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Identifying continuous Gabor frames on locally compact Abelian groups

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Abstract. In this paper, we establish some necessary and sufficient conditions for constructing continuous Gabor frames in $L^2(G)$, where G is a second countable locally compact abelian (LCA) group. More precisely, we reformulate the generalized Zak transform defined by A. Weil on LCA groups and later proposed by Gröchenig in the case of integer-oversampled lattices, however our approach is regarding the assumption that both translation and modulation groups are closed subgroups. Moreover, we discuss the possibility of such a generalization and apply several examples to demonstrate the necessity of standing conditions in the results. Finally, by using the generalized Zak transform and fiberization technique, we characterize the continuous Gabor frames of $L^2(G)$ in terms of a family of frames in $l^2(\widehat{H^\perp})$ for a closed co-compact subgroup H of G.

1. Introduction

The Zak transform is one of the fundamental tools in both pure and applied mathematics, and was originally introduced by Gelfand [10] due to some problems in differential equations. This transform was studied by Weil on locally compact abelian(LCA) groups [21] and by Zak in solid state physics [22]. Later on, it has been considered by many of authors for identifying and characterizing Gabor frames in $L^2(G)$ [2, 4, 6, 11]. Most studies in this field are associated with a discrete, co-compact (uniform lattice) subgroup. In particular, Gröchenig [11] presented some new aspects of Zak transform to analyze uniform lattice Gabor systems on LCA groups. In recent years this aspect has been extended to closed subgroups for the characterization of continuous Gabor frames. Indeed, by considering a closed subgroup H of an LCA group G and applying the Zak transform associated with H some equivalent conditions for the existence of continuous Gabor frames in the form of $\{E_\gamma T_\lambda g\}_{\lambda\in H,\gamma\in H^\perp}$ have been obtained [2, 16].

The main purpose of this paper is to achieve some characterization results regarding continuous Gabor frames as $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ in $L^2(G)$, for closed subgroups $\Lambda \subseteq G$ and $\Gamma \subseteq \widehat{G}$, in which Γ is not necessarily the annihilator of Λ. To this end, we extend and reformulate the idea of integer oversampling for uniform lattices [11]. However, our formulation relies on the assumption that both the translation and modulation groups

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are only closed subgroups and remove some other limited conditions. Moreover, we discuss the existing conditions for such a generalization. Finally, by using the generalized Zak transform and fiberization method, we give some characterizations of continuous Gabor frames in $L^2(G)$ in term of a family of frames in $l^2(\widehat{H^\perp})$ for a closed co-compact subgroup H of G.

This paper is organized as follows. In Section 2, we present some basic facts about locally compact abelian groups and the required definitions of continuous frame theory. Then we provide a sufficient condition for the existence of continuous Gabor frames in $L^2(G)$. In section 3, we extend the idea of integer oversampling. Moreover, we present some existing conditions for this generalization. Applying the generalized Zak transform, we obtain some equivalent conditions for a Gabor system to be a frame family, orthonormal basis, minimal system or a complete family in $L^2(G)$. Finally, in section 4, we use the fiberization method to build a relationship between continuous Gabor frames in $L^2(G)$ and a family of frames in $l^2(\widehat{H^\perp})$ for a closed co-compact subgroup H of G.

2. Notations and preliminaries

Let G be a second countable locally compact abelian (LCA) group. It is known that such a group carries a translation invariant regular Borel measure so called Haar measure and is denoted by μ_G, which is unique up to a positive constant. We will use the addition as the group operation and equip discrete groups with the counting measure. Let $\widehat{G}$ denote the dual group of G, then the Pontryagin duality theorem states that the character group of $\widehat{G}$ is topologically isomorphic with G, i.e., $\widehat{\widehat{G}} \cong G$. The Fourier transform $\mathcal{F} : L^1(G) \longrightarrow C_0(\widehat{G})$ is defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_G f(x)\overline{\xi}(x)\, d\mu_G(x) \quad (\xi \in \widehat{G}).$$

We can recover a function from its Fourier transform, by the Fourier inversion Theorem. Let $f \in L^1(G)$ such that $\widehat{f} \in L^1(\widehat{G})$, then

$$\mathcal{F}^{-1}\widehat{f}(x) = f(x) = \int_{\widehat{G}} \widehat{f}(\xi)\xi(x)\, d\mu_{\widehat{G}}(\xi) \quad (a.e\ x \in G).$$

The Fourier transform can be extended from $L^1(G) \cap L^2(G)$ to an isometric isomorphism between $L^2(G)$ and $L^2(\widehat{G})$, which is known as the Plancherel transform. See [9, 14, 15].

The mathematical theory for Gabor analysis in $L^2(G)$ is based on two classes of operators on $L^2(G)$. The translation by $\lambda \in G$, which is defined as $T_\lambda f(x) = f(x - \lambda)$, for all $x \in G$. Also, the modulation by $\gamma \in \widehat{G}$, is defined by $E_\gamma f(x) = \gamma(x)f(x)$, $x \in G$. These classes of operators are unitary on $L^2(G)$ and satisfy the following relations;

$$T_\lambda E_\gamma = \overline{\gamma(\lambda)} E_\gamma T_\lambda, \;\; \mathcal{F}T_\lambda = E_{-\lambda}\mathcal{F} \; and \; \mathcal{F}E_\gamma = T_\gamma \mathcal{F}.$$

For a subgroup Λ of an LCA group G its annihilator defined by

$$\Lambda^\perp := \{\gamma \in \widehat{G}; \gamma(\lambda) = 1, \; for\ all\ \lambda \in \Lambda\},$$

which is a closed subgroup of $\widehat{G}$. We denote a closed subgroup Λ of G by $\Lambda \leq G$. It is shown that $\widehat{\Lambda} \cong \frac{\widehat{G}}{\Lambda^\perp}$ and $\widehat{(\frac{G}{\Lambda})} \cong \Lambda^\perp$ [9]. These relations show that for a closed subgroup Λ the quotient $\frac{G}{\Lambda}$ is compact if and only if $\Lambda^\perp$ is discrete. See [9, 14, 19] for more details.

