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Robust yield test for a normal production process

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ABSTRACT
Testing the performance of a production process is a very serious and important topic in statistical quality control. This article presents a robust yield test to investigate the performance of an industrial production process in the presence of outliers. For this purpose, a new robust estimator of S_{pk} is introduced to test the production yield for any normal distribution in the presence of various numbers of outliers. Moreover, a Monte Carlo simulation method to estimate the decision-making components is proposed for testing the production yield based on the yield index S_{pk} by normal data. Meanwhile, this article discusses how well the proposed Monte Carlo method can be used for some non-normal data. Numerical computations of the simulation and real data analyses are provided to explain the proposed method.

KEYWORDS
yield analysis; testing capability; Monte Carlo simulation; outliers; robustness

1. Introduction

Process capability indices (PCIs) have been extensively applied in the industrial processes to evaluate whether the quality of products meets the preset specification limits (SLs). The two most popular indices C_p and C_{pk} are introduced as

$$C_p = \frac{USL - LSL}{6\sigma},$$

and

$$C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\},$$

where LSL is the lower SL, USL is the upper SL, μ is the process mean, and σ is the process standard deviation; see Kane (1986). The PCI C_p measures the process variation related to the preset SLs. The PCI C_{pk} considers the magnitudes of the process variation as well as the degree of the process centering, which measures the performance of the process based on the production yield. For situations with two-sided specifications in the process capability analysis, $Yield = P(LSL \leq X \leq USL)$ is a process yield, in which X is a process quality characteristic. If X is normally distributed, then the process yield can be exactly defined as $Yield = \Phi[(USL-\mu)/\sigma] - \Phi[(LSL-\mu)/\sigma]$, where Φ is the cumulative distribution function of the standard normal distribution; see Pearn and Chuang (2004). Moreover, for a normally distributed process with the fixed C_{pk}, the bounds on the process yield can be obtained as (see Boyles, 1991) Q2

$$2\Phi(3C_{pk}) - 1 \leq Yield \leq \Phi(3C_{pk}).$$

It must be noted that the PCI C_{pk} presents a proximate measurement rather than a precise measurement of the process yield. To acquire a precise measurement, Boyles (1994) proposed the PCI

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-\mu}{\sigma}\right) + \frac{1}{2}\Phi\left(\frac{\mu - LSL}{\sigma}\right)\right\}, \quad (1)$$

for a normal process, in which Φ^{-1} represents the inverse function of Φ. It is obvious that there is a one-to-one relationship between the PCI S_{pk} and the production yield, which can be given by

$$Yield = 2\Phi(3S_{pk}) - 1. \quad (2)$$

For the normal process, the expected number of nonconformities corresponding to a capable process with $S_{pk} = 1.00$ will be 2700 parts per million. Therefore, the conventional PCI S_{pk} is widely proposed in the normal industrial production processes to present a precise measurement of the process yield. The exactitude of the natural estimator (NE) of S_{pk}

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was checked in Pearn and Chuang (2004) based on the simulation method to acquire the relative mean square error and relative bias for several quality requirements. Pearn, Wang, and Yen (2006) developed a generalized yield index, and they obtained a lower confidence bound for the production yield. The relationship between the distribution of the overall index S_{pk}^{T} and process parameters was investigated in Pearn and Cheng (2010). A fuzzy-based procedure was presented in Afshari, Sadeghpour Gildeh, and Ahmadi Nadi (2020) to evaluate the overall yield with the presence of a determined degree of vagueness.

In general, using PCIs is necessary to measure how much of the process productions meet the preset capability requirement. Since outliers mask the attributable causes of the variation, one may face unreliable results on using the PCIs. Robust time series methods were considered in Prasad and Bramorski (1998) to define new sets of PCIs for a wide range of industrial processes. The second author (Jabbari Nooghabi, 2020) introduced the PCIs based on the parametric model of outliers. The motivation for using a Monte Carlo (MC) technique was discussed in Parchami, Iranmanesh, and Sadeghpour Gildeh (2022) to test the quality of a manufacturing process based on Yongting's index. Iranmanesh, Parchami, and Sadeghpour Gildeh (2022) proposed the statistical fuzzy quality test for analyzing the manufacturing process based on fuzzy SLs. The MC simulation method was presented in Iranmanesh, Parchami, and Jabbari Nooghabi (2023) to test the quality of a production process based on the popular capability index C_{pk}. However, the presence of outliers might have a serious effect on statistical analyses and decisions in testing the performance of the production process. Therefore, the Bootstrap technique for statistical inferences was applied in Naya et al. (2016) to investigate the capability of the process in the presence of outliers.

The aim of the present study is to introduce a new robust estimator of S_{pk} to test the process performance based on the production yield. For this aim, the MC technique is used to estimate the components of the decision-making for testing the process yield based on the PCI S_{pk}. A simulation study as well as a real data example is provided to show the performance of testing the production yield. This article is structured as follows. Section 2 introduces the non-robust and robust estimators of S_{pk}. Section 3 provides a comparison study between the proposed robust estimator and different estimators of S_{pk}. Section 4 incorporates hypotheses testing for analyzing the performance of the production process on the basis of the proposed robust estimator of S_{pk}. Section 5 proposes the MC simulation procedure to estimate the decision-making components in testing the production yield based on the yield index S_{pk}. The proposed MC simulation method is also extended based on some non-normal processes. Numerical and simulation results are presented in Section 6. Section 7 compares the proposed method with different methods. Finally, conclusions and future works are detailed.

2. Non-robust and robust estimators

During the last three decades, the statistical term "Outlier" has been discussed in the literature. An outlier is an observation in the distribution of data that swerves so much from the other observed data points (Jabbari Nooghabi, 2020). When a dataset is contaminated with a single or few outliers, we face a serious problem in parameter estimations. In this regard, robust estimation methods can be very applicable to investigate the yield/capability of the process.

Generally, two simple robust estimators of location and scale parameters are the median and the median absolute deviation (MAD), respectively. The MAD is one of the alternatives of the robust estimator of the standard deviation. Therefore, Hampel (1974) defined the MAD as

$$MAD = \underset{i}{median}|X_i - M|, \tag{3}$$

where M is the sample median, which is used as a robust estimator of μ. In this study, $1.4826 \times MAD$, which is commonly called the standardized MAD, is considered as a consistent robust estimator of σ at the normal distribution (Rousseeuw and Croux 1993).

Also, the simpler but less robust estimator of the scale parameter is the interquartile range (IQR), that can be computed with the IQR function.

Definition 1. Let IQR be the difference between the 75th and 25th percentiles of the sample data ($IQR = Q_3 - Q_1$). Then, an IQR-based threshold is defined as follows:

$$\begin{cases} T_{\min} = Q_1 - 1.5 \times IQR, \\ T_{\max} = Q_3 + 1.5 \times IQR, \end{cases} \tag{4}$$

where T_{min} and T_{max} are the minimum and maximum threshold for detecting the outliers. Generally, the data point which lies outside of the interval $[T_{\min}, T_{\max}]$, is considered as an outlier (Yang, Rahardja, and Fränti 2019).

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In order to estimate the yield index S_{pk}, one can consider the following definition as a non-robust estimator of S_{pk} (see Lee et al. 2002).

Definition 2. Let $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/(n-1)$ be the sample mean and the sample standard deviation, respectively. Then, the NE of S_{pk} is defined as

$$\hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL - \bar{X}}{S}\right) + \frac{1}{2}\Phi\left(\frac{\bar{X} - LSL}{S}\right)\right\}, \quad (5)$$

where LSL and USL are the SLs, as expressed in Section 1.

