06	2052	6.22E−05	4.03E−03	1.29E−03	3.08E−04	37.0	3.69E−03	4.94E−02	1.81E−02	2.91E−02	0.511	0.238	0.481	0.228
100	1	1	1.48E−06	1811	1.27E−04	5.42E−03	1.66E−04	1.47E−04	41.8	7.39E−03	7.64E−02	9.30E−03	1.47E−02	0.638	0.395	0.692	0.422
100	1	2	1.42E−06	1840	8.28E−05	3.96E−03	2.63E−04	1.40E−04	41.3	4.90E−03	6.35E−02	9.59E−03	1.39E−02	0.626	0.295	0.674	0.305
100	1	3	1.37E−06	1890	6.14E−05	2.92E−03	3.65E−04	1.36E−04	41.8	3.60E−03	5.41E−02	9.75E−03	1.34E−02	0.622	0.233	0.665	0.235
100	2	1	1.49E−06	1873	1.31E−04	5.17E−03	1.66E−04	1.47E−04	40.6	7.58E−03	7.16E−02	9.33E−03	1.46E−02	0.609	0.397	0.654	0.425
100	2	2	1.48E−06	1849	8.55E−05	3.69E−03	2.65E−04	1.45E−04	39.9	4.91E−03	5.84E−02	9.63E−03	1.42E−02	0.599	0.294	0.632	0.304
100	2	3	1.41E−06	1943	5.93E−05	2.90E−03	3.60E−04	1.38E−04	40.5	3.57E−03	5.09E−02	9.70E−03	1.36E−02	0.592	0.233	0.620	0.235
100	3	1	1.56E−06	1825	1.29E−04	4.91E−03	1.64E−04	1.52E−04	37.8	7.52E−03	6.37E−02	9.30E−03	1.48E−02	0.568	0.399	0.588	0.427
100	3	2	1.50E−06	1880	8.58E−05	3.71E−03	2.68E−04	1.45E−04	38.4	4.91E−03	5.46E−02	9.66E−03	1.42E−02	0.569	0.293	0.582	0.303
100	3	3	1.44E−06	1927	5.60E−05	2.90E−03	3.62E−04	1.41E−04	38.6	3.48E−03	4.78E−02	9.70E−03	1.38E−02	0.562	0.233	0.573	0.235
100	5	5	1.48E−06	1900	3.18E−05	1.99E−03	5.43E−04	1.42E−04	33.3	2.20E−03	3.26E−02	9.62E−03	1.37E−02	0.471	0.166	0.435	0.159
100	10	10	1.73E−06	1383	1.42E−05	1.87E−03	9.71E−04	1.63E−04	21.0	1.16E−03	2.08E−02	9.24E−03	1.55E−02	0.297	0.100	0.219	0.085
500	1	1	3.31E−07	1618	1.29E−04	4.29E−03	7.33E−06	3.33E−05	40.9	7.53E−03	6.91E−02	1.98E−03	3.34E−03	0.668	0.399	0.744	0.434
500	1	2	3.16E−07	1653	9.40E−05	3.14E−03	1.21E−05	3.18E−05	41.2	5.16E−03	5.85E−02	2.09E−03	3.20E−03	0.665	0.294	0.744	0.314
500	1	3	2.87E−07	1692	6.45E−05	2.20E−03	1.63E−05	2.87E−05	41.4	3.70E−03	4.98E−02	2.11E−03	2.87E−03	0.661	0.234	0.739	0.246
500	2	1	3.24E−07	1649	1.26E−04	3.85E−03	7.28E−06	3.25E−05	40.8	7.43E−03	6.54E−02	1.97E−03	3.26E−03	0.659	0.398	0.742	0.433
500	2	2	3.05E−07	1672	8.97E−05	2.75E−03	1.19E−05	3.05E−05	40.6	5.09E−03	5.46E−02	2.08E−03	3.05E−03	0.650	0.296	0.731	0.315
500	2	3	2.80E−07	1695	5.94E−05	2.07E−03	1.63E−05	2.81E−05	40.8	3.57E−03	4.76E−02	2.11E−03	2.83E−03	0.649	0.233	0.725	0.245
500	3	1	3.47E−07	1660	1.32E−04	4.23E−03	7.51E−06	3.49E−05	40.1	7.59E−03	6.60E−02	2.00E−03	3.49E−03	0.648	0.397	0.723	0.432
500	3	2	2.96E−07	1703	9.17E−05	2.70E−03	1.18E−05	2.97E−05	39.9	5.12E−03	5.29E−02	2.08E−03	2.98E−03	0.636	0.298	0.708	0.318
500	3	3	2.89E−07	1786	6.28E−05	2.12E−03	1.64E−05	2.89E−05	40.7	3.64E−03	4.68E−02	2.11E−03	2.90E−03	0.633	0.234	0.705	0.246
500	5	5	2.67E−07	1764	3.45E−05	1.22E−03	2.48E−05	2.67E−05	38.5	2.25E−03	3.38E−02	2.12E−03	2.67E−03	0.594	0.164	0.645	0.169
500	10	10	2.57E−07	1818	1.41E−05	5.26E−04	4.57E−05	2.56E−05	33.6	1.12E−03	1.75E−02	2.11E−03	2.55E−03	0.488	0.094	0.469	0.093

(AIC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC), and consistent Akaike information criterion (CAIC), defined as

$$AIC = 2k - 2\ln(lik), \qquad HQIC = 2k\ln(\ln(n)) - 2\ln(lik),$$
$$BIC = k\ln(n) - 2\ln(lik), \qquad CAIC = \frac{2kn}{n-k-1} - 2\ln(lik),$$

where k is the number of parameters and lik denotes the likelihood function. The AIC, BIC, HQIC, and CAIC for the basic model and the 3-CME model are computed, to investigate the performance of the models when lower and upper outliers exist in the sample.