Also, we recall here the Weil's formula, which presents a relationship between the integrable functions on G and the quotient $\frac{G}{\Lambda}$, for a closed subgroup Λ of G. More precisely, consider the canonical map

$\pi_\Lambda : G \longrightarrow \frac{G}{\Lambda}, \pi_\Lambda(x) = x + \Lambda$ from G onto $\frac{G}{\Lambda}$ and let two out of the three Haar measures on G, Λ and $\frac{G}{\Lambda}$ are given. Then the third one can be normalized in a unique approach such that

$$\int_G f(x)\, d\mu_G(x) = \int_{\frac{G}{\Lambda}} \int_\Lambda f(x+\lambda)\, d\mu_\Lambda(\lambda)\, d\mu_{\frac{G}{\Lambda}}(\dot{x}), \tag{1}$$

where $f \in L^1(G)$ and $\dot{x} := \pi_\Lambda(x)$. If (1) holds, then the respective dual measures on $\widehat{G}, \Lambda^\perp \cong \widehat{\frac{G}{\Lambda}}$ and $\frac{\widehat{G}}{\Lambda^\perp} \cong \widehat{\Lambda}$ satisfy

$$\int_{\widehat{G}} \widehat{f}(\xi)\, d\mu_{\widehat{G}}(\xi) = \int_{\frac{\widehat{G}}{\Lambda^\perp}} \int_{\Lambda^\perp} \widehat{f}(\xi+\gamma)\, d\mu_{\Lambda^\perp}(\gamma)\, d\mu_{\widehat{G}/\Lambda^\perp}(\dot{\xi}). \tag{2}$$

In fact, for every two given measures on G, $\widehat{G}$, Λ, $\Lambda^\perp$, and $\frac{\widehat{G}}{\Lambda^\perp}$, so that these two are not the dual measures, then the other measures can be uniquely determined so that (1) and (2) hold simultaneously. For a closed subgroup Λ of G, a Borel section or a fundamental domain is a Borel measurable subset X of G such that every $y \in G$ can be uniquely written as $y = \lambda + x$, where $\lambda \in \Lambda$ and $x \in X$. We equip the Borel section X of G with the restricted Haar measure $\mu_G|_X$. In [6] it is shown that the mapping $x \longmapsto x + \Lambda$ from (X, μ_G) into $(\frac{G}{\Lambda}, \mu_{\frac{G}{\Lambda}})$ is measure-preserving, and the mapping $Q(f) = f'$, defined by

$$f'(x+\Lambda) = f(x), \quad x + \Lambda \in \frac{G}{\Lambda},\ x \in X \tag{3}$$

is an isometry from $L^2(X, \mu_G)$ onto $L^2(\frac{G}{\Lambda}, \mu_{\frac{G}{\Lambda}})$. Also, if Λ is a discrete subgroup, then $\mu_G(X)$ is finite if and only if Λ is co-compact, i.e., Λ is a uniform lattice [5]. For more information of harmonic analysis on locally compact abelian groups, we refer the reader to the classical books [9, 14, 15, 19].

2.1. Frame theory

The major aspect of this paper is related to continuous frames which was introduced in [1]. In what follows, we give some basic definitions and notations of continuous frames.

Definition 2.1. *Let $\mathcal{H}$ be a complex Hilbert space, and let $(M, \sum_M, \mu_M)$ be a measure space, where $\sum_M$ denotes the σ-algebra and μ_M the non-negative measure. A family of vectors $\{f_k\}_{k\in M}$ is called a frame for $\mathcal{H}$ with respect to $(M, \sum_M, \mu_M)$ whenever:*
a) the mapping $M \longrightarrow \mathbb{C}, k \longmapsto \langle f, f_k\rangle$ is measurable for all $f \in \mathcal{H}$,
b) there exist constants $A, B > 0$ such that

$$A \parallel f \parallel^2 \leq \int_M |\langle f, f_k\rangle|^2\, d\mu_M(k) \leq B \parallel f \parallel^2, \quad (f \in \mathcal{H}). \tag{4}$$

The constants A and B in (4) are called frame bounds. If $\{f_k\}_{k\in M}$ is weakly measurable and the upper bound in inequality (4) holds, then $\{f_k\}_{k\in M}$ is said to be a Bessel family with bound B. A frame $\{f_k\}_{k\in M}$ is said to be tight if $A = B$, if furthermore $A = B = 1$ then $\{f_k\}_{k\in M}$ is called a Parseval frame. Also, a family $\{f_k\}_{k\in M}$ is called minimal if $f_j \notin \overline{span}\{f_k\}_{k\neq j}$ for all $j \in M$.

For a Bessel family $\{f_k\}_{k\in M}$ of $\mathcal{H}$, the synthesis operator $T : L^2(M, \mu_M) \longrightarrow \mathcal{H}$ is defined by $T\{c_k\}_{k\in M} = \int_M c_k f_k d\mu_M(k)$, in which the integral is defined in the weak sense and is a bounded linear operator. Its adjoint operator $T^* : \mathcal{H} \longrightarrow L^2(M, \mu_M)$ the *analysis operator*, is obtained by $T^*f = \{\langle f, f_k\rangle\}_{k\in M}$. The frame operator $S : \mathcal{H} \longrightarrow \mathcal{H}$ is defined as $S = TT^*$. We remark that the frame operator is the unique operator satisfying $\langle Sf, g\rangle = \int_M \langle f, f_k\rangle\langle f_k, g\rangle\, d\mu_M(k), (f, g \in \mathcal{H})$ and is well-defined, bounded and self-adjoint for any Bessel system $\{f_k\}_{k\in M}$. Also, it is invertible if and only if $\{f_k\}_{k\in M}$ is a frame [1, 18]. Also, for more information in regard to continuous frame theory on LCA groups, see [3, 7, 16, 17, 20].