In what follows, by inspiration of the provided ideas in Besseris (2014) and Aslam et al. (2019), in order to estimate the yield index S_{pk}, we define two new robust estimators of S_{pk}, which are useful in the presence of outliers.

Definition 3. Let M and IQR be the sample median and the sample IQR, respectively. Then, the IQR-based estimator (IE) of S_{pk} is introduced as

$$\hat{S}^I_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL - M}{IQR/3}\right) + \frac{1}{2}\Phi\left(\frac{M - LSL}{IQR/3}\right)\right\}, \quad (6)$$

where the superscript "I" denotes that the IQR of the observations is used as a robust estimator of 3σ.

Definition 4. Let M and $1.4826 \times MAD$ be the sample median and the sample standardized MAD, respectively. Then, an MAD-based estimator (ME) of S_{pk}, which is used as the robust estimator of S_{pk}, is introduced in the following:

$$\hat{S}^r_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL - M}{1.4826 \times MAD}\right) + \frac{1}{2}\Phi\left(\frac{M - LSL}{1.4826 \times MAD}\right)\right\}, \quad (7)$$

where the superscript "r" denotes that the standardized MAD of the observations is used as a more robust estimator of σ.

It is obvious that the variance of M can be computed by $Var(M) = \frac{\pi\sigma^2}{2n}$ at normal distributions, in which the number π ($\pi \approx 3.14159$) is a mathematical constant that is the ratio of a circle's circumference to its diameter (see more details about the asymptotic distribution of the sample median in Ferguson, 1996). By inspiration of the presented idea in Bellio and Ventura (2005), in order to estimate an approximate $(1-\alpha)100\%$ confidence interval for the median of the population, one can consider the following interval:

$$[L_1, L_2] = M \pm z_{1-\alpha/2}\sqrt{\frac{\pi(1.4826 \times MAD)^2}{2n}}, \quad (8)$$

where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$th quantile of the standard normal distribution. It should be noted that the median of a normal distribution is equivalent to its mean. Therefore, several different possible points from the above interval are considered to apply a robust confidence interval for the location parameter μ, for implementing the steps of the MC simulation in Section 5.

3. A comparison study between the non-robust and robust estimation methods

In this section, we are going to a comparison between NE, IE, and ME statistics based on the mean square error (MSE) and the mean absolute deviation (MAD_{mean}) criterions. In order to investigate the robustness of different estimation methods against outliers, we consider m random samples, each of size n, from the normal distribution with various numbers of outliers (n_p). The MSE and MAD_{mean} of the non-robust and robust estimators of S_{pk} for various numbers of outliers are presented in Table 1 and Figure 1. It should be mentioned that the MSE and MAD_{mean} by using m replications, are calculated as

$$MSE = \frac{1}{m}\sum_{i=1}^{m}(\theta - \hat{\theta}_i)^2, \quad (9)$$

and

$$MAD_{mean} = \frac{1}{m}\sum_{i=1}^{m}|\theta - \hat{\theta}_i|, \quad (10)$$

where $\hat{\theta}_i$ is the estimator of θ for $i = 1, \ldots, m$. In order to compute the presented MSE and MAD_{mean} in Table 1 and Figure 1, we perform Algorithm 1 to generate the sample data sets based on considering $m = 1000$, $LSL = 0.0$, $USL = 10.0$, $\mu = 5$, $\sigma = 4$, $\delta = 20$, and $\gamma = 3$ for different sample sizes, $n = 24(44)156$, with various numbers of outliers ($n_p = 0$, 1, 2, 3, and 4). Note that, by inspiration of Jabbari Nooghabi (2020), we consider that the outliers are occurred from the same distribution but with different parameters in this simulation method.

Algorithm 1 Simulation method for calculating MSE and MAD_{mean} based on generating m samples each of size n with n_p outliers

Require:

(1) $n \geq 1$, $n_p \geq 0$, $m \geq 1$, $\sigma, \gamma > 0$, $\mu, \delta, LSL, USL \in \mathbb{R}$.
(2) Probability density for the random variable X.
(3) Probability density for the random outlier.

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Table 1. Results of the *MSE* and MAD_{mean} of the PCI S_{pk}, for different estimators (NE, IE, and ME) with various numbers of outliers based on the preset SLs [0.0, 10.0].

Parameters	Sample size	Number of outliers	Non-robust method Natural estimator (NE)	Robust method IQR-based estimator (IE)	Robust method MAD-based estimator (ME)
$(\mu, \sigma) = (5, 4)$	n	n_p	MSE (MAD_{mean})	MSE (MAD_{mean})	MSE (MAD_{mean})
Result of simulation based on a normal distribution	24	0	0.004 (.050)	0.321 (0.510)	0.012 (0.082)
		1	0.019 (0.126)	0.277 (0.471)	0.011 (0.081)
		2	0.034 (0.180)	0.236 (0.427)	0.010 (0.079)
		3	0.047 (0.214)	0.158 (0.346)	0.010 (0.084)
		4	0.057 (0.238)	0.103 (0.267)	0.014 (0.100)
	68	0	0.001 (0.028)	0.279 (0.510)	0.004 (0.048)
		1	0.005 (0.064)	0.263 (0.495)	0.004 (0.047)
		2	0.012 (0.101)	0.243 (0.476)	0.003 (0.045)
		3	0.018 (0.131)	0.220 (0.450)	0.003 (0.048)
		4	0.025 (0.156)	0.206 (0.435)	0.004 (0.050)
	112	0	0.001 (0.022)	0.273 (0.512)	0.002 (0.037)
		1	0.003 (0.044)	0.260 (0.499)	0.002 (0.038)
		2	0.006 (0.072)	0.243 (0.482)	0.002 (0.036)
		3	0.010 (0.098)	0.240 (0.480)	0.002 (0.037)
		4	0.014 (0.117)	0.230 (0.469)	0.002 (0.037)
	156	0	0.001 (0.019)	0.264 (0.506)	0.002 (0.031)
		1	0.002 (0.034)	0.259 (0.501)	0.002 (0.031)
		2	0.004 (0.056)	0.252 (0.495)	0.002 (0.031)
		3	0.007 (0.077)	0.244 (0.486)	0.001 (0.031)
		4	0.010 (0.094)	0.238 (0.481)	0.002 (0.031)

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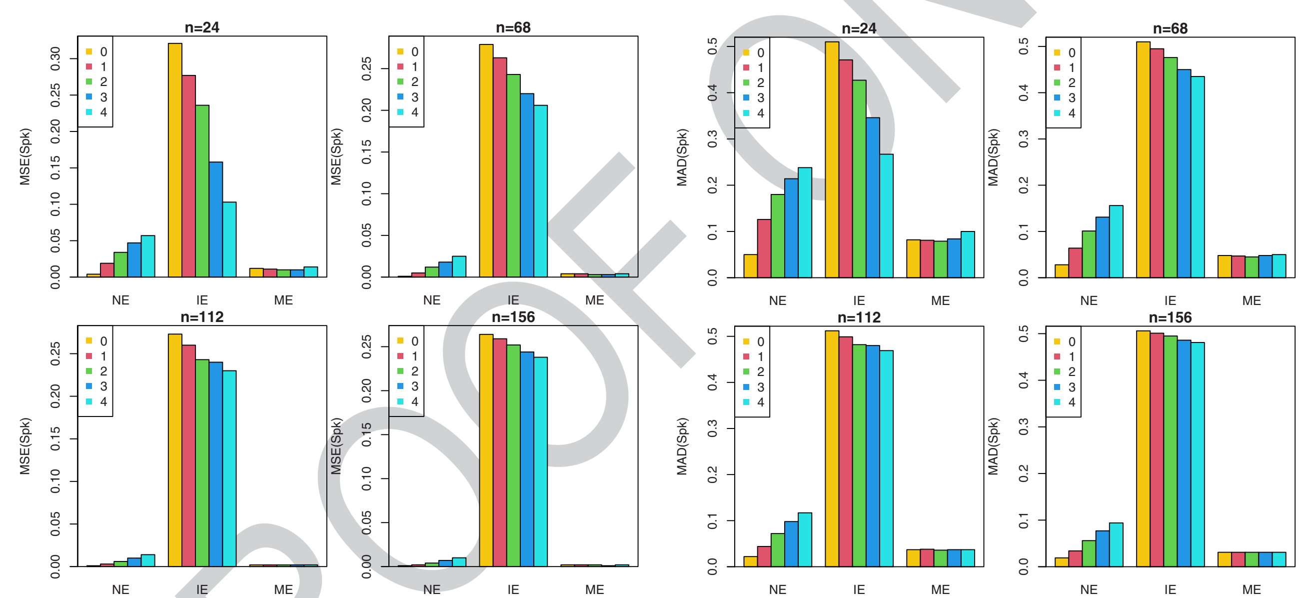

