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GENERAL DECAY AND BLOW UP OF SOLUTIONS FOR A PLATE VISCOELASTIC $p(x)$-KIRCHHOFF TYPE EQUATION WITH VARIABLE EXPONENT NONLINEARITIES AND BOUNDARY FEEDBACK

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Abstract. In this paper, we consider a plate viscoelastic $p(x)-$Kirchhoff type equation with variable-exponent nonlinearities of the form

$$u_{tt} + \Delta^2 u - \left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u - \int_0^t g(t-s)\Delta^2 u(s)ds$$
$$+\beta\Delta^2 u_t + |u_t|^{m(x)-2}u_t = |u|^{q(x)-2}u,$$

associated with initial and boundary feedback. Under appropriate conditions on $p(\cdot), m(\cdot)$ and $q(\cdot)$, general decay result along the solution energy is proved. By introducing a suitable auxiliary function, it is also shown that regarding negative initial energy and a suitable range of variable exponents, solutions blow up in a finite time.

Mathematics Subject Classification (2020): 35B35, 35B44.

Key words: General decay, blow-up, viscoelastic, p(x)-Kirchhoff type equation.

1. Introduction.

Let Ω be a bounded domain of $R^n (n \geq 1)$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Here Γ_0 and Γ_1 are closed and disjoint with positive measures. Consider the following plate viscoelastic $p(x)-$Kirchhoff type equation

$$u_{tt} + \Delta^2 u - \left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u - \int_0^t g(t-s)\Delta^2 u(s)ds + \beta\Delta^2 u_t$$

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$$+|u_t|^{m(x)-2}u_t = |u|^{q(x)-2}u, \ x \in \Omega, t > 0 \tag{1}$$

$$\begin{cases} u(x,t) = 0, & x \in \Gamma_0, t > 0 \\ \Delta u(x,t) = \int_0^t g(t-s)\Delta u(s)ds - \beta \Delta u_t, & x \in \Gamma_1, t > 0 \end{cases} \tag{2}$$

$$u(x,0) = u_0, \ u_t(x,0) = u_1(x), \ x \in \Omega \tag{3}$$

where $\Delta_{p(x)}$ is called $p(x)-$Laplacian operator defined as

$$\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u),$$

and $\beta, a, b > 0$. Here, we have the following conditions on the variable exponents:
$(A1)$ the exponents $p(\cdot)$, $m(\cdot)$ and $q(\cdot)$ are given measurable functions on $\overline{\Omega}$ such that:

$$2 < p_1 \le p(x) \le p_2 < \infty,$$

$$2 < m_1 \le m(x) \le m_2 < \infty,$$

$$2 < q_2 \le q(x) \le q_2 < \infty,$$

with

$$p_1 := \operatorname*{ess\,inf}_{x\in\overline{\Omega}} p(x), \quad p_2 := \operatorname*{ess\,sup}_{x\in\overline{\Omega}} p(x),$$

$$m_1 := \operatorname*{ess\,inf}_{x\in\overline{\Omega}} m(x), \quad m_2 := \operatorname*{ess\,sup}_{x\in\overline{\Omega}} m(x),$$

$$q_1 := \operatorname*{ess\,inf}_{x\in\overline{\Omega}} q(x), \quad q_2 := \operatorname*{ess\,sup}_{x\in\overline{\Omega}} q(x).$$

$(A2)$ The kernel of memory, $g : R^+ \to R^+$, is a differentiable and non-increasing function satisfying

$$g(0) > 0, \quad 1 - \int_0^{+\infty} g(s)ds = l > 0.$$

Torrejón and Yang [37] studied the following equation

$$u_{tt} - M(\|\nabla u\|^2_{L^2(\Omega)})\Delta u - \int_0^t a'(t-\tau)N(\|\nabla u\|^2_{L^2(\Omega)})\Delta u(\tau)d\tau + h(u_t) = f(u), \tag{4}$$

and when $h \equiv 0$, they showed that under appropriate assumptions on the kernel of the memory a', the functions, M, N, the right-hand side f and the data, solutions to (4) are unique, global in time, and their derivatives are weakly convergent to zero in $L^2(\Omega)$, as t tends to $+\infty$. Many existence and blow-up results for Kirchhoff type equations with various initial-boundary value conditions have been proved in the literature. For example, when the initial energy has an upper bound, Wu and Tsai [38] proved the existence and blow-up of solutions for the equation (4) with $M = m_0 + bs^\gamma, h(u_t) = a|u_t|^{\nu-2}u_t + a|u_t|^{m-2}u_t$ and $f(u) = |u|^{p-2}u$. Yang and Gong [39] studied equation (4) with $M = 1 + bs^\gamma, h(u_t) = u_t$ and $f(u) = |u|^{p-2}u$. They proved under certain assumptions on the kernel g and the initial data, solutions blow-up in a finite time with positive initial energy. In another study, Peyravi and

Tahamtani obtained a blow-up result for this model with a strong damping term Δu_t in [22].

In the plate model of Kirchhoff type equations, Pişkin [23] considered the following extensible beam equation with nonlinear damping and source term

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u + |u_t|^{p-1}u_t = |u|^{q-1}u.$$

He established the existence of the solution by the Banach contraction mapping principle and decay result by using Nakao's inequality. Moreover, under suitable conditions on the initial datum, the blow up of solutions has been proved. In this regards we refer to [19, 40, 18, 8, 20, 1, 26].