Let P be a countable or an uncountable index set, $g_p \in L^2(G)$ for all $p \in P$ and H be a closed co-compact subgroup of G. The translation invariant system generated by $\{g_p\}_{p\in P}$ with translation along the closed co-compact subgroup H is as $\{T_h g_p\}_{h\in H, p\in P}$. Also for a topological space T, let the B_T denote Borel algebra of

T. Then, consider the following standing assumptions of [16, 17];

(I) $(P, \sum_P, \mu_P)$ is a σ-finite measure space,

(II) the mapping $p \longmapsto g_p, (P, \sum_P) \longrightarrow (L^2(G), B_{L^2(G)})$ is measurable,

(III) the mapping $(p, x) \longmapsto g_p(x), (P \times G, \sum_P \bigotimes B_G) \longrightarrow (\mathbb{C}, B_{\mathbb{C}})$ is measurable.

The family $\{g_p\}_{p\in P}$ is called admissible or when g_p is clear from the context, it is simply stated that the measure space P is admissible. The nature of these assumptions are presented in [17]. Every closed subgroup P_j of G with the Haar measure is admissible if $p \longmapsto g_p$ is continuous.

A Gabor system in $L^2(G)$ with the window function $g \in L^2(G)$ is a family of functions of the form $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$, where $\Gamma \subseteq \widehat{G}$ and $\Lambda \subseteq G$. In the following, we derive some sufficient conditions for $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ to constitute a frame for $L^2(G)$ that is a generalization of a known result, see for example Theorem 11.4 of [13] and Corollary 3.5 of [7]. The proof of this result is straightforward and so will be omitted.

Proposition 2.2. *Let $(\Lambda, \mu_\Lambda) \subseteq G$ be an admissible measure space, $\Gamma \leq \widehat{G}$ be a closed and co-compact subgroup and $g \in L^2(G)$. If for all $\alpha \in \Gamma^\perp$ we have $supp g \bigcap supp T_\alpha g = \emptyset$, up to a set of measure zero in G and there exist constants $A, B > 0$ such that*

$$A \leq \int_\Lambda |T_\lambda g(x)|^2 \, d\mu_\Lambda(\lambda) \leq B, \quad a.e.\ (x \in G). \tag{5}$$

Then $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a frame for $L^2(G)$.

Also, the converse of the above result was proven by M. S. Jakobsen et.al., see Corollary 5.6 of [16].

3. Generalized Zak transform

In this section, we address the generalized Zak transform in relation to the continuous Gabor systems on LCA groups. We extend the idea of integer oversampling, proposed in [11], from several aspects. In fact, we deal with closed subgroups instead of uniform lattices and remove some other limited assumptions. Especially, we discuss a new aspect with regards to the existence conditions and provide equivalent conditions for a continuous Gabor system to be a frame family, orthonormal basis, complete and minimal family.

Definition 3.1. *Let Λ be a closed subgroup of G. The Zak transform of a function $f \in L^2(G)$ with respect to Λ is the mapping $Z_\Lambda f$, defined on $G \times \widehat{G}$ as*

$$Z_\Lambda f(x, \xi) = \int_\Lambda f(x + \lambda)\xi(\lambda) \, d\mu_\Lambda(\lambda).$$

It is known that, the continuous Zak transform can be extended to a unitary operator from $L^2(G)$ onto $L^2(M_\Lambda)$, where $M_\Lambda := \frac{G}{\Lambda} \times \frac{\widehat{G}}{\Lambda^\perp}$. The next lemma states the basic properties of continuous Zak transform. See [2, 11].

Lemma 3.2. *Let Λ be a closed subgroup of G and $f \in L^2(G)$. Then*

(I) Quasi-periodicity: $Z_\Lambda f\,(x + \lambda, \gamma + \omega) = \overline{\omega(\lambda)} Z_\Lambda f\,(x, \omega)$, for all $\lambda \in \Lambda$, $\gamma \in \Lambda^\perp$, $x \in G$ and $\omega \in \widehat{G}$

(II) Diagonalization: if $(\gamma, \lambda) \in \Lambda^\perp \times \Lambda$, then $E_\gamma T_\lambda f \in L^2(G)$, and $Z_\Lambda E_\gamma T_\lambda f = E_{\lambda,\gamma} Z_\Lambda f$, where $E_{\lambda,\gamma}(x, \omega) = \gamma(x)\omega(\lambda)$ for all $(x, \omega) \in G \times \widehat{G}$.

Now, let G be an LCA group, $\Lambda \leq G$ and $\Gamma \leq \widehat{G}$ be closed subgroups. Also, let there exist a closed subgroup $H \leq \Lambda$ so that $H^{\perp} \leq \Gamma$. In case Λ, Γ and H are uniform lattices so that the quotients $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^{\perp}}$ are finite, Gröchenig [11] briefly presented the idea of studying the Gabor system $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$ in regard to the Zak transform on H. In what follows, motivated by that idea, we consider Λ, Γ and H closed subgroups and let the quotients $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^{\perp}}$ to be countable. So, in this case we can choose $\lambda_i \in \Lambda$ so that $\Lambda = \cup_{i=1}^{\infty} (\lambda_i + H)$ and each coset of $\frac{\Lambda}{H}$ contains only one λ_i. Moreover, there exist $\gamma_j \in \Gamma$ so that $\Gamma = \cup_{j=1}^{\infty} \left(\gamma_j + H^{\perp}\right)$ and each coset of $\frac{\Gamma}{H^{\perp}}$ contains only one γ_j. Then the frame operator of the Gabor system $\{E_\gamma T_\lambda g\}_{\gamma\in\Gamma,\lambda\in\Lambda}$, for a well-fitted window function g (e.g., $g \in C_c(G)$), can be written as follows

$$\begin{aligned} Sf &= \int_\Gamma \int_\Lambda \langle f, E_\gamma T_\lambda g\rangle E_\gamma T_\lambda g \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\gamma) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{H^\perp}\int_H \langle f, E_\omega T_h T_{\lambda_i} E_{\gamma_j} g\rangle E_\omega T_h T_{\lambda_i} E_{\gamma_j} g \, d\mu_H(h)\, d\mu_{H^\perp}(\omega) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{H^\perp}\int_H \langle f, E_\omega T_h g_{ij}\rangle E_\omega T_h g_{ij} \, d\mu_H(h)\, d\mu_{H^\perp}(\omega) \end{aligned}$$