Figure 1. The *MSE* (left graphs) and the MAD_{mean} (right graphs) of the PCI S_{pk} for different estimators (NE, IE, and ME) with various numbers of outliers (n_p = 0, 1, 2, 3, and 4).

Ensure: *MSE* and MAD_{mean} of three estimators $\hat{S}_{pk}, \hat{S}^I_{pk}$, and $\hat{S}^r_{pk}$ (NE, IE, and ME).

for $i = 1$ to m **do**

Generate independently $X_{1,i}, ..., X_{n-n_p,i} \sim N(\mu, \sigma^2)$.

Generate independently $X_{n-n_p+1,i}, ..., X_{n,i} \sim N(\mu + \delta, \gamma\sigma^2)$.

Combine two samples $(X_{1,i}, ..., X_{n-n_p,i})$ and $(X_{n-n_p+1,i}, ..., X_{n,i})$,

to achieve $\{X_{1,i}, ..., X_{n-n_p,i}, X_{n-n_p+1,i}, ..., X_{n,i}\}$.

Compute $\hat{S}_{pk,i}$ based on the sample data set $\{X_{1,i}, ..., X_{n,i}\}$ by Eq. (5).

Compute $\hat{S}^I_{pk,i}$ based on the sample data set $\{X_{1,i}, ..., X_{n,i}\}$ by Eq. (6).

Compute $\hat{S}^r_{pk,i}$ based on the sample data set $\{X_{1,i}, ..., X_{n,i}\}$ by Eq. (7).

end for

Calculate S_{pk} from Eq. (1).

1. Calculate *MSE* of three estimators $\hat{S}_{pk}, \hat{S}^I_{pk}$, and $\hat{S}^r_{pk}$ by Eq. (9).
2. Calculate MAD_{mean} of three estimators $\hat{S}_{pk}, \hat{S}^I_{pk}$, and $\hat{S}^r_{pk}$ by Eq. (10).

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return MSE and MAD_{mean} of three estimators $\hat{S}_{pk}, \hat{S}^{I}_{pk}$, and $\hat{S}^{r}_{pk}$ (NE, IE, and ME).

For instance, in the last column of Table 1, for $n=24$ with one outlier ($n_p = 1$), we generate $m=1000$ samples, each of size $n-n_p=24-1=23$ from the normal distribution $N(\mu=5, \sigma^2=4^2)$ ($X_{1,i}, \dots, X_{23,i} \overset{i.i.d.}{\sim} N(5, 4^2)$ for each $i=1,\dots,1000$), and also for having one outlier in the sample data set, we generate 1000 samples, each of size one from the normal distribution with different parameters ($X_{24,i} \sim N(\mu+20, 3\sigma^2)$ for each $i=1,\dots,1000$). Consequently, by performing Algorithm 1 on the basis of using the ME of S_{pk} ($\hat{S}^{r}_{pk}$) based on the sample data $\{X_{1,i}, \dots, X_{23,i}, X_{24,i}\}$ for $i=1,\dots,1000$, the MSE and MAD_{mean} of $\hat{S}^{r}_{pk,i}$ are computed by the following formulas in this simulation study:

$$MSE = \frac{1}{1000}\sum_{i=1}^{1000}(S_{pk} - \hat{S}^{r}_{pk,i})^2 = 0.011,$$

and

$$MAD_{mean} = \frac{1}{1000}\sum_{i=1}^{1000}|S_{pk} - \hat{S}^{r}_{pk,i}| = 0.081,$$

where S_{pk} is calculated by

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{10-5}{4}\right) + \frac{1}{2}\Phi\left(\frac{5-0}{4}\right)\right\} = 0.417.$$

The results of Table 1 and Figure 1 showed that the sample size has a significant impact on both MSE and MAD_{mean} for the non-robust and robust estimation methods (NE, IE, and ME). By comparing the results of these methods, one can observe that the best result is on the basis of using the ME of S_{pk} ($\hat{S}^{r}_{pk}$), since different numbers of outliers have the least impact on both MSE and MAD_{mean} for this estimator. Hence, with regard to the values of both MSE and MAD_{mean} in this simulation study, one can conclude that the proposed ME of S_{pk} is a more robust estimator to the outliers.

Exclusively, the robust yield test on the basis of using the proposed estimator $\hat{S}^{r}_{pk}$, is presented in the next section. The proposed test is useful to make a reliable decision on evaluating the performance of the normal process, which can be affected by outliers.

4. Decision-making components in testing the production yield

Throughout this study, it is supposed that the quality characteristic is approximately normally distributed and the process is in a state of statistical control. To determine whether the production process is capable or not, one can consider the following hypotheses test:

$$\begin{cases} H_0 : Yield \le c_0 & (process\ is\ incapable), \\ H_1 : Yield > c_0 & (process\ is\ capable), \end{cases} \quad (11)$$

in which the designed constant c_0 is the standard minimal criterion for the process yield. It must be mentioned that the designed constant c_0 is a stringent requirement for the possible mean shift or the variation change which is determined by the producer in the factory/company. In many cases, a minimal criterion 99.73% ($c_0 = 0.9973$) for evaluating the process is suggested. According to the existence a one-to-one relationship between the production yield and the PCI S_{pk} (see Eq. (2)), testing the above hypotheses is equivalent to testing

$$\begin{cases} H_0 : S_{pk} \le s_0, \\ H_1 : S_{pk} > s_0, \end{cases} \quad (12)$$

where $s_0 = \frac{1}{3}\Phi^{-1}\left(\frac{c_0+1}{2}\right)$ is the standard minimal criterion for the PCI S_{pk}, which is determined based on the designed constant c_0.