Tables 9 and 10 show the results under parameters settings 1 and 2, respectively. For all aforementioned discrimination measures, a model with lower values preforms better than the others. For each sample size and the number of lower and upper outliers, it is clear that the AIC, BIC, HQIC, and CAIC for the 3-CME model are smaller than the basic model (2). It shows, as expected, that the 3-CME model provides the better fit to the

Table 8
The posterior risks for the SELF, PLF, and DLF under setting 2.

n	$\sum_{i=1}^{n}\omega_i$	$\sum_{i=1}^{n}\nu_i$	SELF					PLF					DLF				
			$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{\tau}$
20	1	1	1.71E−03	3491	3.33E−03	8.01E−03	6.40E−03	1.38E−02	52.9	4.22E−02	8.08E−02	6.64E−02	1.10E−01	0.574	0.436	0.574	0.518
20	1	2	1.87E−03	3518	2.23E−03	7.38E−03	7.94E−03	1.47E−02	52.9	2.94E−02	7.65E−02	6.07E−02	1.15E−01	0.567	0.351	0.570	0.383
20	1	3	1.83E−03	3596	1.69E−03	6.84E−03	9.55E−03	1.48E−02	53.4	2.30E−02	7.41E−02	5.72E−02	1.19E−01	0.566	0.297	0.577	0.300
20	2	1	2.27E−03	3614	3.20E−03	8.46E−03	7.05E−03	1.72E−02	48.3	4.05E−02	7.18E−02	6.86E−02	1.25E−01	0.504	0.426	0.469	0.505
20	2	2	2.31E−03	3617	2.12E−03	7.89E−03	8.24E−03	1.69E−02	48.3	2.82E−02	6.92E−02	6.07E−02	1.23E−01	0.501	0.344	0.470	0.372
20	2	3	2.34E−03	3759	1.62E−03	7.22E−03	9.48E−03	1.71E−02	49.0	2.23E−02	6.63E−02	5.64E−02	1.26E−01	0.497	0.296	0.474	0.294
20	3	1	3.05E−03	3258	3.26E−03	9.47E−03	7.38E−03	2.12E−02	41.6	4.08E−02	6.56E−02	7.05E−02	1.42E−01	0.440	0.425	0.375	0.506
20	3	2	3.09E−03	3412	2.06E−03	8.91E−03	8.56E−03	2.07E−02	42.4	2.76E−02	6.36E−02	6.12E−02	1.37E−01	0.437	0.342	0.377	0.368
20	3	3	2.79E−03	3628	1.56E−03	8.17E−03	9.40E−03	1.91E−02	43.5	2.19E−02	6.13E−02	5.58E−02	1.32E−01	0.432	0.297	0.379	0.292
25	1	1	1.33E−03	3330	3.42E−03	7.19E−03	5.20E−03	1.09E−02	52.5	4.23E−02	7.90E−02	6.00E−02	8.78E−02	0.584	0.430	0.597	0.523
25	1	2	1.29E−03	3442	2.30E−03	6.18E−03	6.21E−03	1.06E−02	53.1	2.94E−02	7.19E−02	5.42E−02	8.77E−02	0.579	0.342	0.591	0.390
25	1	3	1.35E−03	3483	1.67E−03	5.76E−03	7.17E−03	1.11E−02	53.1	2.24E−02	6.82E−02	5.03E−02	9.14E−02	0.572	0.291	0.583	0.309
25	2	1	1.88E−03	3350	3.49E−03	7.36E−03	5.89E−03	1.43E−02	47.9	4.27E−02	6.93E−02	6.42E−02	1.05E−01	0.519	0.429	0.494	0.529
25	2	2	1.72E−03	3443	2.23E−03	6.66E−03	6.53E−03	1.31E−02	48.3	2.87E−02	6.51E−02	5.54E−02	9.92E−02	0.516	0.342	0.491	0.390
25	2	3	1.58E−03	3698	1.68E−03	5.91E−03	7.33E−03	1.24E−02	49.6	2.25E−02	6.15E−02	5.09E−02	9.73E−02	0.511	0.289	0.492	0.309
25	3	1	2.29E−03	3290	3.45E−03	7.73E−03	6.48E−03	1.67E−02	42.4	4.19E−02	6.11E−02	6.69E−02	1.18E−01	0.449	0.420	0.394	0.522
25	3	2	2.08E−03	3341	2.32E−03	7.15E−03	6.99E−03	1.54E−02	43.0	2.91E−02	5.90E−02	5.78E−02	1.10E−01	0.451	0.338	0.400	0.389
25	3	3	1.92E−03	3545	1.57E−03	6.60E−03	7.59E−03	1.43E−02	43.9	2.13E−02	5.65E−02	5.08E−02	1.06E−01	0.448	0.282	0.399	0.297
50	1	1	6.61E−04	3160	4.02E−03	5.36E−03	3.10E−03	5.61E−03	52.8	4.57E−02	7.25E−02	4.72E−02	4.75E−02	0.606	0.429	0.646	0.567
50	1	3	5.01E−04	3254	1.85E−03	3.64E−03	3.12E−03	4.46E−03	52.7	2.27E−02	5.73E−02	3.39E−02	3.99E−02	0.595	0.277	0.626	0.332
50	2	1	6.64E−04	3226	4.02E−03	4.96E−03	3.25E−03	5.80E−03	49.4	4.57E−02	6.35E−02	4.94E−02	4.96E−02	0.556	0.428	0.572	0.571