The problems with variable exponents arise in many branches of sciences such as flows of electro-rheological fluids, nonlinear viscoelasticity, and image processing [9, 11, 29]. Pişkin [25], proved the blow up of solutions for the following Kirchhoff-type equation:

$$u_{tt} - M(\|\nabla u\|^2)\Delta u + |u_t|^{p(x)-2}u_t = |u|^{q(x)-2}u.$$

In [36], Shahrouzi and Kargarfard considered the following Kirchhoff-type problem:

$$u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta_{m(x)}u + h(x,t,u,\nabla u) + \beta u_t = \phi_{p(x)}(u), \quad in \quad \Omega \times (0,+\infty)$$

$$\begin{cases} u(x,t) = 0, & (x,t) \in \Gamma_0 \times (0,+\infty) \\ M(\|\nabla u\|^2)\frac{\partial u}{\partial n}(x,t) = \alpha u - |\nabla u|^{m(x)}\frac{\partial u}{\partial n}, & (x,t) \in \Gamma_1 \times (0,+\infty) \end{cases}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad in \quad \Omega,$$

where $\phi_{p(x)}(u) = |u|^{p(x)}u$ and $\Delta_{m(x)}$ is $m(x)$-Laplacian operator. They proved the blow up of solutions with positive initial energy and suitable conditions on datum. Recently, Antontsev et. al. [5] investigated the following nonlinear fourth-order Timoshenko equation with variable exponents:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2_{L^2(\Omega)})\Delta u + |u_t|^{p(x)-2}u_t = |u|^{q(x)-2}u,$$

and proved the local existence of the solution under suitable conditions. Moreover, the nonexistence of solutions has been proved with negative initial energy.

Dai and Hao [10] studied the following equation

$$-M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)div(|\nabla u|^{p(x)-2}\nabla u) = f(x,u).$$

Employing a direct variational approach and the theory of the variable exponent Sobolev spaces, they established conditions ensuring the existence and multiplicity of solutions for the problem. Shahrouzi and Ferreira [31] considered the following $r(x)-$Kirchhoff type equation with variable-exponent nonlinearity

$$u_{tt} - \Delta u - \big(a + b\int_\Omega \frac{1}{r(x)}|\nabla u|^{r(x)}dx\big)\Delta_{r(x)}u + \beta u_t = |u|^{p(x)-2}u,$$

and proved that for sufficiently large β and under appropriate conditions on $r(\cdot)$ and $p(\cdot)$, solutions are asymptotically stable. Moreover, they established regarding arbitrary positive initial energy and a suitable range of variable exponents, solutions blow up in a finite time. Recently, Shahrouzi et al. [34] investigated the following initial-boundary value problem which involves the viscoelastic fourth-order $p(x)$-Laplacian operator and the variable-exponent nonlinearities

$$
\begin{aligned}
&|\partial_t u|^{\rho(x)-2}\partial_{tt}u + \mathcal{L}u - \int_0^t \mu(t-s)\Delta_x^2 u(s)ds - \Delta_x \partial_{tt}u + |\partial_t u|^{m(x)-2}\partial_t u \\
&= |u|^{q(x)-2}u, \quad (x,t) \in \Omega \times (0,T) \\
&\begin{cases} u(x,t) = \frac{\partial u(x,t)}{\partial \nu} = 0, & (x,t) \in \Gamma_0 \times (0,+\infty) \\ M(\Delta_x u) - \int_0^t \mu(t-s)\Delta_x u(x,s)ds = 0, & (x,t) \in \Gamma_1 \times (0,+\infty) \end{cases} \\
&u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x), \qquad x \in \Omega,
\end{aligned}
$$

where $\mathcal{L}$ is an operator involving the biharmonic and fourth-order $p(x)$-Laplacian operators which defined as

$$
\mathcal{L}u := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} M\left(\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}\right),
$$

for any $u \in H^2(\Omega)$, where $M(v) = \left(1 + a|v|^{p(x)-2}\right)v$, and a is a positive constant. They proved the global existence of solutions, general decay and blow-up results with positive initial energy as well as negative, under appropriate conditions on initial data. In another study, Shahrouzi et al. [35] proved the global existence, asymptotic stability and finite time blow-up of solutions for the following viscoelastic plate equation involving $(p(x), q(x))$-Laplacian operator

$$
\begin{aligned}
&u_{tt} + \Delta^2 u - \operatorname{div}[(|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2})\nabla u] - \int_0^t g(t-s)\Delta^2 u(s)ds - \xi \Delta u_t \\
&= \alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u.
\end{aligned}
$$

The relevant equations with variable-exponent nonlinearities have also been studied in [2, 3, 4, 7, 16, 17, 24, 30, 32, 33].

Motivated by the above mentioned works, in this paper, we consider a viscoelastic $p(x)-$Kirchhoff type of plate equation (1) with nonlinear boundary conditions (2). Under appropriate conditions on the kernel of memory and variable exponents, we prove general decay and blow up of solutions with negative initial energy. This work improve and extend results in the literature to problems involving fourth-order viscoelastic $p(x)-$Kirchhoff type equation with variable exponent nonlinearities and boundary feedback.

This manuscript is written as follows. In Section 2, we present some definitions and Lemmas about the variable-exponent Lebesgue space, $L^{p(\cdot)}(\Omega)$, the Sobolev space, $W^{1,p(\cdot)}(\Omega)$, to be used for the main results. In Section 3, we prove the general decay of solutions for appropriate initial data. Finally, the blow up of solutions has been proved with negative initial energy and suitable conditions on datum, in the fourth Section.

