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On system reliability estimation in stress-strength setup

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Abstract: In this paper, we consider the estimation of the stress-strength reliability of a coherent system. The distributions of stress and strength random variables are the members of a general class of distributions. For a series-parallel system, the reliability of the stress-strength model is estimated using the maximum likelihood estimation, asymptotic confidence interval, uniformly minimum variance unbiased estimation, and Bayes estimation. Also, simulation studies are performed, and two real data sets are analyzed.

Keywords: Asymptotic confidence interval; Bayes estimation; Maximum likelihood estimation; Stress-strength reliability; Uniformly minimum variance unbiased estimation.
Mathematics Subject Classification (2010): 62N05, 62N02.

1 Introduction

In reliability literature, the stress-strength model described as the reliability of a unit or a system in terms of the stress random variable X and strength random variable Y. The system fails if the stress is greater than the strength, and the probability $R = P(X < Y)$ is the reliability of the system. This idea was first introduced by Birnbaum (1956) and it was developed by Birnbaum and McCarty (1958). The term stress-strength reliability was first used by Church and Harris (1970). For example, this model is used in comparison of the performances of two drugs or two products. Many authors have studied the estimation of R under different distributions and sampling methods. For example see Shawky and Al-Gashgari (2013), Kizilaslan and Nadar (2017), Khan and Khatoon (2020), Yadav et al. (2019), Rao et al. (2019), Abu-Moussa et al. (2021), Alamri et al. (2021), Hemati et al. (2022), and Kohansal (2022).

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The reliability of a multicomponent stress-strength model was developed by Bhattacharyya and Johnson (1974). In their model all n components of the system are exposed to a common stress, and system is alive when at least k $(1 \leq k \leq n)$ components function. This model is called the k-out-of-n:G system and the reliability of this system is denoted by $R_{k,n}$. For example, a four-engine airplane that can fly if at least two of its engines are working is a 2-out-of-4:G system. Most researchers have studied the k-out-of-n:G system. See for example Hassan and Basheikh (2012), Rao et al. (2017), salman and Sail (2018), Jamala et al. (2019), Kohansal (2019), Pandit and Joshi (2019), Hassan et al. (2020), Mezaal et al. (2020), Sauer et al. (2020), Ahmad et al. (2022), Jana and Bera (2022), and Kohansal et al. (2023).

Some authors considered the reliability of coherent systems. Dewanji and Rao (2001) considered the stress-strength reliability for a general coherent system and studied two cases. In the first case, the stress is at the system level, that is the system components are subjected to a common stress and in the other case the stress is at the component level where each component is subjected to a particular stress. Bhattacharya and Roychowdhury (2013) expressed the stress-strength reliability of a system as a function of the stress-strength reliabilities of its individual components. In case of system level stress, Eryilmaz (2010) obtained an expression for the reliability of a general coherent system as a linear combination of the reliabilities of series systems.

The concept of system signature can be used to calculate the stress-strength reliability. Consider a coherent system with n components and with structure function φ. Let $T_1, T_2, \ldots, T_n$ be the component lifetimes and $T = \varphi(T_1, T_2, \ldots, T_n)$ be the system lifetime. If T_is are continuous and have identically independent distributions (iid) and $T_{i:n}$ be the ith ordered lifetime, the following well known result is shown by Samaniego (1985)

$$P(T > t) = \sum_{i=1}^{n} s_i \; P(T_{i:n} > t),$$

where $s_i = P(T = T_{i:n}), i = 1, 2, \ldots, n$. The probability vector $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ is called the signature of the system. For applications of system signatures, see Samaniego (2007). In a coherent system with structure function φ and by using the system signature, Eryilmaz (2008) obtained the reliability of the stress-strength model. He considered a system with n components (or subsystems) when each component consists of m elements. In his paper, $\boldsymbol{X}_i = (X_i^1, X_i^2, \ldots, X_i^m)$ is the random strength vector of ith component, and the elements of the components are subjected to a common random stress. Khanjari (2022) showed that his result given in Theorem 3 is a mistake.

In this paper, we consider a coherent system and study the stress-strength reliability at the system level. For more details about coherent systems see Barlow and Proschan (1971). We assume that the stress and strength are independent random variables. Al-Hussaini (1999) introduced a general class of distributions that can be viewed as the *exponential form of distributions*. If the distribution of the random variable X be a member of this class, the probability density function (pdf), the cumulative distribution function (cdf) and the failure rate function of X are given respectively as

$$\begin{aligned} f_X(x) &= \alpha k_{\boldsymbol{\theta}}^{'}(x) e^{-\alpha k_{\boldsymbol{\theta}}(x)}, \quad F_X(x) = 1 - e^{-\alpha k_{\boldsymbol{\theta}}(x)}, \\ r_X(x) &= \frac{f_X(x)}{1 - F_X(x)} = \alpha k_{\boldsymbol{\theta}}^{'}(x), \quad x > 0, \alpha > 0, \end{aligned}$$

where $\boldsymbol{\theta}$ is a parameter vector, $k_{\boldsymbol{\theta}}(x)$ is a strictly increasing function of x with $k_{\boldsymbol{\theta}}(0) = 0$ and $k_{\boldsymbol{\theta}}(\infty) = \infty$, and $k'_{\boldsymbol{\theta}}(x)$ is derivative of $k_{\boldsymbol{\theta}}(x)$ with respect to x. This class includes exponential, Weibull, Rayleigh, Frechet, Gompertz, Half-logistic, Burr type X, Burr type XII and many other distributions. Khorashadizadeh (2017) studied some reliability properties of the above class.

We first consider the stress and strength distributions as the members of this class and obtain the stress-strength reliability. We then assume that the exponential distribution, as the distribution of stress and strength random variables with different scale parameters, and for the radar system we estimate the stress-strength reliability by using maximum likelihood estimation (MLE), uniformly minimum variance unbiased estimation (UMVUE) and Bayes estimation methods. We also consider the Pareto distribution, which is not a member of the above class, and again estimate the stress-strength reliability by using the same procedures.

In Section 2, we consider a coherent system and study the stress-strength reliability at the system level. Different examples are provided. In Section 3, we consider a radar system and obtain the stress-strength reliability at the system level, when the stress and strength distributions are the members of exponential form of distributions or when they are Pareto distributions. Also, we estimate the system reliability by MLE, UMVUE and Bayes method, and derive asymptotic confidence interval (ACI) when the distributions of the stress and strength are exponential or Pareto. In Section 4, we perform a simulation study and analyze two real data sets.