for all $f \in L^2(G)$ where

$$g_{ij} = T_{\lambda_i} E_{\gamma_j} g. \tag{6}$$

Thus

$$\begin{aligned} Z_H Sf &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{H^\perp}\int_H \langle Z_H f, Z_H\left(E_\omega T_h g_{ij}\right)\rangle Z_H\left(E_\omega T_h g_{ij}\right) \, d\mu_H(h)\, d\mu_{H^\perp}(\omega) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{H^\perp}\int_H \langle Z_H f, E_{\omega,h} Z_H g_{ij}\rangle E_{\omega,h} Z_H g_{ij} \, d\mu_H(h)\, d\mu_{H^\perp}(\omega) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{H^\perp}\int_H \left(\widehat{Z_H f . \overline{Z_H g_{ij}}}\right)(h,\omega) E_{\omega,h} Z_H g_{ij} \, d\mu_H(h)\, d\mu_{H^\perp}(\omega) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} Z_H f . \overline{Z_H g_{ij}} . Z_H g_{ij} \\ &= \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mid Z_H g_{ij} \mid^2\right) Z_H f. \end{aligned}$$

The forthcoming theorem, which collects the above computations, shows that the Zak transform on H diagonalize the Gabor frame operator of $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$. Specifically, the spectrum of the Gabor frame operator equals the range of $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mid Z_H g_{ij} \mid^2$.

Theorem 3.3. *Let g, Λ, Γ and S be as the above and there exists a closed subgroup H of G so that*

$$H \leq \Lambda, \text{ and } H^\perp \leq \Gamma. \tag{7}$$

Moreover, assume that $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are countable. Then, we obtain $Z_H S Z_H^{-1} F = \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mid Z_H g_{ij} \mid^2\right) F$, for all $F \in L^2(M_H)$.

As a special case of Theorem 3.3 we record the following corollaries.

Corollary 3.4. *Let G be an LCA group, $g \in L^2(G)$, $H, \Lambda \leq G$ and $\Gamma \leq \widehat{G}$ be closed subgroups. Then,* *(I)* $Z_H\left(E_\gamma T_\lambda g\right)(x,\omega) = \gamma(\lambda)E_{\lambda,\gamma}(x,\omega)Z_Hg(x,\gamma+\omega)$, *for all* $\lambda \in \Lambda$, $\gamma \in \Gamma$ *and a.e.* $(x,\omega) \in G \times \widehat{G}$.

(II) If the closed subgroup H of G satisfies (7), *then* $Z_H\left(E_\gamma T_\lambda g\right) = E_{\lambda,\gamma}Z_Hg$, *for all* $\lambda \in \Gamma^\perp$ *and* $\gamma \in \Lambda^\perp$.

Proof. To show (*i*), suppose that $g \in L^2(G)$ then

$$\begin{aligned} Z_H\left(E_\gamma T_\lambda g\right)(x,\omega) &= \int_H E_\gamma T_\lambda g(x+h)\omega(h)\,d\mu_H(h) \\ &= \int_H g(x+h-\lambda)\gamma(x)\gamma(h)\omega(h)\,d\mu_H(h) \\ &= \int_H g(x+h)(\gamma+\omega)(h)(\gamma+\omega)(\lambda)\gamma(x)\,d\mu_H(h) \\ &= (\gamma+\omega)(\lambda)\gamma(x)Z_Hg(x,\gamma+\omega) \\ &= \gamma(\lambda)E_{\lambda,\gamma}(x,\omega)Z_Hg(x,\gamma+\omega), \end{aligned}$$

for all $\lambda \in \Lambda$, $\gamma \in \Gamma$, $x \in G$ and $\omega \in \widehat{G}$. The proof of (*ii*) is similar, we only note that in the above computations, we have $\gamma(\lambda) = \gamma(h) = 1$, for all $h \in H$, $\lambda \in \Gamma^\perp$, $\gamma \in \Lambda^\perp$, by using the assumption. □

Corollary 3.5. *Let* g, Λ, Γ, *and H satisfy in conditions of Theorem 3.3, and take the sequence* $\{g_{ij}\}_{i,j=1}^\infty$ *as in* (6). *Then, the following statements hold:*
(I) $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ *is a frame for* $L^2(G)$ *if and only if there exist constants A, B so that* $A \leq \left(\sum_{i=1}^\infty \sum_{j=1}^\infty \mid Z_Hg_{ij}(x,\omega)\mid^2\right) \leq B$, *for a.e.* $(x,\omega) \in G \times \widehat{G}$.
(II) $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ *is a Parseval frame for* $L^2(G)$ *if and only if* $\sum_{i=1}^\infty \sum_{j=1}^\infty \mid Z_Hg_{ij}(x,\omega)\mid^2 = 1$, *for a.e.* $(x,\omega) \in G \times \widehat{G}$.
(III) $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ *is an orthonormal basis for* $L^2(G)$ *if and only if* $\|g\|_{L^2(G)} = 1$ *and* $\sum_{i=1}^\infty \sum_{j=1}^\infty \mid Z_Hg_{ij}(x,\omega)\mid^2 = 1$, *for a.e.* $(x,\omega) \in G \times \widehat{G}$.

Note that Corollary 3.5 and the following Proposition generalize Theorem 11.31 [13] to LCA groups.

Proposition 3.6. *Let* $g \in L^2(G)$, *and H be a closed subgroup of G. Then, the following statements hold:*
(I) $\{E_\gamma T_\lambda g\}_{\lambda\in H,\gamma\in H^\perp}$ *is a complete system in* $L^2(G)$ *if and only if* $Z_Hg \neq 0$, *a.e..*
(II) If H is a uniform lattice, then $\{E_\gamma T_\lambda g\}_{\lambda\in H,\gamma\in H^\perp}$ *is a minimal system in* $L^2(G)$ *if and only if* $\dfrac{1}{\overline{Z_Hg}} \in L^2(M_H)$.