2. Preliminaries. In order to study problem (1)-(3), we need some theories about Lebesgue and Sobolev spaces with variable-exponents (see [6, 11]).

Let $p : \Omega \to [1, \infty]$ be a measurable function, where Ω is a domain of R^n. We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \Big\{ u | \; u \text{ is measurable in } \Omega \text{ and } \int_\Omega |\lambda u(x)|^{p(x)} dx < \infty \text{ for some } \lambda > 0 \Big\}.$$

We equip the Lebesgue space with a variable exponent, $L^{p(\cdot)}(\Omega)$, with the following Luxembourg-type norm

$$\|u\|_{p(x)} := \inf \Big\{ \lambda > 0 \Big| \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1 \Big\}.$$

LEMMA 2.1. ([11]) *Let Ω be a bounded domain in R^n*
(i) the space $(L^{p(\cdot)}(\Omega), \|.\|_{p(\cdot)})$ is a Banach space, and its conjugate space is $L^{q(\cdot)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$.
(ii) For any $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, the generalized Hölder inequality holds

$$|\int_\Omega fg dx| \leq (\frac{1}{p_1} + \frac{1}{q_1}) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)} \leq 2 \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}.$$

The relation between the modular $\int_\Omega |f|^{p(x)} dx$ and the norm follows from

$$\min(\|f\|^{p_1}_{p(\cdot)}, \|f\|^{p_2}_{p(\cdot)}) \leq \int_\Omega |f|^{p(x)} dx \leq \max(\|f\|^{p_1}_{p(\cdot)}, \|f\|^{p_2}_{p(\cdot)}).$$

Let the variable exponent $p(\cdot)$ satisfy the log-Hölder continuity condition

$$|p(x) - p(y)| \leq \frac{A}{log \frac{1}{|x-y|}}, \text{ for all } x, y \in \Omega \text{ with } |x - y| < \delta, \tag{5}$$

where $A > 0$ and $0 < \delta < 1$.

The variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{ u \in L^{p(\cdot)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega) \}.$$

This space is a Banach space concerning the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

Furthermore, let $W_0^{1,p(\cdot)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)}.$$

LEMMA 2.2. (The Poincaré inequality) *Assume that Ω be a bounded domain of $\mathbb{R}^n$ and $p(\cdot)$ satisfies log-Hölder condition, then*

$$\|u\|_{p(x)} \leq c \|\nabla u\|_{p(x)}, \text{ for all } u \in W_0^{1,p(\cdot)}(\Omega), \tag{6}$$

where $c = c(p_1, p_2, |\Omega|) > 0$.

Lemma 2.3. *Let $p(\cdot) \in C(\bar{\Omega})$ and $q : \Omega \to [1,\infty)$ be a measurable function that satisfy*

$$ess\ inf_{x\in\bar{\Omega}}(p^*(x) - q(x)) > 0.$$

Then the Sobolev embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact. Where

$$p^*(x) = \begin{cases} \frac{np_1}{n-p_1}, & if\ p_1 < n, \\ any\ number\ in\ [1,\infty), & if\ p_1 \geq n. \end{cases}$$

If in addition $p(\cdot)$ satisfies log-Hölder condition, then

$$p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & if\ p(x) < n, \\ any\ number\ in\ [1,\infty), & if\ p(x) \geq n. \end{cases}$$

Proposition 2.4. (See [12, 13, 14, 15]) *Let Ω be a bounded domain in R^n, $p \in C^{0,1}(\overline{\Omega})$, $1 < p_1 \leq p(x) \leq p_2 < n$. Then for any $q \in C(\Gamma_1)$ with $1 \leq q(x) \leq \frac{(n-1)p(x)}{n-p(x)}$, there is a continuous trace $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Gamma_1)$, when $1 \leq q(x) << \frac{(n-1)p(x)}{n-p(x)}$, the trace is compact, in particulary the continuous trace $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Gamma_1)$ is compact.*

By using Proposition 2.4, there exist constant C and the embedding

$$H^2_{\Gamma_1}(\Omega) \hookrightarrow L^{p(x)}(\Gamma_1)$$

which implies

$$\|u\|_{p(\cdot),\Gamma_1} \leq C\|\Delta u\|, \quad \forall\ u \in H^2_{\Gamma_1}(\Omega),$$

where $\|u\|_{p(\cdot),\Gamma_1} := \int_{\Gamma_1} |u|^{p(x)} d\Gamma$.

Moreover, embedding

$$H^2_{\Gamma_1}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$$

which implies

$$\|u\|_{p(\cdot)} \leq C\|\Delta u\|, \quad \forall\ u \in H^2_{\Gamma_1}(\Omega). \tag{7}$$