50	2	2	6.11E−04	3312	2.63E−03	4.05E−03	3.18E−03	5.20E−03	49.4	3.07E−02	5.60E−02	3.99E−02	4.46E−02	0.546	0.333	0.557	0.424
50	2	3	6.76E−04	3291	1.84E−03	3.55E−03	3.50E−03	5.46E−03	49.5	2.23E−02	5.16E−02	3.54E−02	4.49E−02	0.548	0.273	0.554	0.331
50	3	1	1.00E−03	3088	3.92E−03	4.71E−03	3.69E−03	7.72E−03	44.7	4.46E−02	5.45E−02	5.08E−02	5.87E−02	0.502	0.422	0.481	0.562
50	3	2	7.19E−04	3298	2.53E−03	3.99E−03	3.32E−03	5.79E−03	45.6	2.99E−02	4.95E−02	4.02E−02	4.76E−02	0.497	0.329	0.476	0.418
50	3	3	6.35E−04	3389	1.68E−03	3.50E−03	3.26E−03	5.31E−03	46.1	2.13E−02	4.59E−02	3.38E−02	4.41E−02	0.494	0.271	0.471	0.324
100	1	1	2.95E−04	2982	4.44E−03	4.26E−03	1.79E−03	2.62E−03	52.4	4.87E−02	6.85E−02	3.45E−02	2.30E−02	0.625	0.437	0.693	0.598
100	1	2	2.36E−04	2975	2.95E−03	3.22E−03	1.41E−03	2.13E−03	52.0	3.32E−02	5.82E−02	2.55E−02	1.93E−02	0.616	0.343	0.678	0.452
100	1	3	2.11E−04	3043	2.12E−03	2.54E−03	1.38E−03	1.98E−03	52.2	2.41E−02	5.02E−02	2.27E−02	1.84E−02	0.612	0.274	0.662	0.354
100	2	1	3.82E−04	3019	4.50E−03	3.88E−03	2.19E−03	3.21E−03	50.5	4.86E−02	6.16E−02	3.68E−02	2.64E−02	0.593	0.434	0.644	0.598
100	2	2	2.48E−04	3141	2.90E−03	2.97E−03	1.54E−03	2.28E−03	50.6	3.23E−02	5.23E−02	2.70E−02	2.07E−02	0.581	0.337	0.625	0.447
100	2	3	2.49E−04	3226	2.17E−03	2.50E−03	1.43E−03	2.22E−03	50.9	2.48E−02	4.68E−02	2.30E−02	1.97E−02	0.577	0.281	0.613	0.363
100	3	1	4.48E−04	3086	4.48E−03	3.39E−03	2.28E−03	3.49E−03	48.0	4.82E−02	5.36E−02	3.72E−02	2.76E−02	0.552	0.429	0.584	0.594
100	3	2	2.82E−04	3081	2.88E−03	2.72E−03	1.62E−03	2.47E−03	48.1	3.22E−02	4.69E−02	2.78E−02	2.16E−02	0.552	0.336	0.576	0.447
100	3	3	3.13E−04	3175	2.10E−03	2.29E−03	1.65E−03	2.56E−03	48.2	2.39E−02	4.20E−02	2.38E−02	2.13E−02	0.542	0.276	0.559	0.358
100	5	5	2.19E−04	3111	1.12E−03	1.67E−03	1.36E−03	2.02E−03	41.4	1.43E−02	2.90E−02	1.83E−02	1.86E−02	0.448	0.193	0.415	0.238
100	10	10	2.45E−04	2330	4.36E−04	1.62E−03	1.65E−03	2.26E−03	25.8	6.81E−03	1.83E−02	1.44E−02	2.09E−02	0.271	0.114	0.197	0.124
500	1	1	1.01E−04	2700	5.58E−03	2.69E−03	9.01E−04	7.11E−04	51.8	5.53E−02	5.56E−02	1.44E−02	5.23E−03	0.655	0.448	0.752	0.648
500	1	2	1.06E−04	2740	3.97E−03	2.23E−03	5.85E−04	5.62E−04	52.0	3.93E−02	4.95E−02	1.15E−02	4.04E−03	0.651	0.359	0.743	0.515
500	1	3	3.74E−05	2722	2.65E−03	1.70E−03	3.64E−04	3.59E−04	51.9	2.75E−02	4.29E−02	8.23E−03	3.43E−03	0.649	0.290	0.738	0.405
500	2	1	8.34E−05	2772	5.34E−03	2.58E−03	8.41E−04	6.14E−04	51.8	5.29E−02	5.31E−02	1.42E−02	4.77E−03	0.643	0.441	0.735	0.634
500	2	2	4.20E−05	2742	3.85E−03	2.00E−03	3.76E−04	3.87E−04	51.1	3.79E−02	4.71E−02	1.09E−02	3.58E−03	0.638	0.353	0.728	0.510
500	2	3	3.64E−05	2786	2.63E−03	1.45E−03	2.89E−04	3.46E−04	50.8	2.70E−02	3.89E−02	8.00E−03	3.29E−03	0.626	0.286	0.711	0.405
500	3	1	8.50E−05	2708	5.80E−03	2.45E−03	9.50E−04	6.56E−04	50.1	5.64E−02	5.09E−02	1.55E−02	5.19E−03	0.627	0.448	0.713	0.656
500	3	2	3.92E−05	2741	3.76E−03	1.99E−03	4.04E−04	3.76E−04	49.9	3.79E−02	4.51E−02	1.04E−02	3.61E−03	0.623	0.356	0.708	0.508
500	3	3	3.24E−05	2771	2.47E−03	1.43E−03	2.00E−04	3.18E−04	50.0	2.61E−02	3.75E−02	7.33E−03	3.12E−03	0.618	0.282	0.694	0.395
500	5	5	3.24E−05	2919	1.33E−03	8.07E−04	1.63E−04	3.16E−04	48.2	1.55E−02	2.67E−02	5.49E−03	3.08E−03	0.572	0.202	0.624	0.270
500	10	10	2.79E−05	2895	4.92E−04	3.53E−04	9.77E−05	2.76E−04	40.2	7.33E−03	1.39E−02	3.71E−03	2.73E−03	0.457	0.118	0.441	0.145