For making a decision, one can use the components of the decision-making—such as the p value, the critical value, the probability of type II error, and the power function of the robust yield test—to assess the production yield based on the PCI S_{pk}. Herein, we intend to consider the following test function:

$$\phi(x_1, x_2, \dots, x_n) = \begin{cases} 1, & \hat{s}^{r}_{pk} > s, \\ 0, & otherwise, \end{cases} \quad (13)$$

where s is the critical value in testing hypotheses (Iranmanesh, Parchami, and Sadeghpour Gildeh 2022). Suppose the probability that an incapable process is mistakenly considered capable, is α (the significant level of the proposed robust yield test). Then, $\phi(x_1, x_2, \dots, x_n)$ rejects the null hypothesis H_0 ($S_{pk} \le s_0$) if $\hat{s}^{r}_{pk} > s$ (i.e., the process is capable). Also,

$$\alpha = P\left(\hat{S}^{r}_{pk} > s | S_{pk} = s_0\right), \quad (14)$$

is the probability of type I error. Hence, we can write

$$1 - \alpha = P\left(\hat{S}^{r}_{pk} \le s | S_{pk} = s_0\right). \quad (15)$$

Therefore, the critical value of this robust yield test is the $(1-\alpha)$ th quantile of $\hat{S}^{r}_{pk}$ distribution such that

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$S_{pk} = s_0$. On the other hand, the p value of this robust yield test is given by

$$p - value = P\left(\hat{S}^r_{pk} > \hat{s}^r_{pk} | S_{pk} = s_0\right) = E\left[I\left(\hat{S}^r_{pk} > \hat{s}^r_{pk} | S_{pk} = s_0\right)\right], \quad (16)$$

where $\hat{s}^r_{pk}$ is the observed ME of S_{pk} on the basis of observations $x_1, ..., x_n$, which is computed by Eq. (7), while the indicator function of an event A is indicated by $I(A)$. Also, the probability of type II error can be calculated as

$$\beta(s^*_{pk}) = P\left(\hat{S}^r_{pk} \leq s | S_{pk} = s^*_{pk}\right) = E\left[I\left(\hat{S}^r_{pk} \leq s \,|\, S_{pk} = s^*_{pk}\right)\right], \quad (17)$$

for any arbitrary point $s^*_{pk} > s_0$. Hence, for every $S_{pk} > 0$

$$\Pi(S_{pk}) = 1 - \beta(S_{pk}) = P\left(\hat{S}^r_{pk} > s | S_{pk}\right) = E\left[I\left(\hat{S}^r_{pk} > s \,|\, S_{pk}\right)\right], \quad (18)$$

is the power function of the proposed robust yield test based on the yield index S_{pk}.

5. Monte Carlo simulation method

In this section, we intend to propose the MC simulation method (MC method) based on the proposed robust estimator $\hat{S}^r_{pk}$ for estimating the decision-making components.

5.1. Methodology

To simulate the components of the decision-making for the statistical testing with null hypothesis $H_0 : Yield \leq c_0$ versus alternative hypothesis $H_1 : Yield > c_0$, at the considered significance level, α, we propose the following procedure as the MC-method:

Step 1. Test the null hypothesis $H_0 : S_{pk} \leq s_0$ versus alternative hypothesis $H_1 : S_{pk} > s_0$, which is equivalent for testing $H_0 : Yield \leq c_0$ versus $H_1 : Yield > c_0$, to specify whether a production process is capable or not. Then, based on the existence of a one-to-one relationship between the production yield and the index S_{pk}, compute the standard minimal criterion s_0 by substituting c_0 and s_0 to Eq. (2) as

$$s_0 = \frac{1}{3}\Phi^{-1}\left(\frac{c_0 + 1}{2}\right), \quad (19)$$

where c_0 is the designed constant for testing the process yield.

Step 2. By Eq. (7), calculate the observed value of the robust estimator $\hat{s}^r_{pk}$ based on the observations $x_1, ..., x_n$.

Step 3. Compute the sequence $\{\mu_1, \mu_2, ..., \mu_h\}$ to cover several different possible points from the interval $[L_1, L_2]$ by the formula

$$\mu_j = L_1 + \frac{j-1}{h-1}(L_2 - L_1), j = 1, 2, ..., h, \quad (20)$$

where L_1 and L_2 are computable by Eq. (8) based on the observations $x_1, x_2, ..., x_n$. The positive natural number h specifies the number of the μ_js that can be selected according to the sensitivity of the applicant. It must be noted that the greater the spread of the observed data, the greater h is recommended in the simulation.

Step 4. Follow the below (i)–(vi) parts for $j = 1, ..., h$:

(i) calculate the unknown root σ_j on the basis of μ_j, from

$$\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL - \mu_j}{\sigma_j}\right) + \frac{1}{2}\Phi\left(\frac{\mu_j - LSL}{\sigma_j}\right)\right\} = s_0, \quad (21)$$

(ii) considering $k = 10^3$, simulate k random samples with size n from $N(\mu_j, \sigma^2_j)$,

(iii) by Eq. (7), estimate $\hat{s}^{r[1]}_{pk,j}, \hat{s}^{r[2]}_{pk,j}, ..., \hat{s}^{r[k]}_{pk,j}$ for every simulated sample in Part (ii),

(iv) according to Eq. (3), the critical value based on 1000 simulated samples is equivalent to the $(1-\alpha)$ th quantile of $\hat{S}^r_{pk}$ distribution,

$$s_j = \hat{s}^{r(\lceil k(1-\alpha)\rceil)}_{pk,j}, \quad (22)$$

where $\hat{s}^{r(1)}_{pk,j}, \hat{s}^{r(2)}_{pk,j}, ..., \hat{s}^{r(k)}_{pk,j}$ are the ordered indices from Part (iii), and $\lceil t \rceil$ is the smallest integer greater than or equal to t,

i. the simulated p value is equal to

$$p - value_j = \frac{1}{k}\sum_{l=1}^{k} I\left(\hat{s}^{r[l]}_{pk,j} > \hat{s}^r_{pk} \,|\, \mu = \mu_j, \sigma = \sigma_j\right), \quad (23)$$

where the estimated indices are indicated by $\hat{s}^{r[1]}_{pk,j}, ..., \hat{s}^{r[k]}_{pk,j}$ and σ_j is the obtainable root of the equation $S_{pk} = s_0$ from Part (i),

i. the simulated β for every point $s^*_{pk} > s_0$, is

$$\beta_j(s^*_{pk}) = \frac{1}{k}\sum_{l=1}^{k} I\left(\hat{s}^{r*[l]}_{pk,j} \leq s_j \,|\, \mu = \mu_j, \sigma = \sigma^*_j\right), \quad (24)$$

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693 694 695 696 697 698 699 700 701 702 703 704 705 706 707 708 709 710 711 712 713 714 715 716 717 718 719 720 721 722 723 724 725 726 727 728 729 730 731 732 733 734 735 736 737 738 739 740 741 742 743 744 745 746 747 748 749 750

where σ_j^* is the obtainable root from the following equation

$$\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL-\mu_j}{\sigma_j^*}\right)+\frac{1}{2}\Phi\left(\frac{\mu_j-LSL}{\sigma_j^*}\right)\right\}=s_{pk}^*. \tag{25}$$

Likewise, the estimated indices based on μ_j and σ_j^* are indicated by $\hat{s}_{pk,j}^{r*[1]},\ldots,\hat{s}_{pk,j}^{r*[k]}$.

Step 5. The MC critical value of this proposed robust yield test is equivalent to

$$s_{MC}=\frac{1}{h}\sum_{j=1}^{h}s_j. \tag{26}$$

Step 6. The null hypotheses in Eqs. (11) and (12) are rejected at the significance level of α, if $\hat{s}_{pk}^r > s_{MC}$ (i.e., the process is capable); otherwise, the process is incapable.

