Harmonic Analysis on Locally Compact Quantum Groups

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Definition

A locally compact group is a group G with a locally compact Hausdorff topology for which the multiplication and the inverse maps are continuous. That is,

$$(x, y) \mapsto xy^{-1} : G \times G \longrightarrow G$$

is continuous.

An important role in the theory of LCG is played by the so-called Haar Measure: A nonzero left invariant Radon measure λ on G which is nonzero on nonempty open sets.

$L^\infty(G)$

$$\mathrm{L}^\infty(G) := \{f : f \text{ is } \lambda - \text{essentially bounded}\}.$$

is a von -bbNeumann algebra equipped with the norm $\|\cdot\|_{\lambda-esssup}$, and the multiplication

$$\Pi : (\phi, \psi) \mapsto \phi\psi : \mathrm{L}^\infty(G)\bar{\otimes}\mathrm{L}^\infty(G) \longrightarrow \mathrm{L}^\infty(G).$$

Associativity: $\Pi(\Pi \otimes \iota) = \Pi(\iota \otimes \Pi)$.
It has also a comultiplication

$$\Gamma : \phi \mapsto \Gamma(\phi)(s,t) = \phi(st) : \mathrm{L}^\infty(G) \to \mathrm{L}^\infty(G)\bar{\otimes}\mathrm{L}^\infty(G).$$

Coassociativity: $(\Gamma \otimes \iota)\Gamma = (\iota \otimes \Gamma)\Gamma$.

$$\mathrm{L}^1(G) := \{f : f \text{ is measurable} \int_G |f|\, d\lambda < \infty\}.$$

is a $^*-$Banach algebra equipped with the norm $\|\cdot\|_1$, and the convolution multiplication and the involution given by

$$(f * g)(x) := \int_G f(y) g(y^{-1}x)\, d\lambda(y), \qquad f^*(x) := \Delta(x^{-1})\overline{f(x^{-1})}.$$

It is worth to note that:

$$(f * g)(x) = (f \otimes g)(\Gamma(x)).$$

$$C_b(G) := \{f\colon G \longrightarrow \mathbb{C}, f \text{ is boundend and continuous}\}.$$

$$LUC(G) := \{f \in C_b(G), f \text{ is right uniformly continuous}\}.$$

$$WAP(G) := \{f \in C_b(G), f \text{ is weakly almost periodic}\}.$$

$$AP(G) := \{f \in C_b(G), f \text{ is almost periodic}\}.$$

All are C^*-algebras with the following inclusion relations:

$$AP(G) \subseteq WAP(G) \subseteq LUC(G) \subseteq C_b(G).$$

We have also an important C^*-subalgebra of $C_b(S)$:

$$C_0(G) := \{f \in C_b(G), f \text{ vanishes at infinity}\}.$$

$$C_0(G) \subseteq WAP(G).$$

$M(G)$

$$M(G) := \{\mu : \mu \text{ is a regular complex measure on } G\}.$$

Then

$$M(G) = C_0(G)^*,$$

is a unital $*^-$Banach algebra.

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- Quantum group compactifications
- Some problems

Definition

A Hopf von Neumann Algebra is a pair $(\mathfrak{M}, \Gamma)$ where $\mathfrak{M}$ is a von Neumann algebra and $\Gamma : \mathfrak{M} \to \mathfrak{M}\bar{\otimes}\mathfrak{M}$ is a normal, unital $*$-homomorphism satisfying $(\iota \otimes \Gamma)\Gamma = (\Gamma \otimes \iota)\Gamma$.