simulated data. Thus, we prefer the 3-CME model for modeling exponential sample when lower and upper outliers exist in the sample.

4. Real example analysis

In here, two examples of an actual insurance claim datasets were considered for illustrative purposes. In both analyses, it is assumed that $N = 100\,000$ with a burn-in period of $20\,000$.

4.1. Example 1: mvi data

In the first explanation, we select a sample of size 2000 from the motor vehicle insurance (mvi) data available in the R package "**gamlss.data**". The mvi data, initially studied by [12], contain 67 143 observations related to the motor vehicle insurance policies from an insurance company over 12 months period in 2004–05. We exploit the Chi-Squared goodness of fit test to see how well the ordinary exponential distribution and our proposed 3-CME

Table 9
The AIC, BIC, HQIC and CAIC for the basic model (2) and the 3-CME model under presumed parameters $\alpha = 0.01, \theta = 40, \beta = 0.01$.

n	$\sum_{i=1}^{n} \omega_i$	$\sum_{i=1}^{n} \nu_i$	The basic model (2)				The 3-CME model			
			AIC	BIC	HQIC	CAIC	AIC	BIC	HQIC	CAIC
20	1	1	286.314	287.309	286.508	286.536	246.242	251.220	238.436	250.528
20	1	2	312.757	313.753	312.952	312.980	259.336	264.314	251.530	263.622
20	1	3	330.315	331.311	330.510	330.538	271.977	276.956	264.172	276.264
20	2	1	283.633	284.628	283.826	283.854	241.683	246.662	233.878	245.970
20	2	2	312.605	313.600	312.798	312.826	254.761	259.740	246.956	259.048
20	2	3	330.391	331.387	330.584	330.612	268.114	273.093	260.308	272.400
20	3	1	284.135	285.131	284.330	284.358	237.479	242.458	229.672	241.764
20	3	2	313.332	314.327	313.526	313.554	251.138	256.117	243.332	255.424
20	3	3	329.030	330.026	329.224	329.252	263.105	268.084	255.298	267.390
25	1	1	348.092	349.311	348.430	348.266	302.956	309.051	295.294	306.114
25	1	2	381.282	382.501	381.620	381.456	316.730	322.824	309.068	319.888
25	1	3	403.197	404.415	403.534	403.370	329.238	335.333	321.576	332.396
25	2	1	346.587	347.806	346.924	346.760	298.568	304.663	290.906	301.726
25	2	2	382.839	384.058	383.178	383.014	312.907	319.001	305.244	316.064
25	2	3	402.895	404.114	403.234	403.070	325.422	331.516	317.760	328.580
25	3	1	345.310	346.528	345.648	345.484	294.933	301.027	287.270	298.090
25	3	2	380.680	381.898	381.018	380.854	308.526	314.621	300.864	311.684
25	3	3	403.097	404.316	403.434	403.270	321.195	327.289	313.532	324.352
50	1	1	655.861	657.773	656.588	655.943	584.635	594.195	577.362	585.998
50	1	2	705.699	707.611	706.428	705.783	600.275	609.835	593.002	601.638
50	1	3	742.787	744.699	743.516	742.871	613.654	623.215	606.382	615.018
50	2	1	653.550	655.462	654.278	653.633	582.241	591.801	574.968	583.604
50	2	2	707.461	709.373	708.190	707.545	596.667	606.227	589.394	598.030
50	2	3	742.501	744.413	743.228	742.583	611.077	620.637	603.806	612.442
50	3	1	651.391	653.303	652.118	651.473	578.776	588.336	571.504	580.140
50	3	2	707.144	709.056	707.872	707.227	593.027	602.587	585.754	594.390
50	3	3	741.379	743.291	742.106	741.461	607.412	616.973	600.140	608.776
100	1	1	1243.840	1246.450	1244.898	1243.885	1148.000	1161.020	1141.052	1148.636
100	1	2	1320.520	1323.120	1321.570	1320.557	1162.230	1175.250	1155.280	1162.864
100	1	3	1382.770	1385.380	1383.826	1382.813	1179.790	1192.810	1172.842	1180.426
100	2	1	1239.010	1241.620	1240.068	1239.055	1145.980	1159.010	1139.036	1146.620
100	2	2	1321.440	1324.040	1322.490	1321.477	1160.020	1173.050	1153.078	1160.662
100	2	3	1381.210	1383.820	1382.266	1381.253	1177.070	1190.100	1170.124	1177.708
100	3	1	1236.290	1238.890	1237.340	1236.327	1139.780	1152.810	1132.834	1140.418
100	3	2	1319.160	1321.770	1320.214	1319.201	1156.230	1169.260	1149.284	1156.868
100	3	3	1383.870	1386.470	1384.924	1383.911	1173.140	1186.170	1166.194	1173.778
100	5	5	1466.590	1469.200	1467.648	1466.635	1196.040	1209.060	1189.090	1196.674
100	10	10	1592.890	1595.490	1593.942	1592.929	1247.120	1260.140	1240.172	1247.756
500	1	1	5777.550	5781.760	5779.214	5777.568	5636.950	5658.020	5630.594	5637.061
500	1	2	5914.830	5919.040	5916.474	5914.828	5656.560	5677.630	5650.214	5656.681
500	1	3	6047.250	6051.460	6048.894	6047.248	5672.590	5693.660	5666.234	5672.701
500	2	1	5780.420	5784.640	5782.074	5780.428	5636.630	5657.710	5630.294	5636.761
500	2	2	5915.770	5919.980	5917.414	5915.768	5652.810	5673.880	5646.454	5652.921
500	2	3	6056.360	6060.570	6058.014	6056.368	5672.150	5693.220	5665.794	5672.261
500	3	1	5774.190	5778.410	5775.854	5774.208	5632.130	5653.200	5625.774	5632.241
500	3	2	5913.890	5918.100	5915.534	5913.888	5649.640	5670.720	5643.294	5649.761
500	3	3	6049.550	6053.770	6051.214	6049.568	5669.580	5690.650	5663.234	5669.701
500	5	5	6265.700	6269.910	6267.354	6265.708	5699.310	5720.380	5692.974	5699.441
500	10	10	6673.160	6677.370	6674.814	6673.168	5772.210	5793.280	5765.874	5772.341

