RECENT RESEARCHES ON THE SECOND DUAL OF BANACH ALGERAS

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Outline

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- Arens products on A^{**}

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- The size of topological centers.

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- The size of topological centers.
- Quantum groups.
- Amenability and weak amenability of A^{**}.
- Some research problems.

Extensions of a Bilinear Mapping

Definition ([, Arens 1951])

A bounded bilinear mapping $f : \mathcal{X} \times \mathfrak{Y} \longrightarrow \mathcal{Z}$ *has two extensions*

$$f^{***} : \mathcal{X}^{**} \times \mathfrak{Y}^{**} \to \mathcal{Z}^{**}, \quad f^{r****r} : \mathcal{X}^{**} \times \mathfrak{Y}^{**} \to \mathcal{Z}^{**}.$$

in which the adjoint $f^* : \mathcal{Z}^* \times \mathcal{X} \longrightarrow \mathfrak{Y}^*$ *of* f *is defined by*

$$\langle f^*(z^*, x), y \rangle = \langle z^*, f(x, y) \rangle.$$

We define f^{**} *and* f^{***} *of* f *by* $f^{**} = (f^*)^*$ *and* $f^{***} = (f^{**})^*$.
We also denote by f^r *the flip map of* f.
If $f^{***} = f^{r****r}$ *then* f *is said to be (Arens) regular.*

Arens Products

Definition

For A =Banach algebra with $\pi : A \times A \longrightarrow A$ we write
$\Box = \pi^{***}$ *=the first Arens product.*
$\Diamond = \pi^{r****r}$ *=the second Arens product.*
Then $(A^{**}, \Box)$ and $(A^{**}, \Diamond)$ are Banach algebras.
A is said to be Arens regular if $\Box = \Diamond$.

Example

1. (Sherman 1950; Takeda 1954; Civin Yood 1961)
   Every C^*-algebra is Arens regular.
2. (Young 1975; Lau and Losert 1988)
   $L^1(G)$ is Arens regular if and only if G is finite.

Topological Centers

Definition

The left topological center:

$$\begin{aligned} Z_1(\mathcal{A}^{**}): &= \{m \in \mathcal{A}^{**} : n \mapsto m\Box n \text{ is } \mathrm{w}^* - \mathrm{w}^* \text{ continuous}\} \\ &= \{m \in \mathcal{A}^{**} : m\Box n = m\Diamond n \text{ for all n} \in \mathrm{A}^{**}\}. \end{aligned}$$

The right topological center:

$$\begin{aligned} Z_2(\mathcal{A}^{**}): &= \{m \in \mathcal{A}^{**} : n \mapsto n\Diamond m \text{ is } \mathrm{w}^* - \mathrm{w}^*\text{continuous}\} \\ &= \{m \in \mathcal{A}^{**} : n\Box m = n\Diamond m \text{ for all n} \in \mathrm{A}^{**}\}. \end{aligned}$$

A is Arens regular if and only if $Z_1(\mathcal{A}^{**}) = A^{**}$ if and only $Z_2(\mathcal{A}^{**}) = A^{**}$.

Strong Irregularity

Definition

A is said to be left (resp. right) strongly irregular $Z_1(\mathcal{A}^{**}) = A$ (resp. $Z_1(\mathcal{A}^{**}) = A$).

Example

1. (Lau and Losert 1988; Neufang 2004)
   $L^1(G)$ is strongly Arens irregular.
2. (Losert, Neufang, Pachl and Steprvans 201?1?2?3)
   $M(G)$ is strongly irregular.

Factorization Property and Mazur property

1. Property F_κ: The dual $\mathcal{A}^*$ of a Banach algebra $\mathcal{A}$ has the property F_κ (κ is a cardinal number) if for every net $\{h_\alpha\}_{\alpha\in I}$ in $Ball(\mathcal{A}^*)$ with $|I| = \kappa$, there exist a family $\{H_\alpha\}_{\alpha\in I}$ in $Ball(\mathcal{A}^{**})$ and one single functional $h \in \mathcal{A}^*$ such that $h_\alpha = H_\alpha \cdot h$ $(\alpha \in I)$.
2. Property M_κ: A Banach space $\mathcal{X}$ has the property M_κ if every $w^* - \kappa-$continuous element of $\mathcal{X}^{**}$ is actually w^*-continuous.

Factorization Property and Mazur property

Theorem ([, Neufang 2005])

Let $\mathcal{A}$ be a Banach algebra satisfying M_κ and whose dual $\mathcal{A}^*$ has the property F_κ, for some $\kappa \geq \aleph_0$. Then $\mathcal{A}$ is left strongly Arens irregular, i.e. $Z_1(\mathcal{A}^{**}) = \mathcal{A}$. If in addition $\mathcal{A}$ has also a BAI then $Z_1((\mathcal{A}^* \cdot \mathcal{A})^*) = RM(\mathcal{A})$.

Corollary ([, Lau 1986; Lau and Losert 1988; Lau and Ulger 1996; Neufang 2005])

$L^1(G)$ is strongly Arens irregular and $Z_1(LUC(G)^*) = M(G)$.

Corollary ([, Raisi Toosi; Kamyabi Gol and Ebrahimi Vishki 2012])

$L^1(G/H)$ is strongly Arens irregular, for the homogenous space G/H, where H is a compact subgroup of G.

Definition

A Hopf von Neumann Algebra is a pair $(\mathfrak{M}, \Gamma)$ *where* $\mathfrak{M}$ *is a von Neumann algebra and* $\Gamma : \mathfrak{M} \to \mathfrak{M}\bar{\otimes}\mathfrak{M}$ *is a normal, unital* $*$*-homomorphism satisfying* $(\iota \otimes \Gamma)\Gamma = (\Gamma \otimes \iota)\Gamma$.