We recall the Young's inequality

$$XY \leq \theta X^{q(x)} + C(\theta, q(x))Y^{q'(x)}, \quad X, Y \geq 0, \quad \theta > 0, \quad \frac{1}{q(x)} + \frac{1}{q'(x)} = 1, \tag{8}$$

where $C(\theta, q(x)) = \frac{1}{q'(x)}(\theta q(x))^{-\frac{q'(x)}{q(x)}}$. In special case when $\theta = \frac{1}{q(x)}$, we have from (8)

$$XY \leq \frac{X^{q(x)}}{q(x)} + \frac{Y^{q'(x)}}{q'(x)}. \tag{9}$$

We set

$$H^2_{\Gamma_1}(\Omega) = \{v \in H^2(\Omega) : v = 0\ on\ \Gamma_0\}.$$

DEFINITION 2.5. Let $u_0, u_1 \in H^2_{\Gamma_1}(\Omega) \times L^2(\Omega)$. A function $u(x,t)$ is called a weak solution of the problem (1)-(3) defined on $[0,T]$ $(0 < T < \infty)$ if

$$u \in C^1\big((0,T); H^2_{\Gamma_1}(\Omega) \cap W^{1,p(\cdot)}(\Omega)\big),$$
$$u_t \in L^2\big((0,T); H^2_{\Gamma_1}(\Omega) \cap L^{m(x)}(\Omega)\big),$$

and satisfy

$$\begin{aligned}
&\int_\Omega v u_{tt} dx + \int_\Omega v\Delta^2 u dx - \left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega v\Delta_{p(x)} u dx \\
&- \int_\Omega \int_0^t g(t-s) v\Delta^2 u(s) ds dx + \beta \int_\Omega v\Delta^2 u_t dx + \int_\Omega v|u_t|^{m(x)-2}u_t dx \\
&= \int_\Omega v|u|^{q(x)-2}u dx, \quad \forall v \in C^1\big((0,T); H^2_{\Gamma_0}(\Omega) \cap W^{1,p(\cdot)}(\Omega)\big) \\
&u(x,0) = u_0, \quad u_t(x,0) = u_1,
\end{aligned}$$

with compatibility boundary condition

$$\Delta u_0 - \int_0^t g(t-s)\Delta u_0(s) ds + \beta \Delta u_1 = 0, \quad \text{on } \Gamma_1.$$

For the sake of completeness, the local existence result for the problem (1)-(3) is stated as follows. This theorem could be proved by the Faedo-Galerkin approximation method and the compactness method with the Banach fixed point theorem that has been used in the works of Rahmoune [27, 28].

THEOREM 2.6. (Local existence) *Suppose that (A1) and (A2) are satisfied; then the problem (1)-(3) has at least one weak solution.*

The energy of the system defined by

$$E(t) = \frac{1}{2}I(t) + a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx, \tag{10}$$

where

$$I(t) = \|u_t\|^2 + (1 - \int_0^t g(s)ds)\|\Delta u\|^2 + b\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 + (g * \Delta u)(t),$$

and

$$(g * u)(t) = \int_0^t g(t-s)\|u(t) - u(s)\|^2 ds.$$

Moreover, by using the boundary conditions and (A2) for any solution of problem (1)-(3), the energy functional satisfies

$$\frac{dE(t)}{dt} = \frac{1}{2}(g' * \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|^2 - \beta\|\Delta u_t\|^2 - \int_\Omega |u_t|^{m(x)}dx \le 0. \tag{11}$$

3. General decay. In this section, we prove a general decay result for the solution energy. We show that the solutions decay uniformly to zero with arbitrary rates the same as the ones of the memory kernel. To this end, we make the following assumptions:

(A3) There exists a non-increasing differentiable function $\xi : R^+ \to R^+$ such that

$$\xi(0) \geq 0, \quad g'(t) \leq -\xi(t)g(t), \quad \int_0^\infty \xi(t)dt = +\infty,$$

(A4) variable exponents and kernel of memory satisfy

$$q_2 \leq m_1 \leq \min\{p_1, \frac{2p_1^2}{p_2}\}.$$

Our main result in this section reads in the following theorem:

THEOREM 3.1. *Let the conditions (A1)-(A4) be satisfied. Then the energy $E(t)$ of the problem (1)-(3) satisfies the following general estimate for the two positive constants k and K:*

$$E(t) \leq KE(0)e^{-k\int_0^t \xi(s)ds}, \quad \text{for all } t \geq 0. \tag{12}$$

To prove the above theorem, for sufficiently small $\varepsilon > 0$ we define

$$F(t) = E(t) + \varepsilon\phi(t), \tag{13}$$

where

$$\phi(t) = \int_\Omega (uu_t - \frac{\beta}{2}|\Delta u|^2)dx.$$

LEMMA 3.2. *Under the assumptions of Theorem 3.1, the functional $\phi(t)$ satisfies, along the solution, the estimate*

$$\begin{aligned} \phi'(t) \quad \leq \quad & (C^2 + \frac{\beta}{2\gamma_0})\|\Delta u_t\|^2 + \frac{1-l}{4\gamma_1}(g * \Delta u)(t) + \frac{m_2 - 1}{m_2}\int_\Omega |u_t|^{m(x)}dx \\ & + \int_\Omega |u|^{q(x)}dx - [l - \gamma_1(1-l) - 2\beta\gamma_0 - \frac{\overline{C}}{m_1}]\|\Delta u\|^2 \\ & -a\int_\Omega |\nabla u|^{p(x)}dx - \frac{b}{p_2}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2. \end{aligned} \tag{14}$$

Proof. Differentiating $\phi(t)$ with respect to t we have

$$\phi'(t) = \|u_t\|^2 + \int_\Omega uu_{tt}dx - \beta\int_\Omega \Delta u\Delta u_t dx.$$