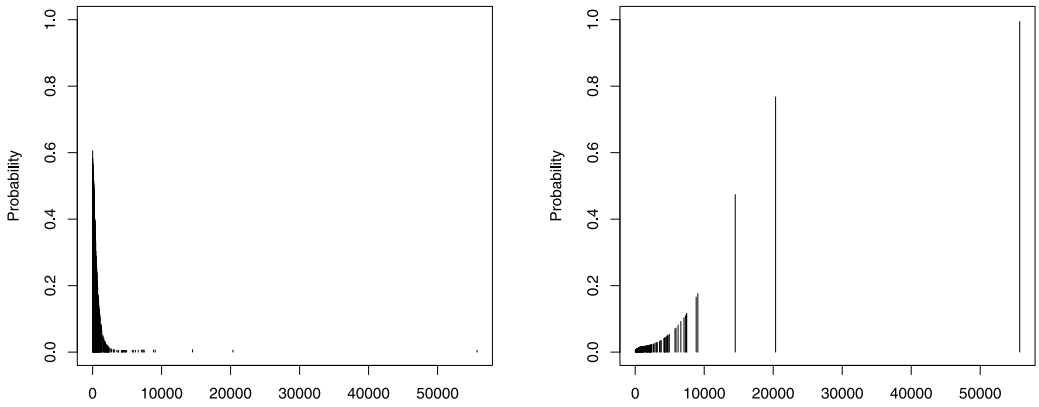
model can describe the real data example. For instance, the p-values of Chi-Squared test for mvi data are 0.2459 and $2.486E^{-10}$ for the 3-CME and exponential distributions, respectively, showing that the mvi data do follow the

Table 10
The AIC, BIC, HQIC and CAIC for the basic model (2) and the 3-CME model under presumed parameters $\alpha = 0.1, \theta = 50, \beta = 0.05$.