Proof. (*I*); Let $Z_Hg \neq 0$ a.e., to show the Gabor system $\{E_\gamma T_\lambda g\}_{\lambda\in H,\gamma\in H^\perp}$ is complete in $L^2(G)$, it is sufficient to prove $\{E_{\lambda,\gamma}Z_Hg\}_{\lambda\in H,\gamma\in H^\perp}$ is complete in $L^2(M_H)$, by the unitarity of Z_H and Lemma 3.2 (*II*). To this end, let $\Phi \in L^2(M_H)$ such that $\langle\Phi, E_{\lambda,\gamma}Z_Hg\rangle_{L^2(M_H)} = 0$, for all $\lambda \in H$, $\gamma \in H^\perp$. So we can write

$$\int_{\frac{G}{H}}\int_{\frac{\widehat{G}}{H^\perp}} \Phi(\alpha,\beta)\overline{Z_Hg(\alpha,\beta)E_{\lambda,\gamma}(\alpha,\beta)}\,d\mu_{\frac{G}{H}}(\dot{\alpha})\,d\mu_{\frac{\widehat{G}}{H^\perp}}(\dot{\beta}) = \langle\Phi, E_{\lambda,\gamma}Z_Hg\rangle_{L^2(M_H)} = 0,$$

for all $\lambda \in H$, $\gamma \in H^\perp$. Since $\phi.\overline{Z_Hg} \in L^1(M_H)$ and the functions in $L^1(M_H)$ are uniquely determined by their Fourier coefficients. Thus, we have $\Phi.\overline{Z_Hg} = 0$, a.e., and by the assumption $\Phi = 0$ a.e., proving the claim. For the converse, suppose the Gabor system $\{E_\gamma T_\lambda g\}_{\lambda\in H,\gamma\in H^\perp}$ is a complete family in $L^2(G)$. Then $\{E_{\lambda,\gamma}Z_Hg\}_{\lambda\in H,\gamma\in H^\perp}$ is also complete in $L^2(M_H)$. On the contrary, let $\Delta_g = \{(\alpha,\beta) \in M_H : Z_Hg(\alpha,\beta) = 0\}$ has a positive measure. Put $\Phi = \chi_{\Delta_g}$, that χ_{Δ_g} is the characteristic function on Δ_g. Then we obtain $\langle\Phi, E_{\lambda,\gamma}Z_Hg\rangle_{L^2(M_H)} = 0$, for all $\lambda \in H$, $\gamma \in H^\perp$, which is a contradiction. So, $Z_Hg \neq 0$ a.e..

The proof of (*II*) is obtained by an adaptive approach of Theorem 11.31 [13]. □

Remark 3.7. *It is worth of note that for the closed subgroups* Λ,Γ *and H which satisfy* (7) *we have* $\Gamma^\perp \times \Lambda^\perp \subseteq H \times H^\perp \subseteq \Lambda \times \Gamma$. *So, if* $Z_H \neq 0$ *a.e. then the Gabor system* $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ *is complete in* $L^2(G)$. *However, as the following examples show, the Gabor system* $\{E_\gamma T_\lambda g\}_{\lambda\in\Gamma^\perp,\gamma\in\Lambda^\perp}$ *is not necessarily complete. Moreover, in the case of* $\dfrac{1}{\overline{Z_Hg}} \in L^2(M_H)$ *the Gabor system* $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ *is not minimal, in general.*

Example 3.8. *(I) Consider the Gaussian function $\phi(x) = e^{-\pi x^2}$. It has already been proven in [12] that the Gabor system $\{E_{m\alpha}T_{n\beta}\phi\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ is complete for $\alpha\beta \leq 1$ and incomplete for $\alpha\beta > 1$. Moreover, $Z_{\alpha\mathbb{Z}}\phi \neq 0$ a.e. on $[0,\alpha) \times \left[0, \frac{1}{\alpha}\right)$, for all non-zero $\alpha \in \mathbb{R}$. Let $H = 4\mathbb{Z}$, $\Lambda = 2\mathbb{Z}$ and $\Gamma = \frac{1}{8}\mathbb{Z}$, then the closed subgroups H, Λ and Γ satisfy (7). So, the Gabor system $\{E_\gamma T_\lambda \phi\}_{\lambda\in H,\gamma\in H^\perp}$ is complete, and so $Z_H\phi \neq 0$, a.e., by Proposition 3.6. Although, the Gabor system $\{E_\gamma T_\lambda\phi\}_{\gamma\in\Lambda^\perp,\lambda\in\Gamma^\perp}$ is incomplete.*

(II) Fix $0 < \alpha < 1$, set $g(x) = |x|^\alpha$ for $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. It is known that the system $\{T_n E_m g\}_{n,m\in\mathbb{Z}}$ is a Schauder basis for $L^2(\mathbb{R})$ (but not Riesz basis for $L^2(\mathbb{R})$), [8]. So this system is minimal and complete. Take $\Lambda = \frac{1}{2}\mathbb{Z}$, $\Gamma = \frac{1}{4}\mathbb{Z}$ and $H = \mathbb{Z}$, then the closed subgroups H, Λ and Γ satisfy (7). We observe that the Gabor system $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is not minimal, also the Gabor system $\{E_\gamma T_\lambda g\}_{\gamma\in\Lambda^\perp,\lambda\in\Gamma^\perp}$ is not complete in $L^2(\mathbb{R})$.

3.1. The existence conditions

In what follows, for two given closed subgroups Λ and Γ, we discuss the existence of a closed subgroup H which satisfies (7) so that the quotient groups $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are countable. We first note that, for every LCA group G, the condition $\Gamma^\perp \leq \Lambda$ is necessary for the existence of H. Moreover obviously, for countable groups it can be considered as a necessary and sufficient condition; especially, for finite group $G = \mathbb{Z}_L = \{0, ..., L-1\}$, $L \in \mathbb{N}$, it is known that a closed subgroup of G is as $\Lambda = N\mathbb{Z}_{\frac{L}{N}}$ where $N \in \mathbb{N}$ so that N is a divisor of L. Also, consider the closed subgroups $\Gamma = M\mathbb{Z}_{\frac{L}{M}}$ and $H = R\mathbb{Z}_{\frac{L}{R}}$ so that $M, R \in \mathbb{N}$ are some divisors of L. In this case the condition (7) is equivalent to

$$N \mid R \mid L \ \text{ and } \ M \mid L/R. \tag{8}$$

More precisely, (8) is the necessary and sufficient condition for the subgroup $H = R\mathbb{Z}_{\frac{L}{R}}$ of G to satisfy (7).

