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NUMERICAL RADIUS OF BOUNDED OPERATORS WITH ℓ^p-NORM

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ABSTRACT. We study the numerical radius of bounded operators on direct sum of a family of Hilbert spaces with respect to the ℓ^p-norm, where $1 \le p \le \infty$. We propose a new method which enables us to prove validity of many inequalities on numerical radius of bounded operators on Hilbert spaces when the underling space is a direct sum of Hilbert spaces with ℓ^p-norm, where $1 \le p \le 2$. We also provide an example to show that some known results on numerical radius are not true for a space that is the set of bounded operators on ℓ^p-sum of Hilbert spaces where $2 < p < \infty$. We also present some applications of our results.

1. Introduction

Let Λ be a set and $\{(H_\lambda, \langle\cdot,\cdot\rangle)\}_{\lambda\in\Lambda}$ be a family of Hilbert spaces,

$$H = \oplus_{\lambda\in\Lambda} H_\lambda \quad \text{and} \quad 1 \le p < \infty.$$

For each $x = (x_\lambda)_{\lambda\in\Lambda} \in H$, we define

$$||x||_p = \left(\sum_{\lambda\in\Lambda} \|x_\lambda\|^p\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty \quad \text{and} \quad ||x||_\infty = \sup_{\lambda\in\Lambda} \|x_\lambda||.$$

The ℓ^p-sum of $\{H_\lambda\}_{\lambda\in\Lambda}$ is defined by

$$H_p = \{x \in H : ||x||_p < \infty\}, \quad H_\infty = \{x \in H : ||x||_\infty < \infty\}.$$

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The inner product on H is defined by

$$\langle (x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda}\rangle = \sum_{\lambda\in\Lambda} \langle x_\lambda, y_\lambda\rangle_\lambda; \quad (x_\lambda), (y_\lambda) \in H_p.$$

If there is no ambiguity about the indexes, we will remove them. This inner product makes each H_p an inner product space. In special case, when $p = 2$, according to [8], H_2 is a Hilbert space. However, $(H_p, \|\cdot\|_p)$ for $p \neq 2$ is not a Hilbert space, when $|\Lambda| > 1$. For example, if $H_p = \mathbb{C} \oplus \mathbb{C}$ and $x = (1, 0)$ and $y = (0, 1)$, then $\|x\|_p = ||x||_\infty = \|y\|_p = ||y||_\infty = 1$, and

$$\|x + y\|_p = \|x - y\|_p = (|1|^p + |1|^p)^{\frac{1}{p}} = 2^{\frac{1}{p}} \quad \text{and} \quad \|x + y\|_\infty = \|x - y\|_\infty = 2.$$

Hence the parallelogram equation does not hold.

Following [5], numerical range and radius of bounded operators initiated by Stone in his book in [9]. This topic has many applications in various branches of mathematics and physics such as functional analysis, matrix norms, inequalities, numerical analysis, perturbation theory, matrix polynomials, systems theory, quantum physics, etc. The interested reader can refer to [1]–[7] for further information.

In the next section, we will define and study numerical radius of bounded operators on H_p for $1 \le p \le \infty$. We will develop a method which enables us to extend many inequalities in the literature for bounded operators on H_p, when $p \in [1, 2] \cup \{\infty\}$. By providing a counterexample, we show that some known inequalities on numerical radius of bounded operators on Hilbert spaces are not true when the underlying space is H_p with ℓ^p-norm, where $2 < p < \infty$.

2. Main Results

Hereafter, unless otherwise is stated, we will assume that H and H_p are the spaces defined in Section 1, for $1 \le p < \infty$. We also denote by $B(H_p)$, the set of bounded linear operators on H_p. Clearly, $B(H_p)$ is a linear space.