Step 7. The MC p value in the proposed robust yield test is equivalent to

$$p-value_{MC}=\frac{1}{h}\sum_{j=1}^{h}p-value_j. \tag{27}$$

Step 8. The MC probability of type II error for any point $s_{pk}^* > s_0$ can be simulated in the proposed robust yield test by the total average of h calculated probability of type II errors at the fixed point s_{pk}^* in repetitions of Part (vi), that is,

$$\beta_{MC}(s_{pk}^*)=\frac{1}{h}\sum_{j=1}^{h}\beta_j(s_{pk}^*). \tag{28}$$

Remark 1 Note that the power function of the robust yield test in the proposed MC-method for every $S_{pk} > 0$ is equal to

$$\Pi_{MC}(S_{pk})=1-\beta_{MC}(S_{pk}). \tag{29}$$

5.2. Generalization of the robust yield test based on some non-normal processes

Generally, the yield index S_{pk} is applied based on the normal distribution of process output. Also, it has a one-to-one relation with the production yield. Typically, the first idea for non-normality assumptions is using the surrogate PCIs based on Clements's percentile method (Clements, 1989) for non-normal production processes. But based on the lack of the existence of a one-to-one relationship between the proposed Clements's PCIs and the production yield, we prefer to transform non-normal data by Box–Cox method (Box and Cox, 1964) which can be useful to normalize some non-normal variables. Given a random variable X from any distribution with only positive values, we consider the following family of the transformation:

$$Y=\begin{cases}\dfrac{X^{\lambda}-1}{\lambda}, & \lambda\neq 0,\\ \log(X), & \lambda=0,\end{cases} \tag{30}$$

where $\lambda\in\mathbb{R}$ is the Box–Cox transformation parameter. Box and Cox (1964) presented the maximum likelihood and Bayesian methods to estimate the parameter λ.

Remark 2 Note that the methodology of the MC-method is proposed for the normal distribution of the process output. Under the non-normality assumption, if the Box–Cox transformation method is able to transform non-normal data into normal data, then by considering the following points one can generalize the MC-method:

1. The generalization idea is used for the family of distributions with only positive values such as the log-normal, weibull, gamma, and the generalized exponential distribution.
2. Box–Cox transformation method is applied to transform the original data $(x_1,\ldots,x_n)$ into normal data. Hence, one can use each type of goodness-of-fit tests (Shapiro–Wilk test, Lilliefors test, and Robust Jarque–Bera test of normality) to confirm the normality assumption of the transformed observations $(y_1,\ldots,y_n)$.
3. After confirming that the normal distribution model is suitable to fit the transformed data, the observed value of robust estimator $\hat{s}_{pk}^r$ can be calculated based on the transformed observations $(y_1,\ldots,y_n)$ for implementing the steps of the simulation in the MC-method.

The following section presents the obtained results corresponding to the application of the robust yield test using a real dataset.

6. Illustrative example and numerical results

6.1. Numerical results

Capacitors are invaluable parts of electronic devices—such as laptops, biomedical devices, and satellites—that have duties to retain the voltage at a certain level. Capacitors are used for reducing the voltage pulsation. An aluminum foil is an important part that affects the quality of capacitors. The voltage is the most

751 752 753 754 755 756 757 758 759 760 761 762 763 764 765 766 767 768 769 770 771 772 773 774 775 776 777 778 779 780 781 782 783 784 785 786 787 788 789 790 791 792 793 794 795 796 797 798 799 800 801 802 803 804 805 806 807 808

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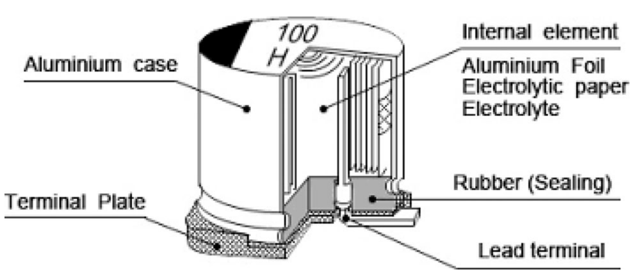

Figure 2. Graph of the components of a capacitor.

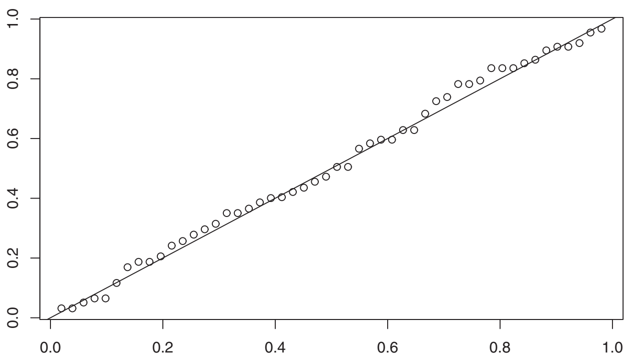

Figure 3. P-P plot for 50 observed voltages of aluminum foils.

significant quality characteristic of the aluminum foil for any capacitor in an electronic device. Figure 2 indicates the details of a capacitor. Fifty random samples are taken from data of voltages for aluminum foils (in terms of a volt) (Tong and Chen 2003). The lower and upper SLs of the voltages of aluminum foils (VAFs) are considered as $[LSL, USL] = [510, 530]$.

Now, we are going to test $H_0 : Yield \leq 0.9973$ versus $H_1 : Yield > 0.9973$ based on the observed random sample $x_1, x_2, ..., x_{50}$, under the significance level of $\alpha = 0.01$. According to Eq. (19), the standard minimal criterion for the index S_{pk} can be computed as $s_0 = 1.00$. Shapiro–Wilk test confirms fitting the normal distribution on the data with $W = 0.9901$ and p value $= 0.9493$. Also, the Lilliefors (corrected Kolmogorov–Smirnov) normality test with $D = 0.0639$ and p value $= 0.8774$, shows that the VAFs variable follows the normal distribution. Moreover, using the Robust Jarque–Bera test of normality, a normal distribution is fitted on the data (X-squared $= 0.4154$, df $= 2$, p value $= 0.8124$). The empirical cumulative distribution function of the data versus theoretical ones (normal P-P (probability–probability) plot) is shown in Figure 3. Therefore, Box–Cox transformation method is not required to transform the VAFs observations into normality. Whenever the alternative hypothesis is that "the lowest value 514.7 is an outlier", the result of the chi-square test for the detection of the outlier shows that one outlier exists. Furthermore, on the basis of the box plot and the histogram, observations are contaminated with one outlier (Figure 4).

Meanwhile, the value of the observed median is 522.55, and the observed standardized MAD is equal to 2.82. Hence, the robust estimator of S_{pk} according to Eq. (7) is

$$\hat{s}^{r}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{USL - 522.55}{2.82}\right) + \frac{1}{2}\Phi\left(\frac{522.55 - LSL}{2.82}\right)\right\} = 0.957.$$

Therefore, $0.957 < s_{MC}$ is the MC critical region of the test based on the PCI S_{pk} at the significance level of 0.01, where the critical value s_{MC} should be simulated based on the MC-method. It must be noted that this proposed method, which is presented in Section 5, is performed based on the observed robust estimator $\hat{s}^{r}_{pk}$, where the mean is changed over the following sequence according to Eq. (20):

522.213 522.309 522.406 522.502 522.598 522.694 522.791 522.887.

The unknown root σ_j is computed in Table 2 for all $h = 8$ considered possible cases from Eq. (21). Therefore, for each case, 1000 independent random samples are simulated with size 50 from $N(\mu_j, \sigma_j^2)$. In every simulated sample, we estimate the index S_{pk} based on observations $x_1, x_2, ..., x_{50}$ according to Eq. (7). After ordering 1000 estimated indices, the 990th index is selected as the critical value, and the results are recorded in Table 2 for all eight considered cases. For instance, in the sixth row of Table 2, the following results are obtained:

- $\mu_6 = 522.694$ is the sixth considered value for μ in the simulation using Step 3 from the proposed MC-method,
- $\sigma_6 = 2.626$ is the computed value of σ for $j = 6$ using Part (i),
- $s_6 = 1.519$ is the simulated critical value using Part (iv),
- p value $_6 = 0.657$ is the simulated p value using Part (v), and
- $\beta_6(1.98) = 0.026$ is the simulated β at the fixed point 1.98 using Part (vi) of Step 4 from the MC-method.