Multiplying equation (1) in u and using boundary conditions we get

$$\begin{aligned}
\phi'(t) &= \|u_t\|^2 - (1-\int_0^t g(s)ds)\|\Delta u\|^2 - a\int_\Omega |\nabla u|^{p(x)}dx - 2\beta\int_\Omega \Delta u\Delta u_t dx \\
&\quad -b\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big) - \int_\Omega u|u_t|^{m(x)-2}u_t dx \\
&\quad +\int_0^t g(t-s)\int_\Omega \Delta u(\Delta u(s)-\Delta u)dxds + \int_\Omega |u|^{q(x)}dx \\
&\leq C^2\|\Delta u_t\|^2 - l\|\Delta u\|^2 - a\int_\Omega |\nabla u|^{p(x)}dx \\
&\quad -\frac{b}{p_2}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 + \int_\Omega |u|^{q(x)}dx - 2\beta\int_\Omega \Delta u\Delta u_t dx \\
&\quad +\int_0^t g(t-s)\int_\Omega \Delta u(\Delta u(s)-\Delta u)dxds - \int_\Omega u|u_t|^{m(x)-2}u_t dx, \qquad (15)
\end{aligned}$$

where condition (A1) and (7) have been used.

To estimate the last three terms on the right-hand side of (15), we use the Young's inequality (9). Consequently, we get

$$|\int_\Omega \Delta u\Delta u_t dx| \leq \gamma_0\|\Delta u\|^2 + \frac{1}{4\gamma_0}\|\Delta u_t\|^2, \qquad (16)$$

$$\begin{aligned}
|\int_\Omega u|u_t|^{m(x)-2}u_t dx| &\leq \int_\Omega \frac{1}{m(x)}|u|^{m(x)}dx + \int_\Omega \frac{m(x)-1}{m(x)}|u_t|^{m(x)}dx \\
&\leq \frac{1}{m_1}\int_\Omega |u|^{m(x)}dx + \frac{m_2-1}{m_2}\int_\Omega |u_t|^{m(x)}dx \\
&\leq \frac{\overline{C}}{m_1}\|\Delta u\|^2 + \frac{m_2-1}{m_2}\int_\Omega |u_t|^{m(x)}dx, \qquad (17)
\end{aligned}$$

where (A1) and embedding inequality have been used.

$$\begin{aligned}
&\int_0^t g(t-s)|\int_\Omega \Delta u(\Delta u(s)-\Delta u)dx| \\
&\leq \gamma_1(\int_0^t g(s)ds)\|\Delta u\|^2 + \frac{1}{4\gamma_1}\int_\Omega \Big(\int_0^t g(t-s)|\Delta u(s)-\Delta u|ds\Big)^2 dx \\
&\leq \gamma_1(1-l)\|\Delta u\|^2 + \frac{1}{4\gamma_1}\int_\Omega \Big(\int_0^t \frac{g(t-s)}{\sqrt{g(t-s)}}\sqrt{g(t-s)}|\Delta u(s)-\Delta u|ds\Big)^2 dx \\
&\leq \gamma_1(1-l)\|\Delta u\|^2 + \frac{1}{4\gamma_1}\Big(\int_0^t g(s)ds\Big)\int_\Omega\int_0^t g(t-s)|\Delta u(s)-\Delta u|^2 dsdx \\
&\leq \gamma_1(1-l)\|\Delta u\|^2 + \frac{(1-l)}{4\gamma_1}(g*\Delta u)(t), \qquad (18)
\end{aligned}$$

where inequality $\int_0^t g(s)ds < \int_0^\infty g(t)dt = 1-l$ has been used.
Hence, by combining (16)-(18) and (15), the proof of Lemma 3.2 is completed. □

LEMMA 3.3. *Under the assumptions of Theorem 3.1, there exists a constant $\alpha > 0$ such that the functional $F(t)$ satisfies, along the solution, the estimate*

$$F'(t) + \varepsilon m_1 F(t) \leq \alpha (g * \Delta u)(t). \tag{19}$$

Proof. Differentiating of $F(t)$ with respect to t, and taking (11) and (14) into account, we deduce

$$\begin{aligned} F'(t) &= E'(t) + \varepsilon\phi'(t) \\ &\leq -[\beta(1-\frac{\varepsilon}{2\gamma_0}) - \varepsilon C^2]\|\Delta u_t\|^2 + \varepsilon \int_\Omega |u|^{q(x)}dx \\ &\quad + \frac{\varepsilon(1-l)}{4\gamma_1}(g*\Delta u)(t) - (1-\frac{m_2-1}{m_2})\int_\Omega |u_t|^{m(x)}dx \\ &\quad + \varepsilon(l - \gamma_1(1-l) - 2\beta\gamma_0 - \frac{\overline{C}}{m_1})\|\Delta u\|^2 - \varepsilon a \int_\Omega |\nabla u|^{p(x)}dx \\ &\quad - \frac{\varepsilon b}{p_2}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2. \end{aligned}$$

For $\varepsilon < \frac{2}{m_1}$, we get

$$\begin{aligned} F'(t) &\leq -\varepsilon m_1 F(t) - [\beta(1-\frac{\varepsilon}{2\gamma_0}) - \varepsilon C^2(\frac{m_1}{2}+1)]\|\Delta u_t\|^2 \\ &\quad -\varepsilon\Big(\beta(\frac{\varepsilon m_1}{2} - 2\gamma_0) + l - \gamma_1(1-l) - \frac{\overline{C}}{m_1} - \frac{m_1}{2}\Big)\|\Delta u\|^2 \\ &\quad -\varepsilon(\frac{m_1}{q_2} - 1)\int_\Omega |u|^{q(x)}dx + \varepsilon(\frac{m_1}{2} + \frac{(1-l)}{4\gamma_1})(g*\Delta u)(t) \\ &\quad -(1-\frac{m_2-1}{m_2})\int_\Omega |u_t|^{m(x)}dx - \varepsilon a(1-\frac{m_1}{p_1})\int_\Omega |\nabla u|^{p(x)}dx \\ &\quad - b\varepsilon(\frac{1}{p_2} - \frac{m_1}{2p_1^2})\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 + \varepsilon^2 m_1 \int_\Omega u u_t dx. \end{aligned}$$