Let $(\mathfrak{M}, \Gamma)$ be a Hopf von Neumann algebra then the unique predual $\mathfrak{M}_*$ of $\mathfrak{M}$ turns into a Banach algebra under the product $*$ given by:

$$(\omega * \omega')(x) = (\omega \otimes \omega')(\Gamma(x)) \quad (\omega, \omega' \in \mathfrak{M}_*, x \in \mathfrak{M}).$$

Definition

A Hopf von Neumann algebra $(\mathfrak{M}, \Gamma)$ *equipped with a pair* (φ, ψ) *where* φ *and* ψ *are a left and right invariant Haar weight on* $\mathfrak{M}$*, respectively, is called a locally compact quantum group.*

1. φ *is a n.s.f. weight on* $\mathfrak{M}$ *which is left invariant, i.e., satisfies*

$$\varphi((\omega \otimes \iota)(\Gamma(x))) = \omega(1)\varphi(x) \qquad (\omega \in \mathfrak{M}_*, x \in \mathcal{M}_\varphi^+),$$

2. ψ *is a n.s.f. weight on* $\mathfrak{M}$ *which is right invariant, i.e., satisfies*

$$\psi((\iota \otimes \omega)(\Gamma(x))) = \omega(1)\psi(x) \qquad (\omega \in \mathfrak{M}_*, x \in \mathcal{M}_\psi^+).$$

For a locally compact group G,

1. $\mathbb{G}_a = (\mathrm{L}^\infty(G), \Gamma_a)$ is a locally compact quantum group, where $\Gamma_a : \mathrm{L}^\infty(G) \to \mathrm{L}^\infty(G)\bar{\otimes}\mathrm{L}^\infty(G), \Gamma_a(f)(s,t) = f(st)$ and φ_a and ψ_a are the left and right Haar integrals on $\mathrm{L}^\infty(G)$, respectively.
2. The induced $*$ on the predual $L^1(G)$ of $L^\infty(G)$ is the usual convolution:

$$\omega * \omega'(t) = \int_G \omega(s)\omega'(s^{-1}t)\,ds \qquad (\omega, \omega' \in L^1(G)).$$

The classical case: $\mathbb{G}_s = (\mathrm{VN}(G), \Gamma_s)$

For a locally compact group G,

1. $\mathbb{G}_s = (\mathrm{VN}(G), \Gamma_s)$ is a locally compact quantum group; where $\Gamma_s : \mathrm{VN}(G) \to \mathrm{VN}(G)\bar{\otimes}\mathrm{VN}(G), \qquad \lambda(t) \to \lambda(t) \otimes \lambda(t)$; with λ is the left regular representation of G on $\mathrm{L}^2(G)$ and $\varphi_s = \psi_s$ is the Plancherel weight on $\mathrm{VN}(G)$.
2. The induced $*$ on the predual $A(G)$ of $VN(G)$ is the usual pointwise product :

$$\omega * \omega'(t) = \omega(t)\omega'(t) \qquad (\omega, \omega' \in A(G)).$$

For a locally compact quantum group $\mathbb{G}$, using the GNS construction, the left Haar weight φ induces a representation (π_φ, H_φ) for $\mathfrak{M}$. Associated with the pair $(\mathbb{G}, \varphi)$ there exists a unique unitary operator $W \in B(H_\varphi \bar{\otimes} H_\varphi)$ – the so-called multiplicative unitary – which presents the coproduct Γ via

$$\Gamma(x) = W^*(1 \otimes x) W, \quad (x \in \mathfrak{M}).$$

Multiplicative Unitary

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- For the case $\mathbb{G}_a$, the unitary operator

$$W_a(\zeta)(x, y) = \zeta(x, x^{-1}y), \quad (\zeta \in L^2(G \times G), x, y \in G),$$

on is $L^2(G \times G)$ is the multiplicative unitary of $\mathbb{G}_a$.

Notation

For a locally compact quantum group $\mathbb{G} = (\mathfrak{M}, \Gamma, \phi, \psi)$ we write:

1. $\mathrm{L}^\infty(\mathbb{G})$ for $\mathfrak{M}$.
2. $\mathrm{L}^1(\mathbb{G})$ for $\mathfrak{M}_*$.
3. $\mathrm{L}^2(\mathbb{G})$ for H_ϕ.

For a locally compact group G,

1. $\mathrm{L}^\infty(\mathbb{G}_a) = L^\infty(G)$.
2. $\mathrm{L}^1(\mathbb{G}_a) = L^1(G)$.