The following result states that ℓ^q-norms on $B(H_p)$ are equal, when

$$p, q \in [1, \infty].$$

Proposition 2.1. *Let $T \in B(H_p)$ and let $1 \le p \le \infty$, then*

$$\|T\|_p = \sup_{\lambda\in\Lambda} \|T_\lambda\|.$$

P r o o f. For each $\epsilon > 0$, there exists $\lambda_0 \in \Lambda$,

$$\sup_{\lambda} \|T_\lambda\| - \epsilon < \|T_{\lambda_0}\|. \tag{2.1}$$

There is x_{λ_0} with $\|x_{\lambda_0}\| = 1$,

$$\|T_{\lambda_0}\| - \epsilon < \|T_{\lambda_0} x_{\lambda_0}\|. \tag{2.2}$$

Define the set $x_0 = (x_\lambda)_{\lambda\in\Lambda}$ as follows

$$x_\lambda = 0 \text{ if } \lambda \neq \lambda_0 \quad \text{and} \quad x_\lambda = x_{\lambda_0} \text{ if } \lambda = \lambda_0.$$

We have

$$\|x_0\|_p = \|x_{\lambda_0}\| = 1 \quad \text{and} \quad \|Tx_0\|_p = \|T_{\lambda_0} x_{\lambda_0}\|.$$

By (2.1) and (2.2),

$$\begin{aligned}\sup_{\lambda} \|T_\lambda\| - 2\epsilon &< \|T_{\lambda_0} x_{\lambda_0}\| \\ &= \|Tx_0\|_p \\ &\leq \sup_{\|x\|_p=1} ||Tx||_p = ||T||_p.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, $\sup_\lambda \|T_\lambda\| \leq \|T\|_p$. Moreover, if $p \neq \infty$, we have

$$\begin{aligned}\|T\|_p &= \sup_{\|(x_\lambda)\|_p=1} \|(T_\lambda x_\lambda)_{\lambda\in\Lambda}\|_p \\ &= \sup_{\|(x_\lambda)\|_p=1} \left(\sum_{\lambda\in\Lambda} \|T_\lambda x_\lambda\|^p\right)^{\frac{1}{p}} \\ &\leq \sup_{\|(x_\lambda)\|_p=1} \left(\sum_{\lambda\in\Lambda} \|T_\lambda\|^p \|x_\lambda\|^p\right)^{\frac{1}{p}} \\ &\leq \sup_{\|(x_\lambda)\|_p=1} \left(\sum_{\lambda\in\Lambda} (\sup_{\lambda} \|T_\lambda\|)^p \|x_\lambda\|^p\right)^{\frac{1}{p}} \\ &= \sup_{\lambda\in\Lambda} \|T_\lambda\|.\end{aligned}$$

For $p = \infty$, we have

$$\begin{aligned}\|T\|_\infty &= \sup_{\|(x_\lambda)_{\lambda\in\Lambda}\|_\infty=1} \|(T_\lambda x_\lambda)_{\lambda\in\Lambda}\|_\infty \\ &= \sup_{\|(x_\lambda)_{\lambda\in\Lambda}\|_\infty=1} \sup_{\lambda} \|T_\lambda x_\lambda\| \\ &\leq \sup_{\|(x_\lambda)_{\lambda\in\Lambda}\|_\infty=1} \sup_{\lambda} \|T_\lambda\| \|x_\lambda\| \\ &\leq \sup_{\lambda} \|T_\lambda\|.\end{aligned}$$

□

The following result follows immediately from Proposition 2.1.

Corollary 2.2. *For each $1 \le p \le \infty$, $B(H_p)$ is a C^*-algebra.*

P r o o f. Define an involution $T=(T_\lambda)_{\lambda\in\Lambda}\mapsto T^*$ on $B(H_p)$, where $T^*=(T^*_\lambda)_{\lambda\in\Lambda}$. Proposition 2.1 guarantees that this map is well-defined. One can easily see that

$$(T)^{**}=T \quad \text{and} \quad (TS)^*=S^*T^* \quad \text{for all } T,S\in B(H_p).$$