By virtue of the additional condition of variable exponents (A4), we choose $\frac{\varepsilon}{2} < \gamma_0 < \frac{\varepsilon m_1}{4}$ and $\gamma_1 = \frac{l}{1-l}$, then we obtain

$$\begin{aligned} F'(t) &\leq -\varepsilon m_1 F(t) - [\beta(1-\frac{\varepsilon}{2\gamma_0}) - \varepsilon C^2(\frac{m_1}{2}+1)]\|\Delta u_t\|^2 \\ &\quad -\varepsilon\Big(\beta(\frac{\varepsilon m_1}{2} - 2\gamma_0) - \frac{\overline{C}}{m_1} - \frac{m_1}{2}\Big)\|\Delta u\|^2 \\ &\quad +\varepsilon(\frac{m_1}{2} + \frac{(1-l)^2}{4l})(g*\Delta u)(t) + \varepsilon^2 m_1 \int_\Omega u u_t dx. \end{aligned} \tag{20}$$

Thanks to Young's inequality (8) and (7), we estimate the last term of (20) as:

$$\varepsilon^2 m_1 |\int_\Omega u u_t dx| \leq \frac{m_1}{2}\|\Delta u\|^2 + \frac{\varepsilon^4 m_1 C^4}{2}\|\Delta u_t\|^2, \tag{21}$$

thus by utilizing (21) into (20), we deduce

$$\begin{aligned} F'(t) &\leq -\varepsilon m_1 F(t) - [\beta(1-\frac{\varepsilon}{2\gamma_0}) - \varepsilon C^2(\frac{\varepsilon^3 m_1 C^2}{2} + \frac{m_1}{2} + 1)]\|\Delta u_t\|^2 \\ &\quad -\varepsilon\Big(\beta(\frac{\varepsilon m_1}{2} - 2\gamma_0) - \frac{\overline{C}\varepsilon^{m_1}}{m_1} - m_1\Big)\|\Delta u\|^2 \\ &\quad +\varepsilon(\frac{m_1}{2} + \frac{(1-l)^2}{4l})(g * \Delta u)(t). \end{aligned} \tag{22}$$

Finally for sufficiently large β and ε small enough, then the proof of Lemma 3.3 is completed. □

To complete the proof of Theorem 3.1, by using (A3) and (11), multiplying (19) in $\xi(t)$ to get

$$\begin{aligned} \xi(t)F'(t) &\leq -\varepsilon m_1 \xi(t)F(t) + \alpha\xi(t)(g * \Delta u)(t) \\ &\leq -\varepsilon m_1 \xi(t)F(t) - \alpha(g' * \Delta u)(t) \\ &\leq -\varepsilon m_1 \xi(t)F(t) - 2\alpha E'(t) \\ &\leq -\varepsilon m_1 \xi(t)E(t) - 2\alpha E'(t). \end{aligned} \tag{23}$$

Now, Let define

$$L(t) = \xi(t)F(t) + 2\alpha E(t),$$

then by using (23) we arrive at

$$\begin{aligned} L'(t) &\leq \xi'(t)F(t) - \varepsilon m_1 \xi(t)E(t) \\ &\leq -\varepsilon m_1 \xi(t)E(t) \\ &\leq -\frac{\varepsilon m_1}{\gamma}\xi(t)L(t), \end{aligned} \tag{24}$$

where $0 \leq L(t) \leq \gamma E(t)$ has been used.

By integrating (24) conclusion of Theorem 3.1 and general decay of solutions of (1)-(3) has been proved.

4. Blow up. In this section, we are going to prove that for appropriate initial data some of the solutions blow up in a finite time. We denote by C various positive constants which may be different at different occurrences. To prove this result for certain solutions with negative initial energy, we set $\beta = 1$. Moreover, it is assumed that:

(B1) variable exponents satisfy

$$\max\{\frac{2}{l}(1+\frac{\overline{C}}{m_1}),\ \frac{2p_2^2}{p_1}\} \leq m_2 \leq q_1,$$

where $\overline{C}$ is a constant mentioned in (17).

Now we are in a position to state and prove our blow-up result as follows:

THEOREM 4.1. *Let the conditions (A1)-(A2) and (B1), are satisfied. Assume that* $E(0) < 0$. *Then the solution to the problem (1)-(3) blows up in finite time* T^*, *and*

$$T^* \leq \frac{1-\sigma}{\eta\sigma\psi^{\frac{\sigma}{1-\sigma}}(0)},$$

where $\sigma < 1$ *and* $\psi(t)$ *is given in (27).*

Proof. Define $H(t) = -E(t)$ and thus by using (11) we arrive at

$$H'(t) = -\frac{dE(t)}{dt} \geq \int_\Omega |u_t|^{m(x)}dx, \tag{25}$$

then negative initial energy and (25) gives $H(t) \geq H(0) > 0$. Also, by definition $H(t)$, it is easy to see that

$$H(t) \leq \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx \leq \frac{1}{q_1}\int_\Omega |u|^{q(x)}dx. \tag{26}$$

Define for $0 < \sigma < 1$

$$\psi(t) = H^{1-\sigma}(t) + \varepsilon\int_\Omega (uu_t + \frac{1}{2}|\Delta u|^2)dx, \tag{27}$$

where ε sufficiently small to be chosen later.