1. $\mathrm{L}^\infty(\mathbb{G}_s) = VN(G)$.
2. $\mathrm{L}^1(\mathbb{G}_s) = A(G)$.

Left invariant mean

Definition

Let $\mathbb{G} = (\mathfrak{M}, \Gamma)$ be a Hopf-von Neumann algebra. A state M of $\mathfrak{M}$ is called a left invariant mean for $\mathbb{G}$ if

$$M((\omega \otimes \iota)(\Gamma(x))) = \omega(1)M(x) \qquad (\omega \in \mathfrak{M}_*,\ x \in \mathfrak{M}).$$

If there is a left invariant mean on $\mathbb{G}$, we call $\mathbb{G}$ left amenable.

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- $\mathbb{G}_a$ is left amenable if and only if G is amenable;
- $\mathbb{G}_s$ is always left amenable.

Left invariant co-mean

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Let G be a locally compact group then

- $\mathbb{G}_a$ is always left co-amenable;
- $\mathbb{G}_s$ is left co-amenable if and only if G is amenable.

For a locally compact quantum group $\mathbb{G}$ we define

$$\mathrm{C}_0(\mathbb{G}) := \overline{\{(\iota \otimes \omega)(W) : \omega \in B(L^2(\mathbb{G}))_*\}}^{\|\cdot\|},$$

and

$$\mathrm{C}_b(\mathbb{G}) := \mathcal{M}(\mathrm{C}_0(\mathbb{G})).$$

Then $\mathrm{C}_0(\mathbb{G})$ is a C^*-subalgebra of $\mathrm{C}_b(\mathbb{G})$.
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For a locally compact quantum group $\mathbb{G}$ we define

$$\mathrm{M}(\mathbb{G}) := \mathrm{C}_0(\mathbb{G})^*.$$

Then $\mathrm{M}(\mathbb{G})$ is a Banach algebra with the product

$$\mu * \nu(x) := (\mu \otimes \nu)(\Gamma(x)), \quad (\mu, \nu \in \mathrm{M}(\mathbb{G}), x \in \mathrm{C}_0(\mathbb{G})).$$

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For a locally compact quantum group $\mathbb{G}$ we define

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$$\begin{aligned} & \mathrm{LUC}(\mathbb{G}_a) = \mathrm{LUC}(G) = \mathrm{L}^\infty(G) \cdot \mathrm{L}^1(G) \\ = & \{f \in C_b(G) : x \mapsto L_x f : G \longrightarrow C_b(G) \text{is norm continuous}\} \\ = & \{f : G \longrightarrow \mathbb{C} : f \text{is right uniformly continuous}\} \end{aligned}$$

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$\mathrm{LUC}(\mathbb{G})$ *and* $\mathrm{RUC}(\mathbb{G})$

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$\mathrm{WAP}(\mathrm{L}^1(G))$ *and* $\mathrm{AP}(\mathrm{L}^1(\mathbb{G}))$

For a locally compact quantum group $\mathbb{G}$ we define

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How about $\mathrm{WAP}(\mathbb{G})$?!!!!

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How about $\mathrm{AP}(\mathbb{G})$?!!!!

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Theorem (Daws, (2010))

If $\mathbb{G} = (\mathfrak{M}, \Gamma)$ *is a a commutative Hopf von Neumann algebra then both of* $\mathrm{AP}(\mathrm{L}^1(\mathbb{G}))$ *and* $\mathrm{WAP}(\mathrm{L}^1(\mathbb{G}))$ *are* C^*-*subalgebras of* $\mathrm{L}^\infty(\mathbb{G})$*.*

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Theorem (Ramezanpour and V., (2009))

For a co-amenable locally compact quantum group $\mathbb{G}$*,* $Z_1(\mathrm{LUC}(\mathbb{G})^*) = \mathrm{M}(\mathbb{G})$ *and* $\mathrm{LUC}(\mathbb{G}) = \mathrm{WAP}(\mathrm{L}^1(\mathbb{G}))$ *if and only if* $\mathbb{G}$ *is compact.*

Example (Losert, (2003))

Let G be either $\mathbb{F}_2$ or SU_3 then $Z_1(UCB(\hat{G})^*) \neq B(G)$, or equivalently $Z_1(LUC(\mathbb{G}_s)^*) \neq M(\mathbb{G}_s)$.