Since for each $T\in B(H_p)$,

$$||TT^*||_p = sup_{\lambda\in\Lambda}||T_\lambda T^*_\lambda||_\lambda = sup_{\lambda\in\Lambda}||T_\lambda||_\lambda||T^*_\lambda||_\lambda = ||T||_p||T^*||_p,$$

the space $B(H_p)$ is a C^*-algebra. □

Definition 2.3. For each $T\in B(H_p)$ where $1\le p<\infty$, we define

$$W_p(T) = \left\{\textstyle\sum_{\lambda\in\Lambda}\langle T_\lambda x_\lambda, x_\lambda\rangle_\lambda : ||(x_\lambda)||_p=1\right\}$$

and

$$\omega_p(T) = \sup\{|\alpha| : \alpha\in W_p(T)\}.$$

Note that our definition is a natural extension of the standard definition of numerical radius for bounded linear operators on Hilbert spaces.

One can easily check that for each $T,S\in B(H_p)$ and $1\le p<\infty$, we have

(i) $W_p(T+S)\subseteq W_p(T)+W_p(S)$,

(ii) if $\alpha\in\mathbb{C}$, then $W_p(\alpha T)=\alpha W_p(T)$,

(iii) $W_p(T^*)=\{\overline{\alpha}:\alpha\in W_p(T)\}$,

(iv) if U is unitary, then $W_p(U^*TU)=W_p(T)$,

(v) $\omega_p(T)=0$ if and only if $T\equiv 0$,

(vi) $T\ge 0$ if and only if $\langle Tx,x\rangle\ge 0$ for each $x\in H_p$,

(vii) T is self-adjoint if and only if $\langle Tx,x\rangle$ is in the extended real line for each $x\in H_p$.

It follows from (i) and (ii) that $\omega_p(T+S)\le\omega_p(T)+\omega_p(S)$ and $\omega_p(\alpha T)=|\alpha|\omega_p(T)$. This together with (v) show that $\omega_p(\ .\ ): B(H_p)\to[0,\infty]$ defines a generalized normed space. By (iii) and (iv), $\omega_p(T)=\omega_p(T^*)$ and $\omega_p(U^*TU)=\omega_p(T)$ provided that U is unitary.

The following result compares $\omega_p(T)$ and $\omega_q(T)$, when $1\le p<q<\infty$ for an operator $T\in B(H_q)$.

Proposition 2.4. *Let $T\in B(H_p)$, $x=(x_\lambda)\in H_p$ and $1\le p<q<\infty$, then*

(a) $|\sum_{\lambda\in\Lambda}\langle T_\lambda x_\lambda, x_\lambda\rangle|\le\omega_p(T)\|(x_\lambda)\|_p^2$.

(b) *$H_p\subseteq H_q$ and if $T\in B(H_q)$, then $\omega_p(T)\le\omega_q(T)$.*

P r o o f.

(a) If $x = 0$, the inequality clearly holds. Suppose that $x \neq 0$, then λ, $\left\|\left(\frac{x_\lambda}{\|(x_\lambda)\|_p}\right)\right\|_p = 1$, hence

$$\left|\sum_{\lambda\in\Lambda} \langle T_\lambda x_\lambda, x_\lambda\rangle\right| \leq \omega_p(T)\|(x_\lambda)\|_p^2. \tag{2.3}$$

(b) Let $1 < p < q$, $(x_\lambda)_{\lambda\in\Lambda} \in H_p$ and $\|x_\lambda\|_p = 1$. So that for each λ, $\|x_\lambda\| \leq 1$. Hence $\|x_\lambda\|^q \leq \|x_\lambda\|^p$. Thus $\sum_{\lambda\in\Lambda} \|x_\lambda\|^q \leq \sum_{\lambda\in\Lambda} \|x_\lambda\|^p$, so that $(x_\lambda)_{\lambda\in\Lambda} \in H_q$. This means that $H_p \subseteq H_q$. Hence $\omega_p(T) \leq \omega_q(T)$ for each $T \in B(H_q)$. □

The following example shows that in some situations the inequality of the right-hand side of Proposition 2.4 is strict.

