From Riemman's Integral to the Amenability of Banach algebras

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Riemman's Integral on $\mathbb{R}$

The main property of the Riemman's integral on $\mathbb{R}$ is the invariant property.

$$\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} f(a+x)\,dx, \qquad (a \in \mathbb{R}).$$

Define $I : C(\mathbb{R}) \longrightarrow \mathrm{C}$ by $I(f) = \int_{\mathbb{R}} f(x)\,dx$ then I is a linear functional satisfying

1. $I(L_a f) = I(f), \quad (a \in \mathbb{R})$, where $L_a(f)(x) = f(a+x)$.
2. $\|I\| \neq 1$,
3. $I(1) \neq 1$.

Invariant mean on $(\mathbb{R}, +)$

For a fixed $t \in \mathbb{R}$, define $M_t : C(\mathbb{R}) \longrightarrow \mathbb{C}$ by

$$M_t(f) = \frac{1}{2t} \int_{-t}^{t} f(x)\, dx$$

then every w^*-cluster point M of $\{M_t\}$ satisfies

1. $M(L_a f) = M(f), \quad (a \in \mathbb{R},)$
2. $\|M\| = 1 = M(1)$; or equivalently,

$$\inf_{a \in \mathbb{R}} f(a) \leq M(f) \leq \sup_{a \in \mathbb{R}} f(a), \quad (f \in C(\mathbb{R})_r).$$

Invariant mean on $(\mathbb{Z}, +)$

For a fixed $n \in \mathbb{Z}$, define $M_n : \ell^\infty(\mathbb{Z}) \longrightarrow \mathbb{C}$ by

$$M_n(f) = \frac{1}{2n} \sum_{k=-n}^{n} f(k)$$

then every w^*-cluster point M of $\{M_n\}$ satisfies

1. $M(L_m f) = M(f), \quad (m \in \mathbb{Z},)$
2. $\|M\| = 1 = M(1)$; or equivalently,

$$\inf_{m \in \mathbb{Z}} f(m) \leq M(f) \leq \sup_{m \in \mathbb{Z}} f(m), \quad (f \in \ell^\infty(\mathbb{Z})_r).$$

Invariant mean on $(\mathbb{T}, \cdot)$

Define $M : C(\mathbb{T}) \longrightarrow \mathbb{C}$ by

$$M(f) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(z)}{z} \, dz$$

then M satisfies

1. $M(L_a f) = M(f), \quad (a \in \mathbb{T}),$
2. $\|M\| = 1 = M(1).$

Topological group

A group $\mathbb{G}$ equipped with a Hausdorff topology is called a topological group if

$$(x, y) \mapsto xy^{-1} : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$$

is jointly continuous. The basic examples are $(\mathbb{R}, +), (\mathbb{Z}, +), (\mathbb{R} \setminus 0, \cdot)$ and $(\mathbb{T}, \cdot)$.

Haar measure

Every locally compact topological group $\mathbb{G}$ enjoys a unique positive left invariant Radon measure λ.

1. $(\mathbb{R}, +)$: $\lambda =$ the Lebesgue measure.
2. $(\mathbb{Z}, +)$: $\lambda =$ the counting measure.
3. $(\mathbb{R} \setminus 0, \cdot)$: $d\lambda(x) = \frac{dx}{|x|}$.
4. $(\mathbb{T}, \cdot)$: $d\lambda(z) = \frac{dz}{2\pi i z}$.

Invariant mean

Let $\mathbb{G}$ be a locally compact topological group. A linear functional $M : C(\mathbb{G}) \longrightarrow \mathbb{C}$ is called an invariant mean of $\mathbb{G}$ if

1. $M(L_a f) = M(f), \quad (a \in \mathbb{G}),$
2. $\|M\| = 1 = M(1)$; or equivalently,

$$\inf_{a\in\mathbb{G}} f(a) \leq M(f) \leq \sup_{a\in\mathbb{G}} f(a), \quad (f \in C(\mathbb{G})_r).$$

$\mathbb{G}$ is said to be amenable if it enjoys an invariant mean.

Amenable Group

1. Every abelian group is amenable.
2. Every compact group is amenable.
3. Every solvable group is amenable.
4. $\mathbb{F}_2$ =the free group of two generator is not amenable.
5. $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$, $(n \geq 2)$, are not amenable.

Towards Amenable Banach Algebras

The Group Algebra $L^1(\mathbb{G})$

For a locally compact group $\mathbb{G}$ equipped with the left Haar measure λ,

$$L^1(\mathbb{G}) := \{f : f : \mathbb{G} \to \mathbb{C} \text{ is integrable}\}$$

is a Banach algebra under the usual vector space operations, the norm $\|f\|_1 = \int_{\mathbb{G}} |f| d\lambda$ and the multiplication

$$f * g(x) = \int_{\mathbb{G}} f(y) g(y^{-1}x) d\lambda(y).$$

Derivations

1. Let A be a Banach algebra and X be a Banach $A-$module. A bounded linear operator $D : A \to X$ is said to be a derivation if $D(ab) = D(a) \cdot b + a \cdot D(b)$.
2. For $x \in X$ the mapping $\delta_x : A \to X$ defined by $\delta_x(a) = ax - xa$ is called an inner derivation.
3. $H^1(A, X)$ =the first cohomological group of A with coefficients in X.

Amenable Banach algebra

A Banach algebra A is said to be amenable if
$H^1(A, X^*) = \{0\}$, for every Banach $A-$module X.
Equivalently,
Every derivation $D : A \to X^*$ is inner, for every Banach $A-$module X.

Johnson Celebrated Theorem, 1972

A locally compact group $\mathbb{G}$ is amenable if and only if the Banach algebra $L^1(\mathbb{G})$ is amenable.

Examples

1. Every commutative C^*-algebra is amenable.
2. $K(H)$, the C^*-algebra of compact operators on H, is amenable.
3. The disc algebra $\mathcal{A}(\mathbb{D})$ is not amenable.
4. The measure algebra $M(\mathbb{G})$ is amenable if and only if $\mathbb{G}$ is amenable and discrete.

Towards New Concepts

1. **Weak Amenability**
2. **Super Amenability**
3. **Essential Amenability**
4. **Approximate Amenability**

Amenability of A^{**}

The second dual A^{**} of a Banach algebra A, equipped with each Arens products $\Box$ and $\Diamond$ is a Banach algebra.

Question:

Is amenability of $(\mathcal{A}^{**}, \Box)$ equivalent to the amenability of $(\mathcal{A}^{**}, \Diamond)$?

Dedicated to Professor M. A. Pourabdollah