Monte Carlo critical values of the considered yield test are determined in Table 2 for eight possible values of the process mean to make a decision on the robust yield test in this study. Moreover, the last row

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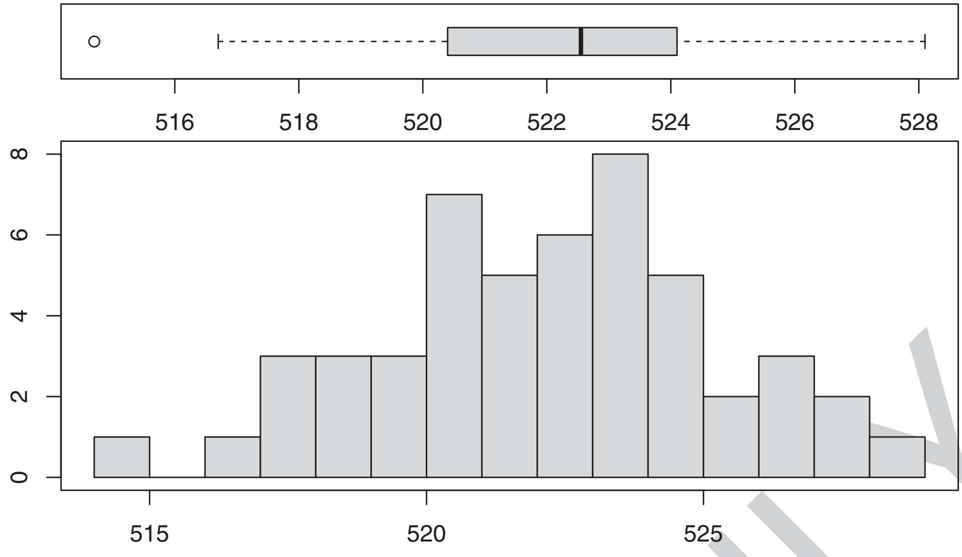

Figure 4. Box plot and histogram for 50 observed voltages of aluminum foils.

Table 2. Results of simulated components of the decision-making for all repetitions of the proposed MC-method, based on VAFs data for the considered SLs [510, 530], at the significance level $\alpha = 0.01$.

j	μ_j	σ_j	s_j	p value $_j$	$\beta(s^*_{pk})_j$
1	522.213	2.798	1.475	0.648	0.019
2	522.309	2.764	1.447	0.653	0.004
3	522.406	2.729	1.530	0.647	0.034
4	522.502	2.695	1.543	0.648	0.037
5	522.598	2.660	1.511	0.660	0.024
6	522.694	2.626	1.519	0.657	0.026
7	522.791	2.591	1.541	0.675	0.030
8	522.887	2.557	1.528	0.669	0.024
Total average			$s_{MC} = 1.512$	$p - value_{MC} = 0.657$	$\beta_{MC}(1.98) = 0.025$

of this table contains the MC critical value, the MC p value, and the MC probability of type II at the fixed point 1.98, respectively. Consequently, in evaluating based on the robust estimator $\hat{S}^r_{pk}$, the average of eight obtained critical values is equivalent to $s_{MC} = 1.512$, which is considered the MC critical value for testing the quality of the production process. Hence, with regard to the preset SLs in this illustrative example, the null hypothesis $H_0 : S_{pk} < s_0$ is not rejected at the significance level of $\alpha = 0.01$, because $\hat{s}^r_{pk} = 0.957 < s_{MC}$, therefore, one can conclude that this process cannot meet the capability requirement at the significance level of 0.01.

6.2. Simulation results

For the purpose of investigating whether outliers can affect decisions in a robust yield test, we present the numerical results of the simulation method by using Algorithm 1 for generating m samples each of size n with n_p outliers. In order to compare the original data (VAFs observations) and the simulated data in testing the production yield, we intend to use the mean and standard deviation of the original observed data (VAFs data) for estimating the mean and standard deviation parameters of the normal distribution, respectively. These two values are obtained $\bar{x} = \sum_{i=1}^{50} x_i/50 = 522.172$ and $s_{n-1} = \sqrt{\sum_{i=1}^{50} (x_i - \bar{x})^2/(50-1)} = 2.974$ from VAFs data, respectively. Firstly, allow us to use Algorithm 1 for generating the sample data sets based on considering $m = 1, LSL = 510, USL = 530, \mu = 522.172, \sigma = 2.974, \delta = 16$, and $\gamma = 3$ for the sample size $n = 50$, with various numbers of outliers ($n_p = 0, 1$, and 2). Under the desired quality condition with $c_0 = 0.9973$ ($s_0 = 1.00$), and $\alpha = 0.01$, the power function of the proposed test $\Pi_{MC}(S_{pk})$ on the basis of the following considered samples is presented in Table 3 for the considered points $S_{pk} = 1.25(0.01)1.86$:

a. The original data (VAFs data, in which the value of the outlier is 514.700).
b. The simulated sample data without outliers,
c. The simulated sample data with the presence of one outlier (the value of the outlier is 537.341).

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Table 3. Results of the simulated MC power function of the proposed test $\Pi_{MC}(S_{pk})$ on the basis of four considered samples in parts (a)–(d) (A comparison between the VAFs data and the simulated samples with the presence of zero, one, and two outliers), under the desired quality condition with $LSL = 510, USL = 530$, $c_0 = 0.9973$ ($s_0 = 1.00$), and $\alpha = 0.01$.

	S_{pk}	(a)	(b)	(c)	(d)	S_{pk}	(a)	(b)	(c)	(d)
Result of	1.25	0.1549	0.1511	0.1547	0.1550	1.56	0.6121	0.6036	0.6115	0.6124
simulation based	1.26	0.1649	0.1598	0.1648	0.1650	1.57	0.6271	0.6181	0.6269	0.6272
on the	1.27	0.1769	0.1715	0.1766	0.1770	1.58	0.6442	0.6360	0.6440	0.6446
considered	1.28	0.1881	0.1830	0.1879	0.1883	1.59	0.6585	0.6508	0.6583	0.6589
normal	1.29	0.2000	0.1940	0.1999	0.2001	1.60	0.6729	0.6647	0.6727	0.6734
distribution	1.30	0.2156	0.2082	0.2155	0.2157	1.61	0.6865	0.6790	0.6861	0.6867
$N(\mu =$	1.31	0.2276	0.2228	0.2276	0.2276	1.62	0.7024	0.6935	0.7015	0.7026
$522.172, \sigma^2 =$	1.32	0.2415	0.2364	0.2414	0.2415	1.63	0.7184	0.7080	0.7180	0.7184
$2.974^2)$	1.33	0.2557	0.2487	0.2555	0.2557	1.64	0.7330	0.7241	0.7329	0.7330
	1.34	0.2712	0.2651	0.2711	0.2712	1.65	0.7474	0.7399	0.7471	0.7476
	1.35	0.2866	0.2795	0.2865	0.2866	1.66	0.7591	0.7525	0.7591	0.7591
	1.36	0.2999	0.2936	0.2998	0.3000	1.67	0.7699	0.7640	0.7696	0.7701
	1.37	0.3146	0.3073	0.3146	0.3146	1.68	0.7820	0.7758	0.7819	0.7820
	1.38	0.3301	0.3212	0.3297	0.3302	1.69	0.7954	0.7883	0.7953	0.7955
	1.39	0.3454	0.3396	0.3453	0.3455	1.70	0.8093	0.8025	0.8090	0.8094
	1.40	0.3624	0.3541	0.3618	0.3626	1.71	0.8211	0.8140	0.8206	0.8213
	1.41	0.3777	0.3710	0.3774	0.3777	1.72	0.8301	0.8249	0.8297	0.8304
	1.42	0.3940	0.3885	0.3937	0.3941	1.73	0.8404	0.8353	0.8401	0.8404
	1.43	0.4108	0.4034	0.4104	0.4110	1.74	0.8499	0.8446	0.8495	0.8499
	1.44	0.4273	0.4201	0.4271	0.4274	1.75	0.8571	0.8519	0.8571	0.8575
	1.45	0.4441	0.4345	0.4439	0.4442	1.76	0.8664	0.8609	0.8661	0.8665
	1.46	0.4615	0.4532	0.4611	0.4617	1.77	0.8755	0.8700	0.8752	0.8756
	1.47	0.4779	0.4704	0.4775	0.4781	1.78	0.8838	0.8790	0.8835	0.8838
	1.48	0.4935	0.4846	0.4931	0.4939	1.79	0.8886	0.8855	0.8886	0.8886
	1.49	0.5115	0.5019	0.5109	0.5118	1.80	0.8959	0.8924	0.8956	0.8959
	1.50	0.5308	0.5210	0.5305	0.5310	1.81	0.9021	0.8984	0.9020	0.9022
	1.51	0.5484	0.5400	0.5483	0.5487	1.82	0.9079	0.9044	0.9077	0.9081
	1.52	0.5657	0.5563	0.5654	0.5660	1.83	0.9140	0.9100	0.9139	0.9140
	1.53	0.5826	0.5745	0.5820	0.5827	1.84	0.9213	0.9183	0.9213	0.9213
	1.54	0.5971	0.5894	0.5971	0.5974	1.85	0.9263	0.9231	0.9260	0.9263
	1.55	0.6121	0.6036	0.6115	0.6124	1.86	0.9315	0.9293	0.9314	0.9316