Example 2.5. Let $\Lambda = \mathbb{N}$ and $H_n = \mathbb{C}$ for each $n \geq 1$. Take some $p > 2$ and let $x_n = \frac{1}{\sqrt{n}}$, then $(x_n)_{n\in\mathbb{N}} \in l^p$. If id is the identity operator on H_p, then

$$\left|\sum_{n=1}^{\infty} \langle id\ x_n, x_n\rangle\right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

So that $\omega_p(id) = \infty$ while $\omega_2(id) = 1$. Hence $\omega_2(id) < \omega_p(id)$.

The following result states that $\left\{\left(B(H_p), ||\ .\ ||_p\right)\right\}_{1\leq p\leq\infty}$ is a nested collection of C^*-algebras.

Corollary 2.6. *For each $1 \leq p < q \leq \infty$, $\left(B(H_p), ||\ .\ ||_p\right)$ is a C^*-subalgebra of $\left(B(H_q), ||\ .\ ||\right)$.*

Proof. Follows from Corollary 2.2 and Proposition 2.4(b). □

Corollary 2.6 together with the fact that there is at most one C^*-norm on a space justifies the correctness of Proposition 2.1, which at first seemed strange.

It is known that the numerical radius of bounded operators defines a norm on $B(H)$, which is also equivalent to the operator norm. The above example shows that this result is not true when underlying space is a direct sum of Hilbert spaces with ℓ^p-norm with $p > 2$. However, when $1 \leq p \leq 2$, the situation is different.

Theorem 2.7. *Let $T \in B(H_p)$ where $1 \leq p \leq 2$, then*

$$\frac{1}{2}\|T\|_p \leq \omega_p(T) \leq \|T\|_p .$$

Moreover, if T is a normal element of $B(H_p)$, then $\omega_p(T) = \|T\|_p$.

P r o o f. Let $\|(x_\lambda)\|_p = 1$. By Proposition 2.1,

$$\begin{aligned}\left|\sum_{\lambda\in\Lambda}\langle T_\lambda x_\lambda, x_\lambda\rangle\right| &\le \sum_{\lambda\in\Lambda}|\langle T_\lambda x_\lambda, x_\lambda\rangle| \\ &\le \sum_{\lambda\in\Lambda}\|T_\lambda x_\lambda\|\|x_\lambda\| \\ &\le \sum_{\lambda\in\Lambda}\|T_\lambda\|\|x_\lambda\|^2 \\ &\le \sum_{\lambda\in\Lambda}\|T\|_p\|x_\lambda\|^2.\end{aligned}$$

For any λ, $\|x_\lambda\| \le 1$, since $1 \le p \le 2$, $\|x_\lambda\|^2 \le \|x_\lambda\|^p$. From the last inequality we have

$$\begin{aligned}\left|\sum_{\lambda\in\Lambda}\langle T_\lambda x_\lambda, x_\lambda\rangle\right| &\le \|T\|_p\sum_{\lambda\in\Lambda}\|x_\lambda\|^p \\ &= \|T\|_p.\end{aligned}$$

By taking supremum over all (x_λ) with $\|(x_\lambda)\|_p = 1$, we see that $\omega_p(T) \le \|T\|_p$.

Take some $\lambda_0 \in \Lambda$ and $x_{\lambda_0} \in H_{\lambda_0}$ with $\|x_{\lambda_0}\| = 1$. Define $x = (x_\lambda)_{\lambda\in\Lambda}$ as follows

$$x_\lambda = 0 \text{ if } \lambda \ne \lambda_0 \quad \text{and} \quad x_\lambda = x_{\lambda_0} \text{ if } \lambda = \lambda_0.$$

Then

$$\langle Tx, x\rangle := \sum_{\lambda\in\Lambda}\langle T_\lambda x_\lambda, x_\lambda\rangle = \langle T_{\lambda_0}x_{\lambda_0}, x_{\lambda_0}\rangle.$$