Lemma 3.9. *Assume that $H \leq \Lambda$ are closed subgroups of G so that $\frac{\Lambda}{H}$ is finite. Then $\frac{\Lambda}{H} \cong \frac{H^\perp}{\Lambda^\perp}$.*

Proof. Applying Proposition 4.2.24 of [19] and the fact that any finite group is self-dual we obtain $\frac{\Lambda}{H} \cong \widehat{\left(\frac{\Lambda}{H}\right)} \cong \frac{H^\perp}{\Lambda^\perp}$. □

Example 3.10. *Suppose that $G = Q_p$, the P–adic numbers group, it is known that every non-trivial closed subgroup H of Q_p is open and compact [9, 14]. Hence, $\frac{G}{H}$ is infinite and discrete, consequently for every two non-trivial closed subgroups Λ and H of Q_p so that $H \leq \Lambda$, the quotient group $\frac{\Lambda}{H}$ is both discrete and compact and so is finite. Similarly, $\frac{\Gamma}{H^\perp}$ is finite, for a subgroup $\Gamma \leq \widehat{G}$. By Lemma 3.9 we have $\dfrac{\Gamma}{H^\perp} \cong \dfrac{H}{\Gamma^\perp}$ and so $\dfrac{H}{\Gamma^\perp}$ is finite as well. That means every non-trivial closed subgroup H of Q_p with the property $\Gamma^\perp \leq H \leq \Lambda$ satisfies the condition (7).*

Example 3.11. *Consider $G = \mathbb{R} \times Z_p$, where Z_p is the group of P–adic integers, and let Λ, Γ be two non-trivial closed subgroups of G and $\widehat{G}$, respectively. So $\Lambda = \alpha\mathbb{Z} \times \Lambda_2$, for some $\alpha \in \mathbb{R}$ and Λ_2 is a closed subgroup of Z_p. We show that for every closed subgroup H which satisfies $\Gamma^\perp \leq H \leq \Lambda$, the quotients $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are finite. Indeed $H \leq \Lambda \lneq \mathbb{R} \times Z_p$ implies that $H = \alpha m\mathbb{Z} \times H_2$, for some $m \in \mathbb{Z}$ and $H_2 \leq \Lambda_2 \leq Z_p$. Hence a similar discussion as in Example 3.10 assures that $\frac{\Lambda_2}{H_2}$ is finite and consequently*

$$\frac{\Lambda}{H} = \frac{\alpha\mathbb{Z}}{\alpha m\mathbb{Z}} \times \frac{\Lambda_2}{H_2}$$

is a finite group. Moreover, $\Gamma^\perp \leq H \leq Z_p$ also follows that $\frac{H}{\Gamma^\perp}$ is finite and so by Lemma 3.9 $\frac{\Gamma}{H^\perp}$ is finite, as well.

In the sequel, we show that, $\Gamma^\perp \leq \Lambda$ is not a sufficient condition for the existence of the desired H with countable quotient groups, in general.

Example 3.12. *Let $G = \mathbb{R}^n$ and $\Lambda = \Gamma = \mathbb{R} \times \mathbb{Z}^{n-1}$. Then $\Gamma^\perp \leq \Lambda$ and for every closed subgroup H so that $\Gamma^\perp \leq H \leq \Lambda$ we can write $H = H_1 \times H_2$ where $H_1 \leq \mathbb{R}$ and $H_2 \leq \mathbb{Z}^{n-1}$. If $H_1 \neq \mathbb{R}$, then $H_1 = \alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$. Thus both $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are uncountable. Moreover, if $H_1 = \mathbb{R}$ i.e. $H = \mathbb{R} \times \mathbb{Z}^{n-1}$ then $\frac{\Gamma}{H^\perp}$ is uncountable.*

Suppose G is a compactly generated group of Lie type, that is isomorphic to one of the form $G = \mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{T}^r \times F$ for $n, m, r \in \mathbb{N}$ and a finite abelian group F. Consider closed subgroups $\Lambda = \Lambda_1 \times \Lambda_2 \times \Lambda_3 \times \Lambda_4 \leq G$ and $\Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3 \times \Gamma_4 \leq \widehat{G}$ so that $\Lambda_1 = \Gamma_1 = \mathbb{R} \times \mathbb{Z}^{n-1}$, and $\Gamma_i = \Lambda_i^\perp$, for $2 \leq i \leq 4$. Then for any closed subgroup H so that $\Gamma^\perp \leq H \leq \Lambda$ we have that $H = H_1 \times \Lambda_2 \times \Lambda_3 \times \Lambda_4$ and so $H^\perp = H_1^\perp \times \Gamma_2^\perp \times \Gamma_3^\perp \times \Gamma_4^\perp$ by lemma 4.2.8 of [19]. Consider $H_1 < \Lambda_1 = \mathbb{R} \times \mathbb{Z}^{n-1}$, thus by Example 3.12, $\frac{\Gamma_1}{H_1^\perp}$ is uncountable and consequently $\frac{\Gamma}{H^\perp}$ is uncountable, as well.

In the next result, we investigate some sufficient conditions for the existence of subgroup H which satisfies (7) so that $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are finite or countable.

Theorem 3.13. *Let Λ and Γ be subgroups of G and $\widehat{G}$, respectively so that $\Gamma^\perp \leq \Lambda$. Then the following assertions hold;*

(I) If Λ and Γ are discrete subgroups, then for every subgroup H such that $\Gamma^\perp \leq H \leq \Lambda$, the quotient groups $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are finite.

(II) If Λ and Γ are open subgroups, then there exists a closed subgroup H which satisfies (7) so that either $\frac{\Lambda}{H}$ or $\frac{\Gamma}{H^\perp}$ is countable.

(III) If G is totally-disconnected and Λ, Γ are open subgroups, then there exists a compact subgroup H which satisfies (7) so that both $\frac{\Lambda}{H}$ and $\frac{\Gamma}{H^\perp}$ are countable.