d. The simulated sample data with the presence of two outliers (the values of outliers are 536.284 and 538.055).

It should be mentioned that the values of outliers in the above parts (a)–(d) are detected according to the threshold in Eq. (4).

The results of Table 3 showed that the presence of outliers in the VAFs data and the simulated data do not have a significant impact on changing the power function of the proposed test $\Pi_{MC}(S_{pk})$ on the basis of the considered samples in parts (a)–(d). For example, in the first row of Table 3, for the considered point $S_{pk} = 1.25$ based on the considered samples in parts (a)–(d), the values of the power function of these four quality tests are approximately equivalent (i.e., because the values of the power function of the proposed test $\Pi_{MC}(S_{pk})$ by changing in the number of outliers close approximately together for each considered point S_{pk}, one can conclude that our proposed method for testing the performance of the production process in the illustrative example (VAFs observations in subsection 6.1) is a robust method).

Furthermore, in order to show that the presence of outliers often does not have a huge impact on making the decision in the proposed robust yield test, we use Algorithm 1 to generate the sample data sets based on considering $m = 200, LSL = 510, USL = 530, \mu = 522.172, \sigma = 2.974$, $\delta = 16$, and $\gamma = 3$ for different sample sizes, n =16, 20, 24, 30, 35, 50, 100, and 150, with various numbers of outliers ($n_p =$ 0, 1, and 2). For this purpose, we should consider $\phi_i(x_1, x_2, ..., x_n)$ as the test function that it rejects the null hypothesis $H_0 : S_{pk} \leq 1.00$ if $\hat{s}^r_{pk,i} > s_{MC,i}$ where $s_{MC,i}$ is the simulated MC critical value in testing hypotheses (Iranmanesh, Parchami, and Sadeghpour Gildeh 2022), for $i = 1, 2, ..., 200$. Therefore, the calculated percentage of rejecting the null hypothesis $H_0 : S_{pk} \leq 1.00$ on the basis of the sample data set which are simulated from the considered normal distribution $N(\mu = 522.172, \sigma^2 = 2.974^2)$ with the presence of various numbers of outliers (0, 1, and 2), are presented in Table 4 under the desired quality condition with $c_0 = 0.9973$ ($s_0 = 1.00$), and $\alpha = 0.01$.

Both Tables 3 and 4 are presented as the confirmations of the correctness and robustness of the proposed yield test with the difference that Table 3 is merely the presentation of a comparison between the VAFs observations and the considered samples in parts (b)–(d), and Table 4 are provided in a general

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1157 1158 1159 1160 1161 1162 1163 1164 1165 1166 1167 1168 1169 1170 1171 1172 1173 1174 1175 1176 1177 1178 1179 1180 1181 1182 1183 1184 1185 1186 1187 1188 1189 1190 1191 1192 1193 1194 1195 1196 1197 1198 1199 1200 1201 1202 1203 1204 1205 1206 1207 1208 1209 1210 1211 1212 1213 1214

Table 4. Results of the percentage of rejecting the null hypothesis H_0 for $m = 200$, with various numbers of outliers and sample sizes $n > 15$, under the desired quality condition with $LSL = 510, USL = 530$, $c_0 = 0.9973$ ($s_0 = 1.00$), and $\alpha = 0.01$.

	n	Number of outliers	Percentage of rejecting H_0
Result of simulation based on the normal distribution $N(522.172, 2.974^2)$	16	0	0.0%
		1	0.0%
		2	0.5%
	20	0	0.5%
		1	0.5%
		2	0.5%
	24	0	0.5%
		1	0.5%
		2	0.0%
	30	0	0.0%
		1	0.0%
		2	0.0%
	35	0	0.5%
		1	0.0%
		2	0.0%
	50	0	0.5%
		1	0.5%
		2	0.0%
	100	0	0.0%
		1	0.0%
		2	0.0%
	150	0	0.0%
		1	0.0%
		2	0.0%
	Average of percentages of rejecting H_0	0	0.25%
		1	0.19%
		2	0.13%

confirmation. Herein, in order to show that the change of our decision is not big based on changing in the number of outliers in the simulated data, the values of the percentage of rejecting the null hypothesis H_0 are presented in Table 4, and because the values of percentages by changing in the number of outliers close approximately together for each n, one can conclude that our proposed method for testing the performance of a production process is the robust method. For example, in the third row of Table 4, for the considered sample size $n = 16$ with the presence of one outlier ($n_p = 2$), the value of the percentage of rejecting the null hypothesis $H_0 : S_{pk} \leq 1.00$ based on the simulated data is 0.5% (this value is very small, i.e., one robust estimator from 200 robust estimators $\hat{S}^r_{pk,i}$ s is greater than the simulated MC critical value $s_{MC,i}$, for $i = 1, 2, ..., 200$, that is one test function from 200 test functions $\phi_i(x_1, x_2, ..., x_n)$ rejects the null hypothesis $H_0 : S_{pk} \leq 1.00$), and one can conclude that the null hypothesis $H_0 : S_{pk} < s_0$ is not rejected at the significance level of $\alpha = 0.01$. Generally, in order to show that the changes of making the decision are not huge, we added the average of percentages of rejecting the null hypothesis $H_0 : S_{pk} \leq 1.00$ in the last row of Table 4 (i.e., by changing in the number of outliers for the simulated data, the values of 0.25%, 0.19%, and 0.13% close approximately together for making a reliable decision). Hence, based on the results of Table 4, one can conclude that the null hypothesis $H_0 : S_{pk} < s_0$ is not rejected at the significance level of $\alpha = 0.01$, and also, this conclusion corresponds to the conclusion in subsection 6.1 that the production process based on the original data (VAFs observations) cannot meet the capability requirement at the significance level of 0.01. Therefore, based on these results, we confirm the correctness and robustness of our proposed yield test.