Hence

$$\begin{aligned}|\langle T_{\lambda_0}x_{\lambda_0}, x_{\lambda_0}\rangle| &= |\langle Tx, x\rangle| \\ &\le \sup_{\|(x_\lambda)\|_p=1}\left|\sum_{\lambda\in\Lambda}\langle T_\lambda x_\lambda, x_\lambda\rangle\right| \\ &= \omega_p(T).\end{aligned}$$

Since λ_0 was arbitrary, by taking supremum on $\|x_\lambda\| = 1$, we see that

$$\omega_\lambda(T_\lambda) \le \omega_p(T) \quad (\forall\lambda \in \Lambda). \tag{2.4}$$

Since $T_\lambda \in B(H_\lambda)$, we have $\frac{1}{2}\|T_\lambda\| \le \omega_\lambda(T_\lambda)$. By Proposition 2.1 and (2.4), we have

$$\frac{1}{2}\|T\|_p \le \omega_p(T).$$

Now suppose that $T = (T_\lambda)_{\lambda\in\Lambda}$ is normal, then for each λ, T_λ is a normal operator on H_λ. Therefore $\|T_\lambda\| = \omega_\lambda(T_\lambda)$. By inequality (2.4), $\|T_\lambda\| \le \omega_p(T)$. By using Proposition 2.1 and taking supremum on λ, $\|T\|_p \le \omega_p(T)$. By the first part of the proof, $\omega_p(T) \le \|T\|_p$. Therefore, $\omega_p(T) = \|T\|_p$. □

The following result is a direct subsequence of Theorem 2.7.

Corollary 2.8. *If $1 \leq p \leq 2$, the function $\omega_p(\ .\ ) : B(H_p) \to [0,\infty)$ defines a norm on $B(H_p)$ which is equivalent to the original norm.*

In what follows, we study the direct sum of Hilbert spaces with *sup*-norm. We also investigate some properties of the numerical radius of bounded linear operator on H_∞. $(H_\infty, \|\,.\,\|_\infty)$. We start by the following definition.

Definition 2.9. For each T $=(T_\lambda) \in B(H_\infty)$, we define

$$\omega_\infty(T) = \sup_{\|(x_\lambda)_{\lambda\in\Lambda}\|_\infty=1} \sup_\lambda |\langle T_\lambda x_\lambda, x_\lambda\rangle|.$$

Lemma 2.10. *Let $T \in B(H_\infty)$, then for every $(x_\lambda)_{\lambda\in\Lambda}$*

$$|\langle Tx, x\rangle| \leq \|(x_\lambda)_{\lambda\in\Lambda}\|_\infty^2 \omega_\infty(T).$$

Proof. Let $\|(x_\lambda)_{\lambda\in\Lambda}\|_\infty \neq 0$. Then $\left\|\left(\frac{x_\lambda}{\|(x_\lambda)_{\lambda\in\Lambda}\|_\infty}\right)\right\| = 1$. So that

$$\left|\left\langle T_\lambda\left(\frac{x_\lambda}{\|(x_\lambda)_{\lambda\in\Lambda}\|_\infty}\right), \frac{x_\lambda}{\|(x_\lambda)_{\lambda\in\Lambda}\|_\infty}\right\rangle\right| \leq \omega_\infty(T).$$

Hence

$$|\langle T_\lambda x_\lambda, x_\lambda\rangle| \leq \|(x_\lambda)_{\lambda\in\Lambda}\|_\infty^2 \omega_\infty(T).$$

If $\|(x_\lambda)_{\lambda\in\Lambda}\|_\infty = 0$, then for each λ,$\|x_\lambda\| = 0$. By Cauchy-Schwartz inequality we have

$$|\langle < T_\lambda x_\lambda, x_\lambda\rangle|^2 \leq \|T_\lambda x_\lambda\|\|x_\lambda\|.$$

It follows that $|\langle T_\lambda x_\lambda, x_\lambda\rangle| = 0$. Therefore

$$|\langle T_\lambda x_\lambda, x_\lambda\rangle| \leq \|(x_\lambda)_{\lambda\in\Lambda}\|_\infty^2 \omega_\infty(T). \qquad \square$$

The following Lemma gives a relation between the numerical radius of T and T_λ.