n	$\sum_{i=1}^{n} \omega_i$	$\sum_{i=1}^{n} \nu_i$	The basic model (2)				The 3-CME model			
			AIC	BIC	HQIC	CAIC	AIC	BIC	HQIC	CAIC
20	1	1	155.520	156.516	155.714	155.742	149.336	154.314	141.530	141.558
20	1	2	172.149	173.145	172.342	172.370	158.282	163.261	150.477	150.504
20	1	3	183.215	184.211	183.410	183.438	166.893	171.872	159.088	159.116
20	2	1	154.761	155.757	154.956	154.984	145.050	150.029	137.244	137.272
20	2	2	171.764	172.760	171.958	171.986	154.257	159.236	146.452	146.480
20	2	3	182.743	183.739	182.936	182.964	162.265	167.244	154.460	154.488
20	3	1	152.479	153.475	152.674	152.702	140.018	144.997	132.212	132.240
20	3	2	170.745	171.740	170.938	170.966	149.227	154.206	141.420	141.448
20	3	3	182.000	182.996	182.194	182.222	157.247	162.226	149.442	149.470
25	1	1	189.844	191.062	190.182	190.018	182.331	188.425	174.668	174.504
25	1	2	207.590	208.809	207.928	207.764	191.766	197.861	184.104	183.940
25	1	3	221.776	222.995	222.114	221.950	200.670	206.764	193.008	192.844
25	2	1	187.993	189.212	188.332	188.168	178.356	184.450	170.694	170.530
25	2	2	207.251	208.469	207.588	207.424	187.633	193.727	179.970	179.806
25	2	3	219.951	221.170	220.290	220.126	196.853	202.947	189.192	189.028
25	3	1	186.631	187.850	186.970	186.806	174.027	180.121	166.365	166.201
25	3	2	205.579	206.798	205.916	205.752	183.698	189.793	176.020	175.856
25	3	3	221.330	222.549	221.668	221.504	193.155	199.249	185.494	185.330
50	1	1	358.500	360.412	359.228	358.583	348.781	358.341	341.510	340.865
50	1	2	383.330	385.242	384.058	383.413	359.964	369.524	352.692	352.047
50	1	2	383.330	385.242	384.058	383.413	359.964	369.524	352.692	352.047
50	1	3	403.226	405.138	403.954	403.309	370.146	379.706	362.874	362.229
50	2	1	357.714	359.626	358.442	357.797	346.134	355.694	338.862	338.217
50	2	2	381.086	382.998	381.814	381.169	356.209	365.769	348.938	348.293
50	2	3	402.198	404.110	402.926	402.281	367.002	376.562	359.730	359.085
50	3	1	355.758	357.670	356.486	355.841	341.813	351.374	334.542	333.897
50	3	2	381.910	383.822	382.638	381.993	353.710	363.271	346.438	345.793
50	3	3	400.739	402.651	401.468	400.823	361.704	371.265	354.432	353.787
100	1	1	693.080	695.685	694.134	693.121	682.068	695.094	675.122	674.109
100	1	2	720.894	723.499	721.948	720.935	692.925	705.951	685.980	684.967
100	1	3	747.366	749.971	748.420	747.407	704.190	717.216	697.244	696.231
100	2	1	692.225	694.830	693.280	692.267	679.204	692.230	672.258	671.245
100	2	2	721.342	723.947	722.396	721.383	689.647	702.672	682.700	681.687
100	2	3	744.747	747.352	745.800	744.787	700.934	713.960	693.988	692.975
100	3	1	688.900	691.505	689.954	688.941	674.917	687.943	667.972	666.959
100	3	2	719.851	722.457	720.906	719.893	687.083	700.108	680.136	679.123
100	3	3	744.339	746.944	745.392	744.379	696.833	709.859	689.888	688.875
100	5	5	784.708	787.313	785.762	784.749	711.391	724.417	704.444	703.431
100	10	10	864.518	867.123	865.572	864.559	739.488	752.514	732.542	731.529
500	1	1	3332.020	3336.230	3333.674	3332.028	3323.800	3344.870	3317.454	3315.808
500	1	2	3373.570	3377.780	3375.214	3373.568	3342.610	3363.680	3336.274	3334.628
500	1	3	3409.460	3413.670	3411.114	3409.468	3354.790	3375.860	3348.434	3346.788
500	2	1	3339.840	3344.050	3341.494	3339.848	3326.320	3347.390	3319.974	3318.328
500	2	2	3370.780	3374.990	3372.434	3370.788	3336.460	3357.530	3330.114	3328.468
500	2	3	3406.140	3410.360	3407.794	3406.148	3350.320	3371.390	3343.974	3342.328
500	3	1	3330.930	3335.140	3332.594	3330.948	3321.550	3342.620	3315.194	3313.548
500	3	2	3367.320	3371.540	3368.974	3367.328	3333.400	3354.470	3327.054	3325.408
500	3	3	3407.600	3411.820	3409.254	3407.608	3349.350	3370.430	3343.014	3341.368
500	5	5	3471.620	3475.830	3473.274	3471.628	3368.740	3389.810	3362.394	3360.748
500	10	10	3612.830	3617.040	3614.494	3612.848	3414.840	3435.910	3408.494	3406.848

Table 11
Estimation results for the mvi data.

Parameter	Estimate	95% CI	CI length
α	0.000472	(0.000308, 0.000801)	0.000496
θ	9.7862	(1.8679, 28.3961)	26.5282
β	0.090099	(0.012928, 0.217629)	0.204701
ρ	0.210908	(0.001888, 0.433967)	0.432079
τ	0.043524	(0.003061, 0.172449)	0.169388



(a) Posterior probability that each observation is a lower outlier (b) Posterior probability that each observation is an upper outlier

Fig. 4. The posterior probabilities that each observation is a lower and upper outliers, corresponding to the mvi dataset.

3-CME distribution at the 5% level of significance. Recently, [31] used the mvi data to illustrate the performance of the contaminated exponential (CE) distribution via the Bayesian approach with the assumption of the presence of upper outliers. By fitting the basic model (2) and the 3-CME model to the dataset, we see that the 3-CME model with AIC = 2294.9, BIC = 2309.4, HQIC = 2300.8 and CAIC = 2295.4 outperforms the ordinary exponential model (2) with AIC = 2338.6, BIC = 2341.4, HQIC = 2339.7 and CAIC = 2338.6. Also based on these discrimination measures, it is clear that the 3-CME model preforms better than the CE model to fit on the mvi data. This motivates us to use the 3-CME model, and statistical methodology with the assumption of the presence of multiple lower and upper outliers.