2 Stress-strength reliability of a coherent system

Consider a coherent system with n components and structure function φ. System components have strengths $Y_1, Y_2, \ldots, Y_n$, which are iid, and each component is subjected to a common stress X which is an independent random variable of the strengths. For ith component, we define the random status as follows

$$Z_i = \begin{cases} 1 & X < Y_i, \\ 0 & X \geq Y_i, \end{cases}$$

that is, ith component is active when the imposed stress is less than its strength. Note that Z_is are not independent. $P(Z_i = 1)$ and $R_\varphi = P(\varphi(Z_1, Z_2, \ldots, Z_n) = 1)$ are the reliabilities of the ith component and the system, respectively. Eryilmaz (2010) obtained R_φ as a linear combination of the reliability of series systems as follows

$$R_\varphi = \sum_{i=1}^{n} a_i P(X < Y_{1:i}),$$

where $Y_{1:i}$ is the first ordered strength variable in $Y_1, Y_2, \ldots, Y_i$. He derived the coefficients a_i for k-out-of-n:F, k-out-of-n:G and linear consecutive k-out-of-n:F systems.

Consider a coherent system with minimal path sets $P_1, P_2, \ldots, P_r$. Define

$$Y_{1:|P_j|} = \min_{i \in P_j} Y_i, \qquad j = 1, 2, \ldots, r,$$

where $|P_j|$ is the number of elements of P_j. By using inclusion-exclusion principle, we have

$$R_\varphi = P(\cup_{j=1}^{r} Y_{1:|P_j|} > X)$$

$$
\begin{aligned}
&= \sum_{k=1}^{r} (-1)^{k-1} \sum_{1\le j_1<j_2<\ldots<j_k\le r} P(Y_{1:|P_{j_1}|} > X, Y_{1:|P_{j_2}|} > X, \ldots, Y_{1:|P_{j_k}|} > X) \\
&= \sum_{k=1}^{r} (-1)^{k-1} \sum_{1\le j_1<j_2<\ldots<j_k\le r} P(Y_{1:|\cup_{l=1}^{k} P_{j_l}|} > X) \\
&= \sum_{k=1}^{r} (-1)^{k-1} \sum_{1\le j_1<j_2<\ldots<j_k\le r} P(\min_{i\in\cup_{l=1}^{k} P_{j_l}} Z_i = 1). \qquad (1)
\end{aligned}
$$

We note that the components of each minimal path set are in series. In a series system with N components, it is well known that

$$P(X < Y_{1:N}) = \int (1 - F_Y(x))^N dF_X(x), \qquad (2)$$

and in view of (1), we can obtain R_φ. Now, we give some examples.

Example 2.1. ***(a)*** *In a 3-components system with minimal path sets* $P_1 = \{1,2\}, P_2 = \{1,3\}$ *and structure function* $\varphi(x_1,x_2,x_3) = \min(x_1, \max(x_2,x_3)) = x_1x_2 + x_1x_3 - x_1x_2x_3$*, which is called a radar system, we have*

$$
\begin{aligned}
R_\varphi &= P(\min_{i\in P_1} Z_i = 1) + P(\min_{i\in P_2} Z_i = 1) - P(\min_{i\in P_1\cup P_2} Z_i = 1) \\
&= P(\min_{i=1,2} Z_i = 1) + P(\min_{i=1,3} Z_i = 1) - P(\min_{i=1,2,3} Z_i = 1) \\
&= P(X < Y_{1:2}) + P(X < Y_{1:2}) - P(X < Y_{1:3}) \\
&= 2P(X < Y_{1:2}) - P(X < Y_{1:3}). \qquad (3)
\end{aligned}
$$

This system is displayed in Figure 1.

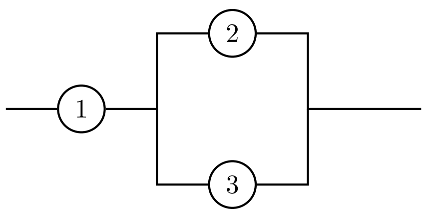

Figure 1: The radar system.

(b) *In a 3-components system with minimal path sets* $\{1\}, \{2,3\}$ *and structure function* $\varphi(x_1,x_2,x_3) = \max(x_1, \min(x_2,x_3)) = x_1 + x_2x_3 - x_1x_2x_3$

$$R_\varphi = P(X < Y_{1:1}) + P(X < Y_{1:2}) - P(X < Y_{1:3}).$$

This system is displayed in Figure 2.

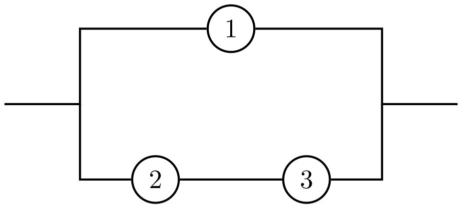

Figure 2: The series-parallel system.

Example 2.2. ***(a)*** *In a 4-components system with minimal path sets* $\{1,2,4\},\{1,3,4\}$, $\{2,3,4\}$ *and structure function* $\varphi(x_1,x_2,x_3,x_4)=x_1x_2x_4+x_1x_3x_4+x_2x_3x_4-2x_1x_2x_3x_4$

$$R_\varphi=3P(X<Y_{1:3})-2P(X<Y_{1:4}).$$

This system is displayed in Figure 3.

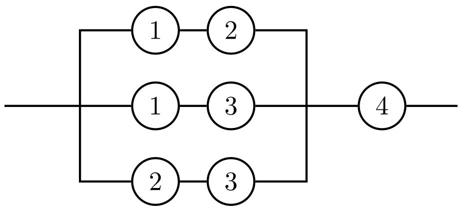

Figure 3: The series-parallel system (2.2.a).

(b) *In a 4-components system with minimal path sets* $\{1,3\},\{1,2,4\}$ *and structure function* $\varphi(x_1,x_2,x_3,x_4)=\min(x_1,\max(x_2,x_3),\max(x_3,x_4))=x_1x_3+x_1x_2x_4-x_1x_2x_3x_4$

$$R_\varphi=P(X<Y_{1:2})+P(X<Y_{1:3})-P(X<Y_{1:4}).$$

This system is displayed in Figure 4.

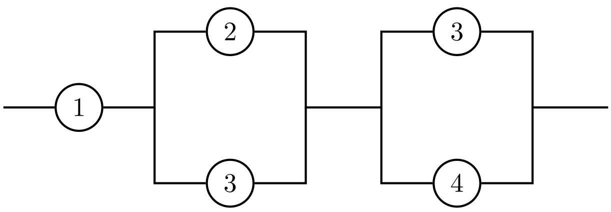

Figure 4: The series-parallel system (2.2.b).