Proposition 2.11. *Let $T \in B(H_\infty)$. Then $\omega_\infty(T) = \sup_\lambda \omega_\lambda(T_\lambda)$.*

Proof. Let $\lambda_0 \in \Lambda$ with $\|x_{\lambda_0}\| = 1$. We define the set $x = (x_\lambda)_{\lambda\in\Lambda}$ by

$$x_\lambda = 0 \text{ if } \lambda \neq \lambda_0, \quad \text{and} \quad x_\lambda = x_{\lambda_0}, \text{ if } \lambda = \lambda_0.$$

Hence $\|(x_\lambda)_{\lambda\in\Lambda}\|_\infty = \|x_{\lambda_0}\| = 1$.

$$|\langle T_{\lambda_0} x_{\lambda_0}, x_{\lambda_0}\rangle| \leq \omega_\infty(T).$$

Since λ_0 was arbitrary with $\|x_{\lambda_0}\| = 1$, we have $|\langle T_{\lambda_0} x_{\lambda_0}, x_{\lambda_0}\rangle| \leq \omega_\infty(T)$. By taking supremum $\|x_{\lambda_0}\| = 1$,

$$\omega_{\lambda_0}(T_{\lambda_0}) = \sup_{\|x_{\lambda_0}\|=1} |\langle T_{\lambda_0} x_{\lambda_0}, x_{\lambda_0}\rangle| \leq \omega_\infty(T).$$

By taking supremum on λ, $\sup_\lambda \omega_\lambda(T_\lambda) \leq \omega_\infty(T)$.

On the other hand, for every $\epsilon > 0$, there is λ' which $x_{\lambda'} \in (x_\lambda)$ with $\|(x_\lambda)_{\lambda\in\Lambda}\|_\infty = 1$,

$$\omega_\infty(T) - \epsilon < |\langle T_{\lambda'} x_{\lambda'}, x_{\lambda'}\rangle| \leq \omega_{\lambda'}(T_{\lambda'})\|x_{\lambda'}\|^2 \leq \sup_\lambda \omega_\lambda(T_\lambda).$$

Now $\epsilon \longrightarrow 0$, then $\omega_\infty(T) \leq \sup_\lambda \omega_\lambda(T_\lambda)$. □

The above result enables us to extend many inequalities on numerical radius of bounded operators on Hilbert spaces for the case underlying space is $B(H_\infty)$. For example, we have the following.

Theorem 2.12.

(a) *Let $T \in B(H_\infty)$, then* $\frac{1}{2}\|T\|_\infty \leq \omega_\infty(T) \leq \|T\|_\infty$.

(b) *If T is a normal elements of $B(H_\infty)$, then $\omega_\infty(T) = \|T\|_\infty$.*

Proof.

(a): Let $(T_\lambda) = T \in B(H_\infty)$. Since for each $\lambda \in \Lambda$,

$$\frac{||T_\lambda||_\lambda}{2} \leq \omega_\lambda(T_\lambda) \leq ||T_\lambda||_\lambda.$$ □

By using Propositions 2.1 and 2.11 and taking supremum over all $\lambda \in \Lambda$, we get to the desired result.

(b): Let T be normal, then for each λ, T_λ is normal and $\|T_\lambda\| = \omega_\lambda(T_\lambda)$. Therefore

$$||T||_\infty = \sup_{\lambda\in\Lambda} ||T_\lambda||_\lambda = \sup_{\lambda\in\Lambda} \omega_\lambda(T_\lambda) = \omega_\infty(T).$$

In 2005, Kittaneh [6] proved that if T is a bounded linear operator on a Hilbert space, then

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|. \tag{2.5}$$

The next result shows that Kittaneh's theorem is true for bounded operators on $B(H_p)$ provided that $1 \leq p < 2$ or $p = \infty$.