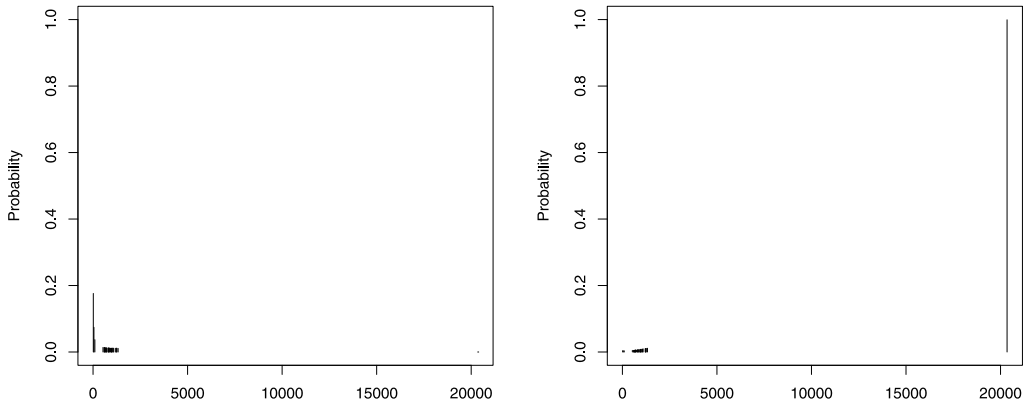
To use the presented Bayesian method, let us consider $a_1 = b_1 = d_1 = 2/3$, $q_1 = t_1 = 0.5$, and $q_2 = t_2 = 3$ and computed the hyperparameter values based on the same strategy which is studied in Section 2.1. Therefore, we see the hyperparameter values $a_2 = 1286$, $b_2 = 13.14$, and $d_2 = 0.02$ for the mvi data. Table 11 shows the parameter estimates, 95% credible interval (CI) of parameters and the length of CI. It can be seen that the Bayes estimates of θ and β are greater and less than one, respectively, indicating the existence of the lower and upper outliers in the mvi data. Fig. 4 demonstrates the posterior probabilities that the observation is a lower and upper outlier. From Fig. 4(a), it can be seen that the smallest observation has the highest posterior probability of being a lower outlier and the rest observations have small posterior probability of being a lower outlier. From Fig. 4(b), it is clear that the largest observation has the highest posterior probability of being an upper outlier and the rests have small posterior probability of being an upper outlier. The 3-CME model can thus identify lower and upper outliers well. These outcomes show the performance of the 3-CME model and its Bayesian analysis. Graphs of the estimated posterior density function for ρ, τ, α, θ, and β are shown in the Appendix A.

4.2. Example 2: random sample of mvi (rsmvi) data

In the second illustration, we select a random sample of size 32 from mvi data (rsmvi) in order to check how well the presented approach works in dealing with small sample size datasets.

Table 12
Estimation results for rsmvi data.

Parameter	Estimate	95% CI	CI length
α	0.001227	(0.000829, 0.001699)	0.000869
θ	50.530	(0.363, 228.979)	228.616
β	0.045556	(0.003909, 0.138051)	0.134142
ρ	0.031588	(0.000031, 0.168631)	0.168600
τ	0.048239	(0.003513, 0.148238)	0.144725

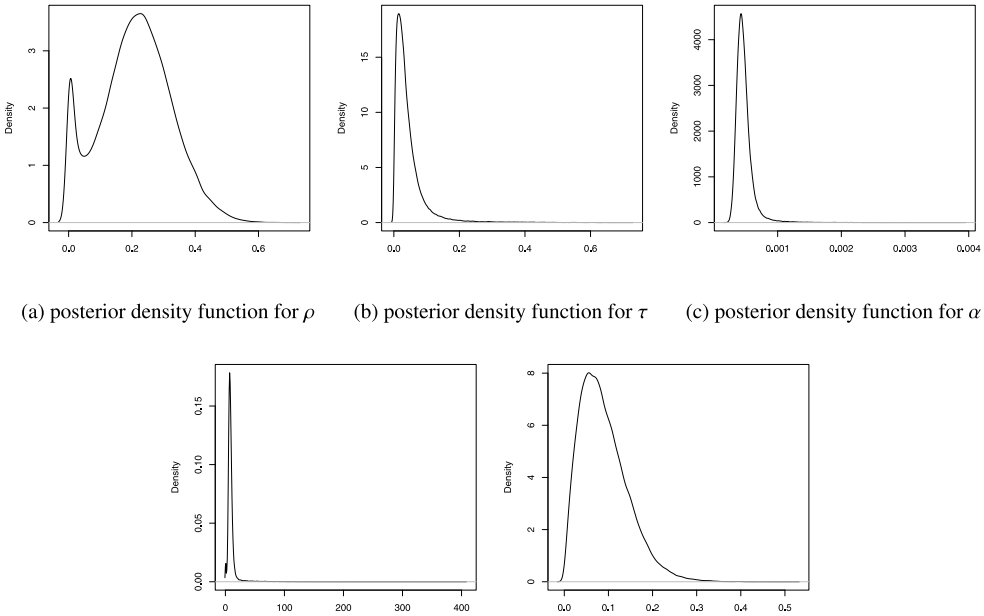


(a) Posterior probability that each observation is a lower outlier(b) Posterior probability that each observation is an upper outlier

Fig. 5. The posterior probabilities that each observation is a lower and upper outlier, corresponding to the rsmvi dataset.

The rsmvi data are: 14.42, 46.68, 96.22, 526.43, 581.6, 608.58, 625.5, 629.01, 647.75, 693.47, 695.12, 695.32, 744.2, 808.32, 816.42, 817.27, 861.45, 881.19, 920.25, 960.54, 965.54, 970.35, 995.93, 1037.78, 1065.45, 1096.27, 1208.53, 1217.273, 1223.02, 1264.362, 1325.93, 20345.1. It seems that the smallest observation (14.42) and the largest observation (20345.1) can be considered as a possible lower and upper outliers, respectively. We fit the basic model (2) and the 3-CME model to the data and see that the 3-CME model with AIC = 518.43, BIC = 525.76, HQIC = 520.86 and CAIC = 520.74 outperforms the ordinary exponential model (2) with AIC = 530.46, BIC = 531.93, HQIC = 530.95 and CAIC = 530.59.