(c) *In a 4-components system with minimal path sets* $\{1,2\},\{1,3\},\{1,4\},\{2,3\}$ *and structure function* $\varphi(x_1,x_2,x_3,x_4)=\min(\max(x_1,x_2),\max(x_2,x_3),\max(x_2,x_4))=x_1x_2+x_1x_3+x_1x_4+x_2x_3-2x_1x_2x_3-x_1x_2x_4-x_1x_3x_4+x_1x_2x_3x_4$

$$R_\varphi=4P(X<Y_{1:2})-4P(X<Y_{1:3})+P(X<Y_{1:4}).$$

This system is displayed in Figure 5.

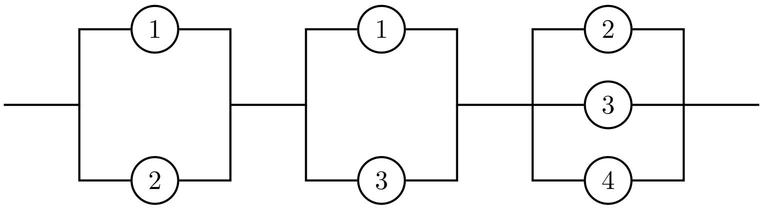

Figure 5: The series-parallel system (2.2.c).

Example 2.3. ***(a)*** *In the bridge system, containing five components and with minimal path sets* $\{1,4\},\{2,5\},\{1,3,5\},\{2,3,4\}$ *and structure function* $\varphi(x_1,x_2,x_3,x_4,x_5) = x_1x_4 + x_2x_5 + x_1x_3x_5 + x_2x_3x_4 - x_1x_2x_3x_4 - x_1x_2x_3x_5 - x_1x_2x_4x_5 - x_1x_3x_4x_5 - x_2x_3x_4x_5 + 2x_1x_2x_3x_4x_5$

$$R_\varphi = 2P(X < Y_{1:2}) + 2\ P(X < Y_{1:3}) - 5\ P(X < Y_{1:4}) + 2\ P(X < Y_{1:5}).$$

This system is displayed in Figure 6.

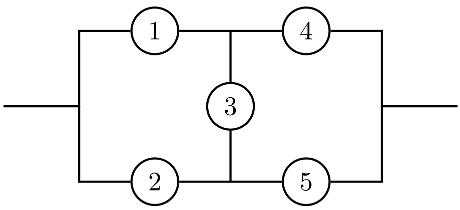

Figure 6: The bridge system.

(b) *In the stereo system, containing five components and with minimal path sets* $\{1,3,4\},\{1,3,5\},\{2,3,4\},\{2,3,5\}$ *and structure function* $\varphi(x_1,x_2,x_3,x_4,x_5) = x_1x_3x_4 + x_1x_3x_5 + x_2x_3x_4 + x_2x_3x_5 - x_1x_2x_3x_4 - x_1x_2x_3x_5 - x_1x_3x_4x_5 - x_2x_3x_4x_5 + x_1x_2x_3x_4x_5$

$$R_\varphi = 4P(X < Y_{1:3}) - 4P(X < Y_{1:4}) + P(X < Y_{1:5}).$$

This system is displayed in Figure 7.

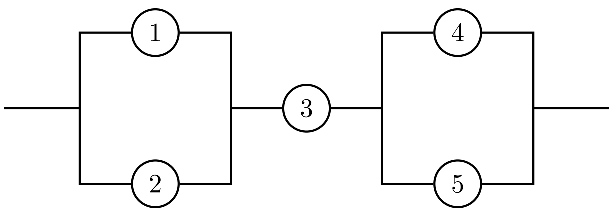

Figure 7: The stereo system.

3 Stress-strength reliability of the radar system

Consider a series system with N component. Let the stress and strength random variables X and Y are independent and their distributions are the members of the class of exponential form of distributions with parameters α and β, respectively. By using (2) and the properties of the function $k_{\boldsymbol{\theta}}(x)$, we have

$$\begin{aligned} P(X < Y_{1:N}) &= \int_0^{+\infty} (e^{-\beta k_{\boldsymbol{\theta}}(x)})^N \alpha k'_{\boldsymbol{\theta}}(x) e^{-\alpha k_{\boldsymbol{\theta}}(x)} dx \\ &= \alpha \int_0^{+\infty} k'_{\boldsymbol{\theta}}(x) e^{-(\alpha+N\beta) k_{\boldsymbol{\theta}}(x)} dx = \frac{\alpha}{\alpha + N\beta} = \psi_N(\alpha, \beta). \end{aligned}$$

Now by using (1), we can obtain R_φ.

From (3) for the radar system we have

$$R_\varphi = 2\psi_2(\alpha, \beta) - \psi_3(\alpha, \beta), \tag{4}$$

note that

$$0 < R_\varphi = \frac{\alpha^2 + 4\alpha\beta}{\alpha^2 + 5\alpha\beta + 6\beta^2} < 1.$$

Also note in general, that R_φ is independent of the functional form of $k_{\boldsymbol{\theta}}(x)$.

3.1 Estimation of R_φ based on exponential distribution

The exponential distribution is used for data that does not age over time, such as the lifetimes of electronic pieces. This distribution with the scale parameter α will be denoted by $E(\alpha)$. The pdf, cdf and failure rate functions of $X \sim E(\alpha)$ are given respectively as

$$f_X(x) = \alpha e^{-\alpha x}, \quad F_X(x) = 1 - e^{-\alpha x}, \quad r_X(x) = \alpha, \qquad x > 0, \alpha > 0.$$

$E(\alpha)$ is a member of exponential form of distributions. Let the stress X and the strength Y are independent random variables and their distributions are $E(\alpha)$ and $E(\beta)$, respectively.

3.1.1 MLE for R_φ

Suppose that $X_1, X_2, \ldots, X_{n_1}$ and $Y_1, Y_2, \ldots, Y_{n_2}$ are two independent random samples from $E(\alpha)$ and $E(\beta)$, respectively. The likelihood function is as follow

$$L(\alpha, \beta) = \alpha^{n_1} e^{-\alpha \sum_{i=1}^{n_1} x_i} \beta^{n_2} e^{-\beta \sum_{i=1}^{n_2} y_i}, \tag{5}$$

by deriving from $l(\alpha, \beta) = \log L(\alpha, \beta)$, MLEs of the scale parameters of α and β are as

$$\hat{\alpha} = \frac{1}{\overline{X}}, \qquad \hat{\beta} = \frac{1}{\overline{Y}}.$$

According to the invariance property of MLE and from (4), we get the MLE of R_φ as

$$\hat{R}_\varphi^M = \frac{4\overline{X}\,\overline{Y} + \overline{Y}^2}{6\overline{X}^2 + 5\overline{X}\,\overline{Y} + \overline{Y}^2}. \tag{6}$$

3.1.2 ACI for R_φ

To obtain the ACI of R_φ, first the asymptotic distribution of $(\hat{\alpha}, \hat{\beta})$ is derived and then the asymptotic distribution of $\hat{R}_\varphi^M$ is obtained. Finally, ACI for R_φ is constructed.