7. Comparison study between the proposed MC-method and various methods

Traditional PCIs, such as, C_p, C_{pk} and $C_{pm} = \frac{USL-LSL}{6\sqrt{\sigma^2+(\mu-T)^2}}$ have many applications in the manufacturing industry, where LSL, USL, μ, and σ are expressed in Section 1, and T is the target value. In this section, we are going to test the quality of inside diameters of piston rings with different PCIs C_p, C_{pk}, C_{pm}, and S_{pk} based on various presented methods at different significance levels. It must be noted that the piston rings data (with the detected values of outliers 73.967 and 74.030) are available by "qcc" package in R software (Scrucca 2004). Shapiro–Wilk test shows that the piston rings data follow normal distribution

1215 1216 1217 1218 1219 1220 1221 1222 1223 1224 1225 1226 1227 1228 1229 1230 1231 1232 1233 1234 1235 1236 1237 1238 1239 1240 1241 1242 1243 1244 1245 1246 1247 1248 1249 1250 1251 1252 1253 1254 1255 1256 1257 1258 1259 1260 1261 1262 1263 1264 1265 1266 1267 1268 1269 1270 1271 1272

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Table 5. Results of comparing some non-robust and robust quality tests with different PCIs based on various methods at different significance levels.

		$\alpha = 0.010$	$\alpha = 0.025$	$\alpha = 0.050$	$\alpha = 0.100$
		Non-robust quality test based on $c = 1.00$ for the preset SLs $[LSL, USL] = [73.96, 74.03]$			
Method (Pearn and Kotz 2006)	Statistical critical value for C_p	1.172	1.142	1.118	1.090
	Decision based on $\hat{c}_p = 1.1586$	Not reject H_0	Reject H_0	Reject H_0	Reject H_0
	Process	Incapable	Capable	Capable	Capable
	p value	0.015	0.015	0.015	0.015
	Probability of type *II* error	$\beta(1.344) = 0.011$	$\beta(1.289) = 0.021$	$\beta(1.239) = 0.042$	$\beta(1.179) = 0.096$
		Non-robust quality test based on $c = 1.00$, for the preset SLs $[LSL, USL] = [73.96, 74.03]$, and $T = 73.999$			
Method (Iranmanesh, Parchami, and Sadeghpour Gildeh 2022)	Monte Carlo critical value for C_{pm}	1.156	1.132	1.109	1.082
	Decision based on $\hat{c}_{pm} = 1.1324$	Not reject H_0	Reject H_0	Reject H_0	Reject H_0
	Process	Incapable	Capable	Capable	Capable
	p value$_{MC}$	0.025	0.025	0.025	0.025
	Probability of type *II* error	$\beta_{MC}(1.295) = 0.030$	$\beta(1.239) = 0.046$	$\beta_{MC}(1.223) = 0.051$	$\beta_{MC}(1.168) = 0.101$
		Non-robust quality test based on $c = 1.00$ for the preset SLs $[LSL, USL] = [73.96, 74.03]$			
Method (Pearn and Chuang 2004)	Monte Carlo critical value for C_{pk}	1.090	1.060	1.035	1.007
	Decision by $\hat{c}_{pk} = 0.954$	Not reject H_0	Not reject H_0	Not reject H_0	Not reject H_0
	Process	Incapable	Incapable	Incapable	Incapable
	p value$_{MC}$	0.689	0.689	0.689	0.689
	Probability of type *II* error	$\beta_{MC}(1.445) = 0.083$	$\beta_{MC}(1.335) = 0.093$	$\beta_{MC}(1.235) = 0.153$	$\beta_{MC}(1.135) = 0.345$
		Robust quality test based on $s_0 = 1.00$ for the preset SLs $[LSL, USL] = [73.96, 74.03]$			
MC-method	Monte Carlo critical value for S_{pk}	1.281	1.228	1.194	1.150
	Decision by $\hat{s}^r_{pk} = 1.002$	Not reject H_0	Not reject H_0	Not reject H_0	Not reject H_0
	Process	Incapable	Incapable	Incapable	Incapable
	p value$_{MC}$	0.511	0.511	0.511	0.511
	Probability of type *II* error	$\beta_{MC}(1.585) = 0.011$	$\beta_{MC}(1.485) = 0.021$	$\beta_{MC}(1.385) = 0.053$	$\beta_{MC}(1.285) = 0.110$

(p value $= 0.7861$). Based on the standard minimal criterion c for the considered PCIs C_u such that $u \in \{p, pk, pm\}$, we consider the following hypotheses

$$\begin{cases} H_0 : C_u \le c, \\ H_1 : C_u > c, \end{cases} \tag{31}$$

and the hypotheses (Iranmanesh, Parchami, and Sadeghpour Gildeh 2022) to test the quality of the production process in this comparison study. It must be clarified that the decision-making components for the proposed MC-method are compared with various presented methods (Iranmanesh, Parchami, and Sadeghpour Gildeh 2022; Pearn and Kotz 2006; Pearn and Chuang 2004) in Table 5. Therefore, four following non-robust and robust quality tests are considered based on the value of the standard minimal criterion 1.00 in Table 5 at different significance levels for inside diameters of piston rings:

1. non-robust quality test based on C_p, for the considered SLs [73.96, 74.03],
2. non-robust quality test based on C_{pm}, for the considered SLs [73.96, 74.03] and target value $T = 73.999$,
3. non-robust quality test based on C_{pk}, for the considered SLs [73.96, 74.03], and
4. robust quality test based on S_{pk}, for the considered SLs [73.96, 74.03].

For each case, the critical value, the p value, and the probability of type II error based on the considered NEs $\hat{C}_p = \frac{USL-LSL}{6S}$, $\hat{C}_{pm} = \frac{USL-LSL}{6\sqrt{S^2+(\bar{X}-T)^2}}$, $\hat{C}_{pk} = \min\left\{\frac{USL-\bar{X}}{3S}, \frac{\bar{X}-LSL}{3S}\right\}$, and the proposed robust estimator $\hat{S}^r_{pk}$ were computed and shown in Table 5 at significance levels of 0.010, 0.025, 0.05, and 0.1. It must be mentioned that the statistical critical value based on the NE $\hat{C}_p$ was calculated by $c\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}$, where c is the standard minimal criterion and $\chi^2_{n-1,\alpha}$ is the lower α-quantile of the chi-square distribution with $n-1$ degrees of freedom (Pearn and Kotz 2006). Moreover,

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the critical values based on the NEs $\hat{C}_{pm}$ and $\hat{C}_{pk}$ were simulated in Iranmanesh, Parchami, and Sadeghpour Gildeh (2022) and Pearn and Chuang (2004) on the basis of the Monte Carlo simulation methods. By comparing the results of Table 5, one can observe that both critical value and p value increase in the proposed MC-method. Therefore, the more reliable result is on the basis of using the proposed MC-method, since this procedure is based on the robust quality test and outliers cannot affect on it.

8. Conclusions and future works

The evaluation of the production process performance was investigated in this article by using the yield index S_{pk}. Testing the performance of a production process is an effective technique for making a decision on the process yield/capability. It was widely discussed that the presence of outliers might have a deleterious effect on statistical analyses and decisions. In this regard, a robust estimator of S_{pk} based on the MAD-based method was introduced to test the production yield. A comparison study was shown that the outliers have the least impact on both mean square error and mean absolute deviation for the proposed robust estimator of S_{pk}. Moreover, an MC simulation method was applied to estimate the components of the decision-making for testing the process yield based on the index S_{pk} by normal data. Meanwhile, this article discussed how well the MC-method can be used for some non-normal data. Numerical results of the simulation and real data analyses were presented to explain the proposed method. As a future study, one can develop the rule of the proposed robust yield test to assess the performance of processes with multiple characteristics.

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