Let $(\mathfrak{M}, \Gamma)$ be a Hopf von Neumann algebra then the unique predual $\mathfrak{M}_*$ of $\mathfrak{M}$ turns into a Banach algebra under the product $*$ given by:

$$(\omega * \omega')(x) = (\omega \otimes \omega')(\Gamma(x)).$$

Locally compact quantum groups

Definition

A Hopf von Neumann algebra $(\mathfrak{M}, \Gamma)$ *equipped with a pair* (φ, ψ) *where* φ *and* ψ *are a left and right invariant Haar weight on* $\mathfrak{M}$*, respectively, is called a locally compact quantum group.*

Example

For a locally compact group G define

1. $\mathbb{G}_a = (\mathrm{L}^\infty(G), \Gamma_a)$ is a locally compact quantum group, where $\Gamma_a(f)(s,t) = f(st)$ and φ_a and ψ_a are the left and right Haar integrals on $\mathrm{L}^\infty(G)$, respectively.
2. $\mathbb{G}_s = (\mathrm{VN}(G), \Gamma_s)$ is a locally compact quantum group; where $\Gamma_s : \mathrm{VN}(G) \to \mathrm{VN}(G)\bar{\otimes}\mathrm{VN}(G), \qquad \lambda(t) \to \lambda(t) \otimes \lambda(t);$ with λ is the left regular representation of G on $\mathrm{L}^2(G)$ and $\varphi_s = \psi_s$ is the Plancherel weight on $\mathrm{VN}(G)$.

Notation

For a locally compact quantum group $\mathbb{G} = (\mathfrak{M}, \Gamma)$ we write:

1. $\mathrm{L}^\infty(\mathbb{G})$ for $\mathfrak{M}$.
2. $\mathrm{L}^1(\mathbb{G})$ for $\mathfrak{M}_*$.
3. $\mathrm{L}^2(\mathbb{G})$ for H.
4. $\mathrm{LUC}(\mathbb{G})$ for $\mathrm{L}^\infty(\mathbb{G}) \cdot \mathrm{L}^1(\mathbb{G})$.
5. $\mathrm{WAP}(\mathbb{G})$ for $\mathrm{WAP}(L^1(\mathbb{G}))$.

Example

For a locally compact group G

1. $\mathrm{L}^\infty(\mathbb{G}_s) = VN(G)$.
2. $\mathrm{L}^1(\mathbb{G}_s) = A(G)$.
3. $\mathrm{LUC}(\mathbb{G}_s) = UCB(\hat{G})$.

Theorem ([, Ramezanpour and Ebrahimi Vishki 2009])

For a co-amenable locally compact quantum group $\mathbb{G}$*,* $Z_1(LUC(\mathbb{G})^*) = M(\mathbb{G})$ *and* $LUC(\mathbb{G}) = WAP(\mathbb{G})$ *if and only if* $\mathbb{G}$ *is compact.*

Example ([, Losert 2003])

Let G be either $\mathbb{F}_2$ or SU_3 then $Z_1(UCB(\hat{G})^*) \neq B(G)$ or equivalently $Z_1(LUC(\mathbb{G}_s)^*) \neq M(\mathbb{G}_s)$.

Derivations

Definition

1. *Let* $\mathcal{A}$ *be a Banach algebra and* $\mathcal{X}$ *be a Banach* $\mathcal{A}-$*module. A bounded linear operator* $D : \mathcal{A} \to \mathcal{X}$ *is said to be a* ***derivation*** *if* $D(ab) = D(a) \cdot b + a \cdot D(b)$.
2. *For* $x \in \mathcal{X}$ *the mapping* $\delta_x : \mathcal{A} \to \mathcal{X}$ *defined by* $\delta_x(a) = ax - xa$ *is called an* ***inner derivation.***
3. $H^1(\mathcal{A}, \mathcal{X})$ *=the* ***first cohomological group*** *of* A *with coefficients in* $\mathcal{X}$.

Amenability and Weak amenability

Definition

A Banach algebra $\mathcal{A}$ is said to be

1. *amenable if $H^1(\mathcal{A}, \mathcal{X}^*) = \{0\}$, for every Banach $\mathcal{A}-$module X.*
2. *weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$.*
3. *$n-$weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$.*

Example

1. (Johnson 1971)
   $L^1(G)$ is amenable if and only G is amenable.
2. (Dales, Ghahramani and Helemskii 2002)
   $M(G)$ is amenable if and only G is amenable and discrete +
   $M(G)$ is weakly amenable if and only G is discrete.
3. (Johnson 1991; Ghahramani and Despic 1998) $L^1(G)$ is always weakly amenable.

$n-$weak amenability of A^{**}

1. Question :
   Does $n-$weakly amenability of $\mathcal{A}^{**}$ force $\mathcal{A}$ to be $n-$weakly amenable?
2. Answer: (Ebrahimi Vishki and Barootkoob 2011) YES, for $n \geq 2$.

Weak amenability of A^{**}

1. Question 1:
   Does weakly amenability of $\mathcal{A}^{**}$ force $\mathcal{A}$ to be weakly amenable?
2. Question 1':
   Is weakly amenability of $(\mathcal{A}^{**}, \Box)$ equivalent to the weakly amenability of $(\mathcal{A}^{**}, \Diamond)$?

1. Question 2:
   Let $\mathcal{A}$ be a unital dual Banach algebra and let $\mathcal{I}$ be a norm closed weak$^*-$dense ideal of $\mathcal{A}$ with a bounded approximate identity. Does the strong irregularity of $\mathcal{I}$ imply the strong irregularity of $\mathcal{A}$

Dedicated to Professor M. A. Pourabdollah