Theorem 3.1. *As $n_1 \to \infty$ and $n_2 \to \infty$, we have*

$$\begin{bmatrix} \sqrt{n_1}(\hat{\alpha}-\alpha) \\ \sqrt{n_2}(\hat{\beta}-\beta) \end{bmatrix} \xrightarrow{d} N_2(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix}). \tag{7}$$

where $\xrightarrow{d}$ shows the convergence in distribution.

Proof. We have

$$\frac{\partial^2 l(\alpha,\beta)}{\partial\alpha^2} = \frac{-n_1}{\alpha^2}, \quad \frac{\partial^2 l(\alpha,\beta)}{\partial\alpha\partial\beta} = 0, \quad \frac{\partial^2 l(\alpha,\beta)}{\partial\beta^2} = \frac{-n_2}{\beta^2}, \tag{8}$$

and the Fisher information matrix is given by

$$I = -\begin{bmatrix} \mathbb{E}(\frac{\partial^2\ l(\alpha,\beta)}{\partial\alpha^2}) & \mathbb{E}(\frac{\partial^2\ l(\alpha,\beta)}{\partial\alpha\partial\beta}) \\ \mathbb{E}(\frac{\partial^2\ l(\alpha,\beta)}{\partial\alpha\partial\beta}) & \mathbb{E}(\frac{\partial^2\ l(\alpha,\beta)}{\partial\beta^2}) \end{bmatrix} = \begin{bmatrix} \frac{n_1}{\alpha^2} & 0 \\ 0 & \frac{n_2}{\beta^2} \end{bmatrix}.$$

Therefore the asymptotic covariance matrix, Σ is as follow

$$\Sigma = I^{-1} = \begin{bmatrix} \frac{\alpha^2}{n_1} & 0 \\ 0 & \frac{\beta^2}{n_2} \end{bmatrix}, \tag{9}$$

and the proof is completed. □

Based on Theorem 3.1, ACIs for α and β are as follows

$$\hat{\alpha} \pm z_{\frac{\alpha}{2}} \frac{\hat{\alpha}}{\sqrt{n_1}}, \quad and \quad \hat{\beta} \pm z_{\frac{\alpha}{2}} \frac{\hat{\beta}}{\sqrt{n_2}}. \tag{10}$$

Theorem 3.2. *If $n_1 \to \infty$ and $n_2 \to \infty$, then*

$$\hat{R}_\varphi^M - R_\varphi \xrightarrow{d} N(0, \mathbb{V}ar(\hat{R}_\varphi^M)), \tag{11}$$

where

$$\mathbb{V}ar(\hat{R}_\varphi^M) = \frac{\alpha^2\beta^2(\alpha^2+12\alpha\beta+24\beta^2)^2}{(\alpha^2+5\alpha\beta+6\beta^2)^4}(\frac{1}{n_1}+\frac{1}{n_2}).$$

Proof. R_φ is a function of α and β, therefore by using Delta method (Lehmann and Casella, 1998), we have

$$\mathbb{V}ar(\hat{R}_\varphi^M) = \begin{bmatrix} \frac{\partial R_\varphi}{\partial\alpha} & \frac{\partial R_\varphi}{\partial\beta} \end{bmatrix} \Sigma \begin{bmatrix} \frac{\partial R_\varphi}{\partial\alpha} \\ \frac{\partial R_\varphi}{\partial\beta} \end{bmatrix} = (\frac{\partial R_\varphi}{\partial\alpha})^2\frac{\alpha^2}{n_1} + (\frac{\partial R_\varphi}{\partial\beta})^2\frac{\beta^2}{n_2}. \tag{12}$$

From (4), the partial derivations of R_φ are as follows

$$\frac{\partial R_\varphi}{\partial\alpha} = \frac{\beta(\alpha^2+12\alpha\beta+24\beta^2)}{(\alpha^2+5\alpha\beta+6\beta^2)^2}, \qquad \frac{\partial R_\varphi}{\partial\beta} = \frac{-\alpha(\alpha^2+12\alpha\beta+24\beta^2)}{(\alpha^2+5\alpha\beta+6\beta^2)^2}. \tag{13}$$

By substituting (13) into (12), the theorem is proved. □

Now, based on Theorem 3.2, the ACI for R_φ is

$$\hat{R}_\varphi^M \pm z_{\frac{\alpha}{2}} \sqrt{\hat{\mathbb{V}ar}(\hat{R}_\varphi^M)}, \tag{14}$$

where

$$\hat{\mathbb{V}ar}(\hat{R}_\varphi^M) = \frac{(24\overline{X^2} + 12\overline{X}\,\overline{Y} + \overline{Y^2})^2}{(6\overline{X^2} + 5\overline{X}\,\overline{Y} + \overline{Y^2})^4}(\frac{1}{n_1} + \frac{1}{n_2}).$$

3.1.3 UMVUE of R_φ

According to (4), R_φ is a linear combination of $\psi_N(\alpha,\beta) = \frac{\alpha}{\alpha+N\beta}, N = 2,3$. As the UMVUE is invariant under the linear combinations, it is sufficient to obtain the UMVUE of $\psi_N(\alpha,\beta)$. Let X_1 and Y_1 be independent random variables such that $X_1 \sim E(\alpha)$ and $Y_1 \sim E(\beta)$. We define $h(X_1,Y_1)$ as follow

$$h(X_1,Y_1) = I(NX_1 < Y_1),$$

where $I(C)$ is the indicator function of the event C. $h(X_1,Y_1)$ is an unbiased estimator of $\psi_N(\alpha,\beta)$, as

$$\begin{aligned}\mathbb{E}(h(X_1,Y_1)) &= \int_0^{+\infty} (\int_{Nx_1}^{+\infty} \beta e^{-\beta y_1} dy_1)\alpha e^{-\alpha x_1} dx_1 \\ &= \alpha \int_0^{+\infty} e^{-(\alpha+N\beta)x_1} dx_1 = \psi_N(\alpha,\beta).\end{aligned}$$

Now consider the complete suffcient statistic $T = (T_1,T_2) = (\sum_{i=1}^{n_1} X_i, \sum_{i=1}^{n_2} Y_i)$. UMVUE of $\psi_N(\alpha,\beta)$ is then obtained by $\mathbb{E}(h(X_1,Y_1)|T)$ as

$$\hat{\psi}_N^U(\alpha,\beta) = P(N\frac{X_1}{T_1}\frac{T_1}{T_2} < \frac{Y_1}{T_2}|T).$$