Theorem 2.13. *Let $T \in B(H_p)$ and let $1 \leq p \leq 2$ or $p = \infty$, then*

$$\frac{1}{4}\|T^*T + TT^*\|_p \leq \omega_p^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|_p. \tag{2.6}$$

Proof. By [6, Kittaneh Theorem 1], for each $T_\lambda \in B(H_\lambda)$ we have

$$\frac{1}{4}\|T_\lambda^*T_\lambda + T_\lambda T_\lambda^*\| \leq \omega_\lambda^2(T_\lambda) \leq \frac{1}{2}\|T_\lambda^*T_\lambda + T_\lambda T_\lambda^*\| \quad (\lambda \in \Lambda). \tag{2.7}$$

By taking supremum of both sides of (2.7), we see that

$$\sup_{\lambda\in\Lambda} \frac{1}{4}\|T_\lambda^*T_\lambda + T_\lambda T_\lambda^*\| \leq \sup_{\lambda\in\Lambda} \omega_\lambda^2(T_\lambda) \leq \sup_{\lambda\in\Lambda} \frac{1}{2}\|T_\lambda^*T_\lambda + T_\lambda T_\lambda^*\| \quad (\lambda \in \Lambda). \tag{2.8}$$

Applying Propositions 2.1 and 2.11, we get to (2.6) for $p = \infty$.

Suppose that $1 \le p \le 2$, then

$$\begin{aligned} \omega_p^2(T) &\le \omega_2^2(T) && \text{by Proposition 2.4,} \\ &\le \frac{1}{2}\|T^*T + TT^*\|_2 && \text{by Kittaneh's theorem,} \\ &= \frac{1}{2}\|T^*T + TT^*\|_p && \text{by Proposition 2.1.} \end{aligned}$$

Let $\lambda_0 \in \Lambda$ with $\|x_{\lambda_0}\| = 1$. Define the set $x = (x_\lambda)_{\lambda\in\Lambda}$ as follows

$$x_\lambda = 0 \ \text{ if } \ \lambda \ne \lambda_0 \quad \text{and} \quad x_\lambda = x_{\lambda_0} \ \text{ if } \ \lambda = \lambda_0.$$

Then

$$\|x\|_p = \|x_{\lambda_0}\| = 1$$

and

$$\begin{aligned} \langle Tx, x\rangle &= \sum_{\lambda\in\Lambda} \langle T_\lambda x_\lambda, x_\lambda\rangle \\ &= \langle T_{\lambda_0} x_{\lambda_0}, x_{\lambda_0}\rangle. \end{aligned}$$

$$\begin{aligned} |\langle T_{\lambda_0} x_{\lambda_0}, x_{\lambda_0}\rangle| &= |\langle Tx, x\rangle| \\ &\le \sup_{\|(x_\lambda)\|_p = 1} \left| \sum_{\lambda\in\Lambda} \langle T_\lambda x_\lambda, x_\lambda\rangle \right| \\ &= \omega_p(T). \end{aligned}$$

Since λ_0 was arbitrary, by taking supremum over all $\|x_\lambda\| = 1$, we see that

$$\omega_\lambda(T_\lambda) \le \omega_p(T)$$

for each $\lambda \in \Lambda$. Hence by applying Kittaneh's theorem once again, we see that

$$\begin{aligned} \frac{1}{4}\|T_\lambda^* T_\lambda + T_\lambda T_\lambda^*\| &\le \omega_\lambda^2(T_\lambda) \\ &\le \omega_p^2(T) \quad (\lambda \in \Lambda). \end{aligned}$$

By applying Theorem 2.1 and by taking supremum on all $\lambda \in \Lambda$, we have

$$\frac{1}{4}\|T^*T + TT^*\|_p \le \omega_p^2(T). \qquad \square$$

Remark 2.14. The method used in Theorem 2.13 can be used to extend some other inequalities on numerical radius of bounded operators on Hilbert spaces to the case the underlying space is a direct sum of Hilbert spaces with ℓ^p-norm for $\le p \le 2$ or $p = \infty$.

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