By taking a derivative of (27) and using (1), we have

$$\begin{aligned}
\psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|^2 + \varepsilon\int_\Omega uu_{tt}dx + \varepsilon\int_\Omega \Delta u\Delta u_t dx \\
&= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon(1-\int_0^t g(s)ds)\|\Delta u\|^2 \\
&\quad -b\varepsilon\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big) + \varepsilon\int_\Omega |u|^{q(x)}dx \\
&\quad -a\varepsilon\int_\Omega |\nabla u|^{p(x)}dx + \varepsilon\int_0^t g(t-s)\int_\Omega \Delta u(\Delta u(s)-\Delta u)dxds \\
&\quad -\varepsilon\int_\Omega u|u_t|^{m(x)-2}u_t dx \\
&\geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon(1-\int_0^t g(s)ds)\|\Delta u\|^2 \\
&\quad -\frac{b\varepsilon}{p_1}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 + \varepsilon\int_\Omega |u|^{q(x)}dx - a\varepsilon\int_\Omega |\nabla u|^{p(x)}dx \\
&\quad +\varepsilon\int_0^t g(t-s)\int_\Omega \Delta u(\Delta u(s)-\Delta u)dxds \\
&\quad -\varepsilon\int_\Omega u|u_t|^{m(x)-2}u_t dx,
\end{aligned} \tag{28}$$

where the condition (A1) has been used.

Now, we use (18) with $\gamma_1 = 1$ to estimate the integral of memory term in the inequality (28), we obtain

$$\begin{aligned}\psi'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon[(1 - \int_0^t g(s)ds) + 1 - l]\|\Delta u\|^2 \\ &- \frac{b\varepsilon}{p_1}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 + \varepsilon\int_\Omega |u|^{q(x)}dx - a\varepsilon\int_\Omega |\nabla u|^{p(x)}dx \\ &- \frac{\varepsilon(1-l)}{4}(g*\Delta u)(t) - \varepsilon\int_\Omega u|u_t|^{m(x)-2}u_t dx. \end{aligned} \quad (29)$$

By using the definition of the $H(t)$ and condition (A1), it follows that

$$\begin{aligned}\varepsilon m_2 H(t) &= \varepsilon m_2 \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx - \frac{\varepsilon m_2}{2}\|u_t\|^2 - \frac{\varepsilon m_2}{2}(1 - \int_0^t g(s)ds)\|\Delta u\|^2 \\ &- \varepsilon m_2 a \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{\varepsilon m_2 b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 \\ &- \frac{\varepsilon m_2}{2}(g*\Delta u)(t) \\ &\leq \frac{\varepsilon m_2}{q_1}\int_\Omega |u|^{q(x)}dx - \frac{\varepsilon m_2}{2}\|u_t\|^2 - \frac{\varepsilon m_2}{2}(1 - \int_0^t g(s)ds)\|\Delta u\|^2 \\ &- \frac{\varepsilon m_2 a}{p_2}\int_\Omega |\nabla u|^{p(x)}dx - \frac{\varepsilon m_2 b}{2p_2^2}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 \\ &- \frac{\varepsilon m_2}{2}(g*\Delta u)(t). \end{aligned} \quad (30)$$

Using (30), we obtain from (29)

$$\begin{aligned}\psi'(t) &\geq \varepsilon m_2 H(t) + (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon(\frac{m_2}{2} + 1)\|u_t\|^2 \\ &+ \varepsilon[(\frac{m_2}{2} - 1)(1 - \int_0^t g(s)ds) - 1 + l]\|\Delta u\|^2 \\ &+ \varepsilon a(\frac{m_2}{p_2} - 1)\int_\Omega |\nabla u|^{p(x)}dx + \varepsilon(1 - \frac{m_2}{q_1})\int_\Omega |u|^{q(x)}dx \\ &+ \varepsilon b(\frac{m_2}{2p_2^2} - \frac{1}{p_1})\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 + \varepsilon(\frac{m_2}{2} - \frac{(1-l)}{4})(g*\Delta u)(t) \\ &- \varepsilon\int_\Omega u|u_t|^{m(x)-2}u_t dx, \end{aligned}$$

Since we have $1-\int_0^t g(s)ds > 1-\int_0^\infty g(t)dt = l$, so we get

$$\begin{aligned}\psi'(t) \geq\ & \varepsilon m_2 H(t) + (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon(\frac{m_2}{2}+1)\|u_t\|^2 \\ & +\varepsilon(\frac{m_2 l}{2}-1)\|\Delta u\|^2 + \varepsilon a(\frac{m_2}{p_2}-1)\int_\Omega |\nabla u|^{p(x)}dx \\ & +\varepsilon b(\frac{m_2}{2p_2^2}-\frac{1}{p_1})\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 + \varepsilon(1-\frac{m_2}{q_1})\int_\Omega |u|^{q(x)}dx \\ & +\varepsilon(\frac{m_2}{2}-\frac{1-l}{4})(g*\Delta u)(t) - \varepsilon\int_\Omega u|u_t|^{m(x)-2}u_t dx. \end{aligned} \tag{31}$$