Suppose $S_1 = \frac{X_1}{T_1}$, $S_2 = \frac{Y_1}{T_2}$ and $V = \frac{T_1}{T_2}$, we have

$$\hat{\psi}_N^U(\alpha,\beta) = P(NVS_1 < S_2|T).$$

It is known that S_1 has Beta distribution with parameters $(1, n_1 - 1)$. The pdf of S_1 is as follow

$$f_{S_1}(s_1) = (n_1 - 1)(1 - s_1)^{n_1-2}, \qquad 0 < s_1 < 1.$$

Similarly S_2 has a Beta distribution with parameters $(1, n_2 - 1)$. Note that S_1 and S_2 are ancillary statistics. By using Basu theorem (Basu, 1955), (S_1,S_2) is independent of T and we have

$$\begin{aligned}f_{S_1,S_2|T}(s_1,s_2|T) &= (n_1-1)(n_2-1)(1-s_1)^{n_1-2}(1-s_2)^{n_2-2}, \\ &\qquad 0 < s_1 < 1, 0 < s_2 < 1.\end{aligned} \tag{15}$$

Now to obtain $\hat{\psi}_N^U(\alpha,\beta)$ we consider two cases. If $NV \leq 1$, we have $S_1 \leq 1$ and

$$\hat{\psi}_N^U(\alpha,\beta) = \int_0^1 (\int_{NVs_1}^1 (n_2-1)(1-s_2)^{n_2-2} ds_2)(n_1-1)(1-s_1)^{n_1-2} ds_1$$

$$= \int_0^1 (1 - NVs_1)^{n_2-1}(n_1 - 1)(1 - s_1)^{n_1-2} ds_1.$$

If $NV > 1$, S_1 should be less than $\frac{1}{NV}$ and hence

$$\begin{aligned} \hat{\psi}_N^U(\alpha,\beta) &= \int_0^{\frac{1}{NV}} (\int_{NVs_1}^1 (n_2 - 1)(1 - s_2)^{n_2-2} ds_2)(n_1 - 1)(1 - s_1)^{n_1-2} ds_1 \\ &= \int_0^{\frac{1}{NV}} (1 - NVs_1)^{n_2-1}(n_1 - 1)(1 - s_1)^{n_1-2} ds_1. \end{aligned}$$

Therefore,

$$\hat{\psi}_N^U(\alpha,\beta) = \begin{cases} Q(V,1), & NV \le 1, \\ Q(V,\frac{1}{NV}), & NV > 1, \end{cases} \tag{16}$$

where

$$Q(V,a) = \int_0^a (1 - NVs_1)^{n_2-1}(n_1 - 1)(1 - s_1)^{n_1-2} ds_1.$$

From (4), we have

$$\hat{R}_\varphi^U = 2\hat{\psi}_2^U(\alpha,\beta) - \hat{\psi}_3^U(\alpha,\beta). \tag{17}$$

3.1.4 Bayes estimation of R_φ

We obtain Bayes estimation of R_φ under the square error loss function. Suppose that the scale parameters α and β are independent random variables and have Gamma prior distributions $\Gamma(\mu,\gamma)$ and $\Gamma(\nu,\lambda)$, respectively. Then the joint prior pdf of (α,β) is as follow

$$\pi(\alpha,\beta) = \frac{\gamma^\mu}{\Gamma(\mu)}\frac{\lambda^\nu}{\Gamma(\nu)}\alpha^{\mu-1}e^{-\gamma\alpha}\beta^{\nu-1}e^{-\lambda\beta}. \tag{18}$$

From (5) and (18) the posterior pdf of (α,β) is as follow

$$\pi(\alpha,\beta|\underset{\sim}{x},\underset{\sim}{y}) = \frac{A^{n_1+\mu}B^{n_2+\nu}}{\Gamma(n_1+\mu)\Gamma(n_2+\nu)}\alpha^{n_1+\mu-1}\beta^{n_2+\nu-1}e^{-(A\alpha+\beta B)},$$

where $A = n_1\overline{x} + \gamma$ and $B = n_2\overline{y} + \lambda$. The Bayes estimator of R_φ is as follow. We first obtain the Bayes estimator of $\psi_N(\alpha,\beta)$ and then based on (4) find the Bayes estimator of R_φ. The Bayes estimator of $\psi_N(\alpha,\beta)$ is the mean of the posterior distribution as follow

$$\begin{aligned} \mathbb{E}(\psi_N(\alpha,\beta)|\underset{\sim}{x},\underset{\sim}{y}) &= \mathbb{E}(\frac{\alpha}{\alpha+N\beta}|\underset{\sim}{x},\underset{\sim}{y}) = \frac{A^{n_1+\mu}B^{n_2+\nu}}{\Gamma(n_1+\mu)\Gamma(n_2+\nu)} \\ &\times \int_0^{+\infty}\int_0^{+\infty} \frac{\alpha}{\alpha+N\beta}\alpha^{n_1+\mu-1}\beta^{n_2+\nu-1}e^{-(A\alpha+\beta B)}d\alpha d\beta \\ &= \frac{A^{n_1+\mu}B^{n_2+\nu}}{\Gamma(n_1+\mu)\Gamma(n_2+\nu)} \times I. \end{aligned} \tag{19}$$

By using the following transformations

$$\left| \begin{array}{c} w = \alpha + N\beta \\ u = \frac{\alpha}{\alpha+N\beta} \end{array} \right., \qquad \left| \begin{array}{c} \alpha = wu \\ \beta = \frac{w(1-u)}{N} \end{array} \right., \qquad |jacobi| = \frac{w}{N},$$

double integral in (19) is equal to

$$\begin{aligned} I &= \int_0^1 \int_0^{+\infty} u[wu]^{n_1+\mu-1}[\frac{w(1-u)}{N}]^{n_2+\nu-1} e^{-[wuA+\frac{w(1-u)}{N}B]} \frac{w}{N} dwdu \\ &= \frac{1}{N^{n_2+\nu}} \int_0^1 u^{n_1+\mu}(1-u)^{n_2+\nu-1} \int_0^{+\infty} w^{n_1+\mu+n_2+\nu-1} e^{-w[\frac{B}{N}-u(\frac{B}{N}-A)]} dwdu \\ &= \frac{\Gamma(n_1+\mu+n_2+\nu)}{N^{n_2+\nu}} \int_0^1 u^{n_1+\mu}(1-u)^{n_2+\nu-1}[\frac{B}{N}-u(\frac{B}{N}-A)]^{-(n_1+\mu+n_2+\nu)} du \\ &= \Gamma(n_1+\mu+n_2+\nu) \frac{N^{n_1+\mu}}{B^{n_1+\mu+n_2+\nu}} \\ &\quad \times \int_0^1 u^{n_1+\mu}(1-u)^{n_2+\nu-1}[1-u(1-\frac{NA}{B})]^{-(n_1+\mu+n_2+\nu)} du. \end{aligned}$$