Proof. (I) Consider a subgroup H of G such that $\Gamma^\perp \leq H \leq \Lambda$. Then, the assumption assures that Λ, Γ and H are uniform lattices. On the other hand, by Proposition 4.2.24 of [19], we have $\widehat{(\frac{H^\perp}{\Lambda^\perp})} \cong \frac{\Lambda}{H}$ and so the duality relationships of $\widehat{(\frac{G}{\Lambda})} \cong \Lambda^\perp$ and $\widehat{\Lambda} \cong \frac{\widehat{G}}{\Lambda^\perp}$ imply that $\frac{\Lambda}{H}$ is both compact and discrete. Hence $\frac{\Lambda}{H}$ is finite, similarly the quotient group $\frac{\Gamma}{H^\perp}$ is finite.

(II) Since Γ is an open subgroup of $\widehat{G}$, the duality relation $\Gamma^\perp \cong \widehat{(\frac{\widehat{G}}{\Gamma})}$ implies that $\Gamma^\perp$ is compact. So, by proposition 3.1.5 of [19], there exists a unit neighborhood V of e (the identity of G) such that $\Gamma^\perp + V \subseteq \Lambda$. If we take $H := \Gamma^\perp + \langle V \rangle$, where $\langle V \rangle$ is the subgroup generated by V, then H satisfies (7) and is open in Λ. Thus, $\frac{\Lambda}{H}$ is discrete and countable but $\frac{\Gamma}{H^\perp}$ is not necessarily countable. On the other hand, the structure of V assure that there exists a compact subgroup N of G such that $N \subseteq V$, [14]. Set $H := \Gamma^\perp + N$. Then H is a compact subgroup of G and $\Gamma^\perp \leq H \leq \Lambda$, i.e., H satisfies (7). Moreover,

$$\frac{\widehat{G}}{H^\perp} = \frac{\widehat{G}}{\Gamma \cap N^\perp} \cong (\widehat{\Gamma^\perp + N})$$

is discrete and so countable. Therefore, $\frac{\Gamma}{H^\perp}$ is also countable, but not necessarily $\frac{\Lambda}{H}$.

(III) Using the proof of (II) there exists a unit neighborhood V of e so that $\Gamma^\perp + V \subseteq \Lambda$. The assumption that G is totally-disconnected deduces that there exists an open compact subgroup K so that $K \subseteq V$, by Theorem 7.7 of [14]. Put $H := \Gamma^\perp + K$. Then H is a compact open subgroup of G and $\Gamma^\perp \leq H \leq \Lambda$. Moreover, H is open in Λ. Thus $\frac{\Lambda}{H}$ is countable. An analogous discussion shows that $\frac{\Gamma}{H^\perp}$ is countable as well. This completes the proof. □

4. Fibrization method

The fiberization technique is closely related to Zak transform methods in Gabor analysis. Let H be a closed and co-compact subgroup of G and $\Omega \subset \widehat{G}$ be a Borel section of $H^\perp$ in $\widehat{G}$, we use the fiberization mapping which was introduced in [6] $\mathcal{T} : L^2(G) \longrightarrow L^2(\Omega, l^2(H^\perp))$, as follows

$$\mathcal{T} f(\omega) = \{\widehat{f}(\omega + \alpha)\}_{\alpha \in H^\perp}, \quad (\omega \in \Omega).$$

The fiberization is an isometric isomorphic operation as was shown in [6]. Furthermore the frame property of translation-invariant and Gabor system can be characterized in terms of fibers.

Theorem 4.1. *[16] Let A and B be two positive constants and let $H \leq G$ be a closed, co-compact subgroup and let $\{g_p\}_{p\in P} \subseteq L^2(G)$, where (P, μ_P) is an admissible measure space. Then the following assertions are equivalent.*

(I) The family $\{T_h g_p\}_{h\in H, p\in P}$ is a frame for $L^2(G)$ with bounds A and B.

(II) For almost every $\omega \in \Omega$, the family $\{\mathcal{T} g_p(\omega)\}_{p\in P}$ is a frame for $l^2(H^\perp)$ with bounds A and B, where Ω is a Borel section of $H^\perp$ in $\widehat{G}$.

The next result shows that the frame property of a Gabor system in $L^2(G)$ under certain assumptions is equivalent with the frame property of a family of associated Zak transforms in $l^2(\widehat{H^\perp})$.

Theorem 4.2. *Let $g \in L^2(G)$, Λ and Γ be closed subgroups of G and $\widehat{G}$ respectively and let H be a closed, co-compact subgroup of G which satisfies (7). Then there exists a sequence $\{g_{ku}\}_{k\in\frac{\Lambda}{H}, u\in\frac{\Gamma}{\Lambda^\perp}}$ in $L^2(G)$ such that following assertions are equivalent.*

(I) $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a frame for $L^2(G)$ with bounds A and B.

(II) $\{Z_{H^\perp}\widehat{g_{ku}}(\omega, .)\}_{k\in\frac{\Lambda}{H}, u\in\frac{\Gamma}{\Lambda^\perp}}$ is a frame for $l^2(\widehat{H^\perp})$ with bounds A and B, for a.e. $\omega \in \Omega$, where Ω is a Borel section of $H^\perp$ in $\widehat{G}$.

Proof. Since $H \leq \Lambda$, so every $\lambda \in \Lambda$ can be written uniquely as $\lambda = t + k$ where $t \in H$ and $k \in \frac{\Lambda}{H}$. Also, $\Lambda^\perp \leq \Gamma$ implies that every $\gamma \in \Gamma$ has a unique form such as $\gamma = \mu + u$ where $\mu \in \Lambda^\perp$ and $u \in \frac{\Gamma}{\Lambda^\perp}$. Thus,

$$\{T_\lambda E_\gamma g\}_{\lambda\in\Lambda,\gamma\in\Gamma} = \{T_t E_\mu g_{ku}\}_{t\in H, k\in\frac{\Lambda}{H}, \mu\in\Lambda^\perp, u\in\frac{\Gamma}{\Lambda^\perp}}$$