On the other hand, similar to (17), by using (25) we have

$$\begin{aligned}|\int_\Omega u|u_t|^{m(x)-2}u_t dx| &\leq \frac{\overline{C}}{m_1}\|\Delta u\|^2 + \frac{m_2-1}{m_2}\int_\Omega |u_t|^{m(x)}dx \\ &\leq \frac{\overline{C}}{m_1}\|\Delta u\|^2 + \frac{m_2-1}{m_2}H'(t) \\ &\leq \frac{\overline{C}}{m_1}\|\Delta u\|^2 + \frac{m_2-1}{m_2}KH^{-\sigma}(t)H'(t), \end{aligned} \tag{32}$$

where K is a sufficient large constant that will be enunciated later.

$$\begin{aligned}\psi'(t) \geq\ & \varepsilon m_2 H(t) + (1-\sigma-\frac{\varepsilon K(m_2-1)}{m_2})H^{-\sigma}(t)H'(t) \\ & +\varepsilon(\frac{m_2}{2}+1)\|u_t\|^2 + \varepsilon(\frac{m_2 l}{2}-\frac{\overline{C}}{m_1}-1)|\Delta u\|^2 \\ & +\varepsilon a(\frac{m_2}{p_2}-1)\int_\Omega |\nabla u|^{p(x)}dx + \varepsilon b(\frac{m_2}{2p_2^2}-\frac{1}{p_1})\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 \\ & +\varepsilon(1-\frac{m_2}{q_1})\int_\Omega |u|^{q(x)}dx + \varepsilon(\frac{m_2}{2}-\frac{1-l}{4})(g*\Delta u)(t). \end{aligned} \tag{33}$$

At this point by using (B1), we deduce

$$\begin{aligned}\psi'(t) \geq\ & (1-\sigma-\frac{\varepsilon K(m_2-1)}{m_2})H^{-\sigma}(t)H'(t) + \varepsilon m_2\big[H(t) + \|u_t\|^2 + |\Delta u\|^2 \\ & +a\int_\Omega |\nabla u|^{p(x)}dx + b\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 + \int_\Omega |u|^{q(x)}dx \\ & +(g*\Delta u)(t)\big]. \end{aligned} \tag{34}$$

Now, suppose that ε sufficiently small and K large enough such that $1-\sigma-\frac{\varepsilon K(m_2-1)}{m_2} > 0$ and (32) holds, then we deduce

$$\begin{aligned}\psi'(t) \geq\ & \varepsilon m_2\big[H(t) + \|u_t\|^2 + \|\Delta u\|^2 + a\int_\Omega |\nabla u|^{p(x)}dx + b\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^2 \\ & +\int_\Omega |u|^{q(x)}dx + (g*\Delta u)(t)]. \end{aligned} \tag{35}$$

Therefore we get

$$\psi(t) \geq \psi(0) > 0, \ for \ all \ t \geq 0.$$

On the other hand, by using the Hölder and Young inequalities, we have

$$|\int_\Omega uu_t dx|^{\frac{1}{1-\sigma}} \leq C(\|u\|^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}}) \leq C\left(\|\Delta u\|^2 + \|u_t\|^2 + H(t)\right). \tag{36}$$

Thus, using the inequality

$$(a_1 + a_2 + \cdots + a_m)^\lambda \leq 2^{\frac{m-1}{\lambda-1}}(a_1^\lambda + a_2^\lambda + \cdots + a_m^\lambda),$$

(for $a_1, a_2, \cdots, a_m \geq 0, \lambda \geq 1$), we have for sufficiently small ε and some $\eta > 0$

$$\begin{aligned}
\psi^{\frac{1}{1-\sigma}}(t) &= \left[H^{1-\sigma}(t) + \varepsilon\int_\Omega uu_t dx + \frac{\varepsilon}{2}\|\Delta u\|^2\right]^{\frac{1}{1-\sigma}} \\
&\leq 4^{\frac{1-\sigma}{\sigma}}\left(H(t) + \varepsilon^{\frac{1}{1-\sigma}}|\int_\Omega uu_t dx|^{\frac{1}{1-\sigma}} + \left(\frac{\varepsilon}{2}\right)^{\frac{1}{1-\sigma}}\|\Delta u\|^{\frac{2}{1-\sigma}}\right) \\
&\leq C\left(\|\Delta u\|^2 + \|u_t\|^2 + H(t)\right) \\
&\leq \eta^{-1}\psi'(t),
\end{aligned}$$

therefore

$$\psi'(t) \geq \eta\psi^{\frac{1}{1-\sigma}}(t). \tag{37}$$

Integrating (37) from 0 to t, we deduce

$$\psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\eta\sigma t}{1-\sigma}}.$$

This shows that solutions blow up in finite time $T^* \leq \frac{1-\sigma}{\eta\sigma\psi^{\frac{\sigma}{1-\sigma}}(0)}$, and proof of Theorem 4.1 has been completed. □

REMARK 4.2. The proof of Theorem 4.1 can be extended for any $\beta > 0$.

REMARK 4.3. Let $u(t)$ be a local weak solution to problem (1)-(3), if $0 < E(t) < E_1$ and $(B1)$ holds, then by taking $H(t) = E_1 - E(t)$ instead of $H(t) = -E(t)$ in the proof of Theorem 4.1 and similar argument as in the proof of Theorem 4.1, $u(t)$ blows up at a finite time.

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