Suppose $G_N = \frac{NA}{B}$ and $D_N = 1 - G_N$. Because of $G_N > 0$, we have $D_N < 1$ and therefore $|D_N| < 1$ or $D_N \leq -1$. In case of $D_N \leq -1$, we have $|D_N| < |D_N - 1|$ and hence $|\frac{D_N}{1-D_N}| < 1$. In case of $|D_N| < 1$, Equation (19) is as follow

$$\begin{aligned} \mathbb{E}(\psi_N(\alpha,\beta)|\underset{\sim}{x},\underset{\sim}{y}) &= \frac{A^{n_1+\mu}B^{n_2+\nu}}{\Gamma(n_1+\mu)\Gamma(n_2+\nu)} \Gamma(n_1+\mu+n_2+\nu) \frac{N^{n_1+\mu}}{B^{n_1+\mu+n_2+\nu}} \\ &\quad \times \int_0^1 u^{n_1+\mu}(1-u)^{n_2+\nu-1}(1-uD_N)^{-(n_1+\mu+n_2+\nu)} du \\ &= G_N^{n_1+\mu} \frac{\Gamma(n_1+\mu+n_2+\nu)}{\Gamma(n_1+\mu)\Gamma(n_2+\nu)} \\ &\quad \times \int_0^1 u^{(n_1+\mu+1)-1}(1-u)^{(n_1+\mu+n_2+\nu+1)-(n_1+\mu+1)-1} \\ &\quad \times (1-uD_N)^{-(n_1+\mu+n_2+\nu)} du. \end{aligned}$$

Using the hypergeometric function of the form

$$ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt, \qquad |z| < 1, $$

which is given in Abramowitz and Stegun (1972), we have

$$\begin{aligned} \mathbb{E}(\psi_N(\alpha,\beta)|\underset{\sim}{x},\underset{\sim}{y}) &= G_N^{n_1+\mu} \frac{n_1+\mu}{n_1+\mu+n_2+\nu} \\ &\quad \times {}_2F_1(n_1+\mu+n_2+\nu, n_1+\mu+1, n_1+\mu+n_2+\nu+1; D_N), \\ &\qquad |D_N| < 1. \end{aligned} \tag{20}$$

The case of $D_N \leq -1$ is as follow. Using the form of

$$ {}_2F_1(a,b,c;z) = (1-z)^{-a} \times {}_2F_1(a, c-b, c; \frac{z}{1-z}), $$

which is given in Abramowitz and Stegun (1972), we have

$$ \mathbb{E}(\psi_N(\alpha,\beta)|\underset{\sim}{x},\underset{\sim}{y}) = G_N^{n_1+\mu} \frac{n_1+\mu}{n_1+\mu+n_2+\nu}(1-D_N)^{-(n_1+\mu+n_2+\nu)} $$

$$\times{}_2F_1(n_1+\mu+n_2+\nu, n_2+\nu, n_1+\mu+n_2+\nu+1; \frac{D_N}{1-D_N}),$$
$$D_N \leq -1,$$

and

$$\begin{aligned}\mathbb{E}(\psi_N(\alpha,\beta)|\underset{\sim}{x},\underset{\sim}{y}) &= G_N^{-(n_2+\nu)}\frac{n_1+\mu}{n_1+\mu+n_2+\nu}\\ &\times{}_2F_1(n_1+\mu+n_2+\nu, n_2+\nu, n_1+\mu+n_2+\nu+1; \frac{D_N}{1-D_N}),\\ &D_N \leq -1.\end{aligned} \tag{21}$$

Using the Equations (20) and (21), and in view of (19), the Bayes estimator of $\psi_N(\alpha,\beta)$ is as follow

$$\hat{\psi}_N^B(\alpha,\beta)=\mathbb{E}(\psi_N(\alpha,\beta|\underset{\sim}{x},\underset{\sim}{y}) = \frac{n_1+\mu}{n_1+\mu+n_2+\nu}$$
$$\times\begin{cases} G_N^{n_1+\mu}\times_2 F_1(n_1+\mu+n_2+\nu, n_1+\mu+1, n_1+\mu+n_2+\nu+1; D_N), \\ \qquad |D_N|<1, \\ G_N^{-(n_2+\nu)}\times_2 F_1(n_1+\mu+n_2+\nu, n_2+\nu, n_1+\mu+n_2+\nu+1; \frac{D_N}{1-D_N}), \\ \qquad D_N\leq -1. \end{cases}$$

Therefore from (4), the Bayes estimator of R_φ is as follow

$$\hat{R}_\varphi^B = 2\hat{\psi}_2^B(\alpha,\beta) - \hat{\psi}_3^B(\alpha,\beta). \tag{22}$$

3.2 Estimation of R_φ based on Pareto distribution

A Pareto distribution is used for data whose failure rate decreases over time. This distribution provides a good model for biomedical issues, such as survival time after a heart transplant. This distribution with the shape parameter α and location parameter γ will be denoted by $Pa(\alpha,\gamma)$. The pdf, cdf and failure rate functions of $X \sim Pa(\alpha,\gamma)$ are given respectively as

$$f_X(x) = \frac{\alpha\gamma^\alpha}{x^{\alpha+1}}, \quad F_X(x) = 1-(\frac{\gamma}{x})^\alpha, \quad r_X(x) = \frac{\alpha}{x}, \qquad x>\gamma, \alpha>0, \gamma>0.$$

This distribution is a member of the class of exponential form of distributions. Let the stress X and the strength Y be independent random variables and their distributions are $Pa(\alpha,\gamma_0)$ and $Pa(\beta,\gamma_0)$, respectively. The common scale parameter γ_0 is known. For a series system with N components and by using (2), we have

$$P(X<Y_{1:N}) = \int_{\gamma_0}^{+\infty}(\frac{\gamma_0}{x})^{N\beta}\frac{\alpha\gamma_0^\alpha}{x^{\alpha+1}}dx = \frac{\alpha}{\alpha+N\beta} = \psi_N(\alpha,\beta),$$

and from (3) for the radar system we have

$$R_\varphi = 2\psi_2(\alpha,\beta) - \psi_3(\alpha,\beta). \tag{23}$$

Suppose that $X_1, X_2, \ldots, X_{n_1}$ and $Y_1, Y_2, \ldots, Y_{n_2}$ are two independent random samples from $Pa(\alpha, \gamma_0)$ and $Pa(\beta, \gamma_0)$, respectively. The likelihood function is as follow

$$L(\alpha, \beta) = \frac{\alpha^{n_1} \gamma_0^{n_1 \alpha}}{(\prod_{i=1}^{n_1} x_i)^{\alpha+1}} \frac{\beta^{n_2} \gamma_0^{n_2 \beta}}{(\prod_{i=1}^{n_2} y_i)^{\beta+1}}. \tag{24}$$