where $\mathcal{G} := \{g_{ku}\} = \{T_k E_u g\}_{k\in\frac{\Lambda}{H}, u\in\frac{\Gamma}{\Lambda^\perp}}$. Therefore, applying the fiberization method along with Theorem 4.1 for co-compact subgroup H of G, the system $\{T_t E_\mu \mathcal{G}\}_{t\in H, \mu\in\Lambda^\perp}$ (or equivalently $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$) is a frame for $L^2(G)$ if and only if $\{\mathcal{T} E_\mu \mathcal{G}(\omega)\}_{\mu\in\Lambda^\perp}$ is a frame in $l^2(H^\perp)$, for a.e $\omega \in \Omega$ where Ω is a Borel section of $H^\perp$ in $\widehat{G}$. On the other hand, we obtain

$$\begin{aligned}
\{\mathcal{T} E_\mu \mathcal{G}(\omega)\}_{\mu\in\Lambda^\perp} &= \{\mathcal{T} E_\mu g_{ku}(\omega)\}_{\mu\in\Lambda^\perp, k\in\frac{\Lambda}{H}, u\in\frac{\Gamma}{\Lambda^\perp}} \\
&= \{\{\widehat{E_\mu g_{ku}}(\omega + \alpha)\}_{\alpha\in H^\perp}\}_{\mu\in\Lambda^\perp, k\in\frac{\Lambda}{H}, u\in\frac{\Gamma}{\Lambda^\perp}} \\
&= \{\{T_\mu \widehat{g_{ku}}(\omega + \alpha)\}_{\alpha\in H^\perp}\}_{\mu\in\Lambda^\perp, k\in\frac{\Lambda}{H}, u\in\frac{\Gamma}{\Lambda^\perp}} \\
&= \{\{\widehat{g_{ku}}(\omega + \alpha)\}_{\alpha\in H^\perp}\}_{k\in\frac{\Lambda}{H}, u\in\frac{\Gamma}{\Lambda^\perp}},
\end{aligned}$$

where the last equality is due to the assumption (7). Consider $\psi_{k,u}(\omega) := \{\widehat{g_{ku}}(\omega+\alpha)\}_{\alpha\in H^\perp}$ for all $k \in \frac{\Lambda}{H}$, $u \in \frac{\Gamma}{\Lambda^\perp}$ and a.e $\omega \in \Omega$. Then the Fourier inversion transform of $\psi_{k,u}(\omega) \in l^2(H^\perp)$ is as follows

$$\begin{aligned}
\mathcal{F}^{-1}(\psi_{k,u}(\omega))(\xi) &= \sum_{\alpha\in H^\perp} \widehat{g_{ku}}(\omega + \alpha)\alpha(\xi) \\
&= Z_{H^\perp}\widehat{g_{ku}}(\omega, \xi)
\end{aligned}$$

for all $k \in \frac{\Lambda}{H}$, $u \in \frac{\Gamma}{\Lambda^\perp}$ and a.e., $\xi \in \widehat{H^\perp}$ and $\omega \in \Omega$. Hence, the assertion (I) is equivalent to the system $\{Z_{H^\perp}\widehat{g_{ku}}(\omega, .)\}_{k\in\frac{\Lambda}{H}, u\in\frac{\Gamma}{\Lambda^\perp}}$ being a frame for $l^2(\widehat{H^\perp})$, a.e $\omega \in \Omega$, as required. □

In the next corollary, we deduce some connections between the results obtained in [16].

Corollary 4.3. *Let $g \in L^2(G)$, Λ be a closed co-compact subgroup of G and Γ be a closed subgroup of $\widehat{G}$ so that $\Gamma^\perp \leq \Lambda$. Then the following assertions are equivalent.*

(I) $\{E_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a frame for $L^2(G)$ with bounds A and B.

(II) $\{\widehat{g}(\alpha+\gamma)\}_{\gamma\in\Gamma}$ *is a frame for* $l^2(\Lambda^\perp)$ *with bounds A and B, for a.e.* $\alpha\in\mathcal{A}$*, where* $\mathcal{A}$ *is a Borel section of* $\Lambda^\perp$ *in* $\widehat{G}$*.*

(III) $A\leq\int_{\mathcal{K}}\left|Z_{\Lambda^\perp}\widehat{g}(\alpha+k,x)\right|^2 d\mu_{\mathcal{K}}(k)\leq B$*, for a.e.* $\alpha\in\mathcal{A}$ *and* $x\in\widehat{\Lambda^\perp}$*, where* $\mathcal{A}$ *is a Borel section of* $\Lambda^\perp$ *in* $\widehat{G}$*,* $\mathcal{K}\subset\Gamma$ *is a Borel section of* $\Lambda^\perp$ *in* Γ*.*

Proof. $(I)\Leftrightarrow(II)$; We note that $(\Gamma,\Sigma_\Gamma,\mu_\Gamma)$ is an admissible measure space, since Γ is a closed subgroup of $\widehat{G}$. Hence, by Proposition 4.5 in [16], (I) is equivalent to $\{\{\widehat{g}(\alpha+\gamma+y)\}_{y\in\Lambda^\perp}\}_{\gamma\in\Gamma}=\{\widehat{g}(\alpha+\gamma)\}_{\gamma\in\Gamma}$ being a frame for $l^2(\Lambda^\perp)$ with bounds A and B, for a.e. $\alpha\in\mathcal{A}$, where $\mathcal{A}$ is a Borel section of $\Lambda^\perp$ in $\widehat{G}$.

$(I)\Leftrightarrow(III)$ Applying the fact that Λ is co-compact and $\Lambda^\perp\cap\Gamma=\Lambda^\perp$, it is sufficient to take a Haar measure μ_Γ on Γ and a unique Haar measure $\mu_{\frac{\Gamma}{\Lambda^\perp}}$ on $\frac{\Gamma}{\Lambda^\perp}$ so that for all $f\in L^1(\Gamma)$ we have

$$\int_\Gamma f(x)\,d\mu_\Gamma(x)=\int_{\frac{\Gamma}{\Lambda^\perp}}\sum_{l\in\Lambda^\perp}f(x+l)\,d\mu_{\frac{\Gamma}{\Lambda^\perp}}(\dot{x}).$$

In addition, consider $\mathcal{K}\subset\Gamma$ a Borel section of $\Lambda^\perp$ in Γ and set a measure on $\mathcal{K}$ isometric to $\mu_{\Gamma/\Lambda^\perp}$ in the sense of (3). Then the desired result is obtained by Theorem 4.6 in [16]. □

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