To use the presented Bayesian method, we take $a_1 = b_1 = d_1 = 2/3$, $q_1 = t_1 = 0.5$, and $q_2 = t_2 = 3$ and computed the hyperparameter values based on the same strategy which is studied in Section 2.1. Therefore, we see the hyperparameter values $a_2 = 556$, $b_2 = 16.26$, and $d_2 = 0.012$ for the rsmvi data. By performing the proposed methodology with hyperparameter values, the Bayesian parameter estimates are calculated. Table 12 shows the parameter estimates, 95% CI of parameters and the length of CI. It can be seen that the Bayes estimates of θ and β are greater and less than one, respectively, indicating the existence of the lower and upper outliers in the rsmvi data. Fig. 5 demonstrates the posterior probabilities that the observation is a lower and upper outliers. As expected, it can be seen that from Fig. 5(a) the largest observation $x = 20345.1$ has the highest posterior probability (0.999) of being an upper outlier and the rest observations have small posterior probability of being upper outliers. Also, Fig. 5(b) shows that the smallest observation $x = 14.42.1$ has the biggest posterior probability (0.176) of being a lower outlier and the rests have small posterior probability of being lower outliers. These outcomes confirm that the 3-CME model can identify lower and upper outliers well, and highlight the performance of the 3-CME model and its Bayesian analysis for analyzing data with small sample size. Graphs of the estimated posterior density function for ρ, τ, α, θ, and β are shown in the Appendix A.



(a) posterior density function for ρ (b) posterior density function for τ (c) posterior density function for α

(d) posterior density function for θ (e) posterior density function for β.

Fig. A.6. Posterior density functions for ρ, τ, α, θ, and β corresponding to the mvi data.

5. Conclusion

In this paper, the 3-CME distributions for modeling data with the presence of multiple lower and upper outliers have been proposed. The Bayesian method has been extended for calculating parameter estimates. For any sample size, it is demonstrated that the effect of lower and upper outliers is automatically reflected in the posterior distribution. Five simulation studies have been conducted by implementing the Gibbs sampler, to show the efficient and impressive performance of the 3-CME model. We have exhibited that the presented model can be selected as an alternative model in dealing with the data with and without lower and upper outliers. In addition, we have demonstrated that the SELF is a preferable loss function for Bayesian inference based on the 3-CME model. By studying two real example, we show that the 3-CME model outperforms the exponential distribution. Outcomes of the simulation and real dataset studies show that upper outliers have the highest posterior probability of upper outlying, but the main points have a relatively small posterior probability of upper outlying. Alternatively, the lower outliers have the highest posterior probability of lower outlying, whereas the main points have a relatively small posterior probability of lower outlying. It indicates that the 3-CME model can identify multiple lower and upper outliers very well. As recommended by the reviewer, it could be interesting to extend our current approach by considering Gamma, Weibull, and Birnbaum–Saunders [25] distributions in follow-up research as well as for order data [23,24]. It could be also interesting to point out that the analysis of reliability characteristics including reliability functions and stress strength reliability for the 3-CME is an open avenue for future work [17].

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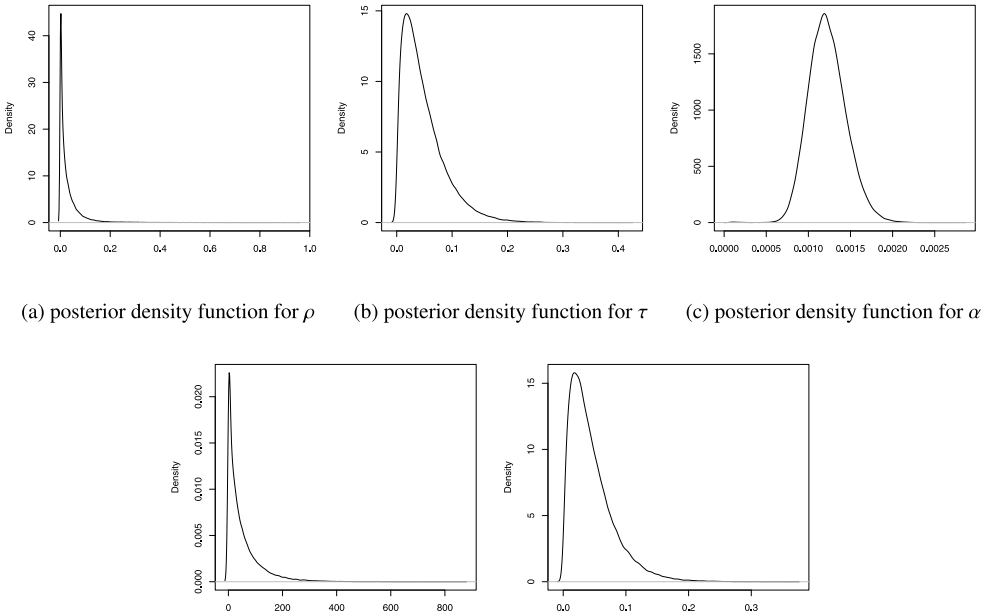
Appendix A

Graphs of the estimated posterior density function for ρ, τ, α, θ and β are shown in Figs. A.6 and A.7 for mvi and rsmvi datasets, respectively.



(a) posterior density function for ρ (b) posterior density function for τ (c) posterior density function for α

(d) posterior density function for θ (e) posterior density function for β.

Fig. A.7. Posterior density functions for ρ, τ, α, θ, and β corresponding to the rsmvi data.

Appendix B. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.matcom.2023.01.037.

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