By taking differentiating from $l(\alpha, \beta) = \log L(\alpha, \beta)$, MLEs of the shape parameters α and β are as follow

$$\hat{\alpha} = \frac{1}{\frac{1}{n_1} \sum_{i=1}^{n_1} \log x_i - \log \gamma_0}, \qquad \hat{\beta} = \frac{1}{\frac{1}{n_2} \sum_{i=1}^{n_2} \log y_i - \log \gamma_0}. \tag{25}$$

According to the invariance property of MLE and from (23), we get the MLE of R_φ as follow

$$\hat{R}_\varphi^M = 2\psi_2^M(\hat{\alpha}, \hat{\beta}) - \psi_3^M(\hat{\alpha}, \hat{\beta}). \tag{26}$$

Because the second order derivatives of $l(\alpha, \beta)$ are such as (8), the asymptotic covariance matrix Σ, the limit distribution of $(\hat{\alpha}, \hat{\beta})$, ACI for α and β, the limit distribution of $\hat{R}_\varphi^M$, and ACI for R_φ, are the same as (9), (7), (10), (11) and (14), respectively. Note that MLEs of α and β are given in (25) and MLE of R_φ is given in (26).

For UMVUE of R_φ, assume that $X_1 \sim Pa(\alpha, \gamma_0)$ and $Y_1 \sim Pa(\beta, \gamma_0)$ are independent random variables and define $g(X_1, Y_1)$ as follow

$$g(X_1, Y_1) = I(\frac{1}{\gamma_0^{N-1}} X_1^N < Y_1).$$

$g(X_1, Y_1)$ is an unbiased estimator of $\psi_N(\alpha, \beta)$ as

$$\begin{aligned} \mathbb{E}(g(X_1, Y_1)) &= \int_{\gamma_0}^{+\infty} (\int_{\frac{1}{\gamma_0^{N-1}} x_1^N}^{+\infty} \frac{\beta \gamma_0^\beta}{y_1^{\beta+1}} dy_1) \frac{\alpha \gamma_0^\alpha}{x_1^{\alpha+1}} dx_1 = \int_{\gamma_0}^{+\infty} (\frac{\gamma_0}{x_1})^{N\beta} \frac{\alpha \gamma_0^\alpha}{x_1^{\alpha+1}} dx_1 \\ &= \alpha \gamma_0^{\alpha + N\beta} \int_{\gamma_0}^{+\infty} x_1^{-(\alpha+N\beta)-1} dx_1 = \psi_N(\alpha, \beta). \end{aligned}$$

Let $M_i = \log \frac{X_i}{\alpha}$, for $i = 1, 2, ..., n_1$ and $N_i = \log \frac{Y_i}{\beta}$, for $i = 1, 2, ..., n_2$. Because $M_i \sim E(\alpha)$ and $N_i \sim E(\beta)$, the statistic $T = (T_1, T_2) = (\sum_{i=1}^{n_1} M_i, \sum_{i=1}^{n_2} N_i)$ is the complete sufficient statistic. UMVUE of $\psi_N(\alpha, \beta)$ is then as follow

$$\begin{aligned} \hat{\psi}_N^U(\alpha, \beta) &= \mathbb{E}(g(X_1, Y_1)|T) = P((\frac{X_1}{\gamma_0})^N < \frac{Y_1}{\gamma_0}|T) = P(N \log \frac{X_1}{\gamma_0} < \log \frac{Y1}{\gamma_0}|T) \\ &= P(NM_1 < N_1|T) = P(NVS_1 < S_2|T), \end{aligned} \tag{27}$$

where $S_1 = \frac{M_1}{T_1}, S_2 = \frac{N_1}{T_2}$ and $V = \frac{T_1}{T_2}$. Because S_1 and S_2 have Beta distributions with parameters $(1, n_1 - 1)$ and $(1, n_2 - 1)$, respectively, S_1 and S_2 are ancillary statistics. By using Basu theorem (Basu, 1955), (S_1, S_2) is independent of T and conditional pdf $(S_1, S_2)|T$ is the same as (15). $\hat{\psi}_N^U(\alpha, \beta)$ in (27) is the same as (16), where

$$V = \frac{T_1}{T_2} = \frac{\sum_{i=1}^{n_1} M_i}{\sum_{i=1}^{n_2} N_i} = \frac{\sum_{i=1}^{n_1} \log \frac{X_i}{\gamma_0}}{\sum_{i=1}^{n_2} \log \frac{Y_i}{\gamma_0}}.$$

Therefore from (23) we have

$$\hat{R}_{\varphi}^{U} = 2\hat{\psi}_{2}^{U}(\alpha,\beta) - \hat{\psi}_{3}^{U}(\alpha,\beta).$$

Now to obtain the Bayes estimator of R_{φ} under the square error loss function, from (24) we have

$$L(\alpha,\beta) = \alpha^{n_1}\beta^{n_2}e^{-((\alpha+1)\sum_{i=1}^{n_1}\log x_i + (\beta+1)\sum_{i=1}^{n_2}\log y_i)}e^{(n_1\alpha+n_2\beta)\log\gamma_0}.$$

Assume that prior distribution of (α,β) be the same as (18). Then the posterior pdf of (α,β) is as follows

$$\pi(\alpha,\beta|\underset{\sim}{x},\underset{\sim}{y}) = \frac{A^{n_1+\mu}B^{n_2+\nu}}{\Gamma(n_1+\mu)\Gamma(n_2+\nu)}\alpha^{n_1+\mu-1}\beta^{n_2+\nu-1}e^{-(A\alpha+B\beta)},$$

where $A = \sum_{i=1}^{n_1}\log x_i - n_1\log\gamma_0 + \gamma$ and $B = \sum_{i=1}^{n_2}\log y_i - n_2\log\gamma_0 + \lambda$. Therefore the Bayes estimator of R_{φ} is the same as (22). Note that in subsection 3.1, the scale parameters α and β and in subsection 3.2 the shape parameters α and β are estimated.

4 Simulation and real data

In this section, we study the performance of different estimations of R_{φ} and consider two real data sets.

4.1 Simulation

In this subsection we present some experimental results about the estimation methods. We consider different values for sample sizes and the stress-strength parameters in the exponential distribution. We assume that $n_1 = n_2 = n = 5, 10, 20, 30, 50, 100$, stress-strength parameters as $(\alpha,\beta) = (4, 0.5), (3.5, 1), (3, 1.5), (2.5, 2), (2, 2.5), (1.5, 3)$ and $(\mu,\gamma,\nu,\lambda) = (2.75, 1, 1.75, 1)$.