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$$\mathrm{WAP}(\mathbb{G}) := \{x \in \mathrm{WAP}(\mathrm{L}^1(\mathbb{G})) : x^*x, xx^* \in \mathrm{WAP}(\mathrm{L}^1(\mathbb{G}))\}.$$

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$\mathrm{WAP}(\mathbb{G})$ *is a (unital)* C^*-*subalgebra of* $\mathrm{L}^\infty(\mathbb{G})$. *More precisely, it is the largest* C^*-*algebra in* $\mathrm{WAP}(\mathrm{L}^1(\mathbb{G}))$.

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Semigroup Compactification (classical form)

Let S be a semitopological semigroup. A pair (ψ, X) is called a semigroup compactification of S, where

1. X is a compact Hausdorff right topological semigroup.
2. $\psi : S \longrightarrow X$ is a continuous homomorphism with dense image.
3. $\psi(S) \subset \Lambda(X)$:=the topological center of X.

Theorem

For every unital C^*-subalgebra $\mathcal{F}$ of $\mathrm{C}_b(S)$, the pair $(\epsilon, S^{\mathcal{F}})$ is a semigroup compactification of S (called $\mathcal{F}-$compactification) where

1. *$S^{\mathcal{F}}$ is the spectrum of $\mathcal{F}$, equipped with the Gelfand topology and the product $\mu \cdot \nu := \mu \circ \mathrm{L}_\nu$ where*
   $\mathrm{L}_\nu f(s) := \nu(L_s f), \ (s \in S, f \in \mathcal{F})$.
2. *$\epsilon :\longrightarrow S^{\mathcal{F}}$ is the evaluation map.*

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Quantum Separate Continuity

For a unital C^*-algebra A we define

$$\mathrm{SC}(A\times A) := \{x \in A^{**}\overline{\otimes}A^{**} : (\mu\otimes\iota)x, (\iota\otimes\mu)x \in A, \text{ for every} \mu \in A^*\},$$

and

$$A\otimes^{sc}A := \{x \in \mathrm{SC}(A\times A) : \; x^*x, xx^* \in \mathrm{SC}(A\times A)\}.$$

Definition

A compact semitopological quantum semigroup is a pair $\mathbb{S} := (A, \Delta)$ where,

1. A is a unital C^*-algebra.
2. $\Delta : A \longrightarrow A\otimes^{sc}A$ is a unital $*-$homomorphism whose normal extension
$$\widehat{\Delta} : A^{**} \longrightarrow A^{**}\overline{\otimes}A^{**}$$
is coassociative.

Suppose that S is a compact semitopological semigroup, then $\mathrm{C}(S)\otimes^{sc}\mathrm{C}(S) = sc(S \times S) :=$
the algebra of bounded separately continuous functions on $S \times S$.
And the pair $(\Delta, \mathrm{C}(S)$ is a compact quantum semitopological semigroup, where

$$\Delta : \mathrm{C}(S) \longrightarrow sc(S \times S) \text{ by } \Delta(f)(s,t) := f(st), \ (f \in \mathrm{C}(S), s, t \in S).$$

Note that the coassocitavity of $\widehat{\Delta}$ follows from the associativity of the multiplication of S.

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WAP$-$*quantum semigroup compactification*

Theorem (Daws 2015)

For a Hopf von Neumann algebra $\mathbb{G} = (\mathfrak{M}, \Gamma)$*, the pair* $(\Gamma|_{\mathrm{WAP}(\mathbb{G})}, \mathrm{WAP}(\mathbb{G}))$ *is the largest compact quantum semitopological semigroup.*

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