Table 1 show that the values of R_{φ} and by using (6), (17) and (22), we derive the MLE, UMVUE and bayes estimator and compute their mean square errors (MSE). We observe that as n increases, the values of MSE for MLE, UMVUE and Bayes estimator are decreasing. Also in the most cases, the MSE values of MLE and UMVUE are less than those of Bayes estimator, so MLE and UMVUE have better results.

Table 1: R_φ, MLE, UMVUE, and Bayes estimations of R_φ and their MSEs

(α,β)	n	R_φ	$\hat{R}_\varphi^M$	$MSE(\hat{R}_\varphi^M)$	$\hat{R}_\varphi^U$	$MSE(\hat{R}_\varphi^U)$	$\hat{R}_\varphi^B$	$MSE(\hat{R}_\varphi^B)$
	5	0.87273	0.84942	0.00917	0.87043	0.00821	0.79130	0.01344
	10	0.87273	0.86216	0.00370	0.87255	0.00339	0.82640	0.00553
(4,0.5)	20	0.87273	0.86717	0.00166	0.87232	0.00158	0.84804	0.00224
	30	0.87273	0.86949	0.00108	0.87291	0.00104	0.85648	0.00134
	50	0.87273	0.87064	0.00062	0.87268	0.00061	0.86271	0.00073
	100	0.87273	0.87174	0.00030	0.87276	0.00030	0.86811	0.00034
	5	0.73427	0.71113	0.01974	0.73066	0.02219	0.69003	0.02646
	10	0.73427	0.72367	0.00944	0.73398	0.00995	0.70443	0.00835
(3.5,1)	20	0.73427	0.72830	0.00466	0.73354	0.00477	0.71174	0.00424
	30	0.73427	0.73109	0.00312	0.73462	0.00317	0.71852	0.00297
	50	0.73427	0.73207	0.00186	0.73420	0.00187	0.72398	0.00183
	100	0.73427	0.73325	0.00091	0.73431	0.00092	0.72889	0.00093
	5	0.60000	0.58775	0.02507	0.59610	0.03081	0.60908	0.07677
	10	0.60000	0.59508	0.01281	0.59967	0.01425	0.64015	0.04094
(3,1.5)	20	0.60000	0.59678	0.00656	0.59911	0.00692	0.64995	0.01829
	30	0.60000	0.59888	0.00446	0.60046	0.00463	0.63901	0.01098
	50	0.60000	0.59897	0.00269	0.59993	0.00274	0.62277	0.00616
	100	0.60000	0.59958	0.00134	0.60005	0.00135	0.60570	0.00249
	5	0.47511	0.47582	0.02574	0.47159	0.03181	0.46418	0.13911
	10	0.47511	0.47692	0.01328	0.47476	0.01480	0.50101	0.12087
(2.5,2)	20	0.47511	0.47534	0.00682	0.47419	0.00721	0.54507	0.11029
	30	0.47511	0.47636	0.00465	0.47561	0.00482	0.58805	0.10147
	50	0.47511	0.47551	0.00280	0.47504	0.00286	0.63633	0.09550
	100	0.47511	0.47540	0.00140	0.47516	0.00141	0.70672	0.08893
	5	0.36090	0.37238	0.02283	0.35807	0.02686	0.27416	0.14719
	10	0.36090	0.36804	0.01146	0.36054	0.01236	0.23754	0.14055
(2,2.5)	20	0.36090	0.36390	0.00577	0.36004	0.00599	0.17732	0.13504
	30	0.36090	0.36394	0.00392	0.36136	0.00401	0.14570	0.13289
	50	0.36090	0.36240	0.00234	0.36084	0.00238	0.09554	0.13029
	100	0.36090	0.36173	0.00116	0.36094	0.00117	0.03901	0.12940
	5	0.25714	0.27504	0.01736	0.25508	0.01863	0.11433	0.09842
	10	0.25714	0.26710	0.00821	0.25682	0.00838	0.04784	0.07835
(1.5,3)	20	0.25714	0.26162	0.00399	0.25643	0.00402	0.01218	0.06882
	30	0.25714	0.26099	0.00268	0.25753	0.00269	0.00262	0.06669
	50	0.25714	0.25918	0.00159	0.25710	0.00159	0.00014	0.06615
	100	0.25714	0.25822	0.00078	0.25717	0.00078	0.00000	0.06612

4.2 Real data

We consider a pair of real data sets that were used by Mirjalili et al. (2016). The First and the second data sets show the breaking strength of jute fiber of gauge lengths 20 and 10 mm, respectively.

First data set (X): {71.46, 419.02, 284.64, 585.57, 456.60, 113.85, 187.85, 688.16, 662.66, 45.58, 578.62, 756.70, 594.29, 166.49, 99.72, 707.36, 765.14, 187.13, 145.96, 350.70, 547.44, 116.99, 375.81, 581.60, 119.86, 48.01, 200.16, 36.75, 244.53, 83.55}.

Second data set (Y): {693.73, 704.66, 323.83, 778.17, 123.06, 637.66, 383.43, 151.48, 108.94, 50.16, 671.49, 183.16, 257.44, 727.23, 291.27, 101.15, 376.42, 163.40, 141.38, 700.74, 262.90, 353.24, 422.11, 43.93, 590.48, 212.13, 303.90, 506.60, 530.55, 177.25}.

For each data set, we used the Kolmogorov-Smirnov (K-S) test to fit the exponential distribution. For the first data set, the MLE, K-S distance between the empirical distribution function and the fitted distribution function and the corresponding p-value are $0.0029, 0.1328$ and 0.6183, respectively. For the second data set, the corresponding values are $0.0027, 0.1750$ and 0.2831, respectively. We see that for these data sets, the exponential distribution has a good fitness. According to (6) and (17), the $\hat{R}_{\varphi}^{M}$ and $\hat{R}_{\varphi}^{U}$ are 0.4350 and 0.4334, respectively. Also by using (22), if $(\mu, \gamma, \nu, \lambda) = (2.75, 1, 1.75, 1)$, we have $\hat{R}_{\varphi}^{B} = 0.7169$.

Discussion and conclusions

In this paper, for the general case of a coherent system, we considered the reliability of the stress-strength model when the stress is at the system level and obtained it for different systems. We derived this reliability for a wide class of distributions, called the exponential form of distributions. If the distributions of the stresses and strengths are exponential or Pareto, by using different methods we also estimated the stress-strength reliability for the radar system. The results of simulation indicate that the MLE and UMVUE are proper estimations. Further researches in this model can be done for other systems and other members of exponential form of distributions.

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