Metrika (2023) 86:605–625
https://doi.org/10.1007/s00184-022-00886-x

A new light on reliability equivalence factors

Jalal Etminan[1] · Mohammad Khanjari Sadegh[1] · Majid Chahkandi[1]

Received: 10 December 2021 / Accepted: 6 August 2022 / Published online: 3 October 2022

Abstract
In the reliability theory, the performance of a system can be improved by different methods, such as redundancy and reduction methods. The redundancy method may not be optimal when some restrictions such as volume and weight are crucial. In the reduction method, system reliability is increased by reducing the failure rate of some of its components by a factor $0 < \rho < 1$ which is called the reliability equivalence factor (REF). This article considers a new light on reliability equivalence factors in a coherent system with independent components. A closed form for ρ is obtained when the reduction method is applied on a single component of the system. Based on this, we also define a new measure of component importance. Various numerical illustrative examples are given to support the new results.

Keyword Reduction method · Redundancy method · Reliability equivalence factors · Importance measure

1 Preliminaries and introduction

In a fixed point of time t, consider a coherent system with n independent components. The following basic concepts and notations are required in the next sections. Suppose

$$X_i(t) = \begin{cases} 1 & if \ T_i > t \\ 0 & otherwise, \end{cases}$$

where T_i is the lifetime of component i and let $p_i(t) = E(X_i(t)) = P(T_i > t) = \bar{F}_i(t)$ be its reliability, $i = 1, \ldots, n$. Also

$$\varphi(\mathbf{X}(t)) = \varphi((X_1(t), X_2(t), \ldots, X_n(t))) = \begin{cases} 1 & if \ T > t \\ 0 & otherwise, \end{cases}$$

✉ Mohammad Khanjari Sadegh
mkhanjari@birjand.ac.ir

[1] Department of Statistics, University of Birjand, Birjand, Iran

and $h(\mathbf{p}(t)) = h(p_1(t), \ldots, p_n(t)) = E\varphi(\mathbf{X}(t)) = P(T > t)$ are called the structure and reliability functions of the system, respectively, and T is the system lifetime. Birnbaum reliability importance of component i is defined as $I_B(i; \mathbf{p}) = \dfrac{\partial h(\mathbf{p})}{\partial p_i}$ which is equal to $P(\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x}) = 1) = h(1_i, \mathbf{p}) - h(0_i, \mathbf{p})$ where $(0_i, \mathbf{x}) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$ and $(0_i, \mathbf{p}) = (p_1, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_n)$.

As a special case of $I_B(i; \mathbf{p})$, the Birnbaum structural importance of component i is given by

$$I_\varphi(i) = I_B(i; \mathbf{p})|_{p_1=\cdots=p_n=\frac{1}{2}} = \frac{1}{2^{n-1}} \sum_{\{(\cdot_i, \mathbf{x})\}} [\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})],$$

where $(\cdot_i, \mathbf{x})$ represents $\mathbf{x}$ without the component i. In fact, $I_\varphi(i)$ is equal to the probability that the system is in such a state that component i is critical for the system when $p_1 = \cdots = p_n = \frac{1}{2}$. In other words, the structural importance of component i is the ratio between the number of component state vectors in which the state of component i dictates the state of the system with that of the total number of other component state vectors. The structural importance of a component actually measures the importance of the position of the component. Note that $I_B(i; \mathbf{p})$ is independent of p_i, see Barlow and Proschan (1975a), Shaked and Shanthikumar (1990) and Kuo and Zuo (2003) for more details. The Birnbaum measure of component importance has been widely studied and referred, see Kuo and Zuo (2003), Barlow and Proschan (1975b) and Shen and Xie (1989). Xie and Shen (1989) defined a general importance measure of component as the increase of the system reliability due to an improvement of the component i as

$$I_{RIP}(i; \mathbf{p}) = h(p_i', \mathbf{p}) - h(\mathbf{p}),$$

where $h(p_i', \mathbf{p})$ is the system reliability after improving the reliability of component i. This importance measure depends on what improvement action is taken on the component level. They have also obtained an interesting relation for $I_{RIP}(i; \mathbf{p})$ as

$$I_{RIP}(i; \mathbf{p}) = (p_i' - p_i) I_B(i; \mathbf{p}). \tag{1}$$

In reliability analyses, any system is assumed to have a finite life that can be extended by using components with high reliability or by adding redundant components to the original components. These techniques are known as “ reduction" and “ redundancy" methods, respectively. Reduction and redundancy are two main approaches in improving system reliability. In redundancy method, some system components are duplicated by redundant components. Whereas in reduction method, system reliability is increased by reducing the failure rates of some of its components by a factor $0 < \rho < 1$. In redundancy allocation, it is generally accepted that redundant components may be inserted into the system via two methods, known as active and standby redundancies. In an active redundancy, the redundant components are inserted in parallel to the original components of the system. In standby redundancy the redundant components start working immediately after component failures, see Bayramoglu Kavlak (2017) and Billinton and Allan (1992).

A standby component has a failure rate while it is in standby. There are three types of standby: hot standby, warm standby and cold standby. Hot standby components are also called active redundant components. A hot standby component has the same failure rate as the active component. A cold standby component has a zero failure rate. In other words, it does not fail while in standby. Warm standby implies that inactive components have a failure rate that is between 0 and the failure rate of active components. Warm standby may include cold standby and hot standby as extreme cases. A warm standby component is not an active component. However, it may fail while in the standby condition. An example of a cold standby component is a spare light bulb in an overhead projector. An example of warm standby is a power plant, which often has at least one extra generating unit spinning so that it can be switched into full operation quickly when needed, see Kuo and Zuo (2003). The standby redundancy methods are plagued by space limitation, complexity and higher development costs. So, the reduction method may be an appropriate alternative.

Råde (1989) introduced the concept of reliability equivalence factors. The reliability equivalence factor (REF) is a factor as $0 < \rho < 1$ by which the failure rates of some system components should be reduced to reach the value of reliability similar to a system that improved by a redundancy method. The reliability equivalence factors are well described in Råde (1989, 1990, 1991), Sarhan (2000, 2002, 2005, 2009), Sarhan et al. (2008). In a real case, the reduction policy can be applied in different ways. If we are designing the system, we can use higher quality components which is equivalent with components with lower failure rate. For example, consider a battery of a mobile phone. We know that the battery life of the smart phone is very important for the users. Thus, many manufacturing companies are working to produce new types of batteries with expected lifetimes which are at least double that of the current battery. If we are in a point of time $t = t_0$, we can replace the original component with a component with more quality or if the original component is failed we can repair it. An imperfect repair is equivalent to reduction policy with $0 \leq \rho \leq 1$. Perfect repair and minimal repair can be considered as special cases of imperfect repair. A perfect repair is equivalent to reduction method with $\rho = 0$ and in a minimal repair the failure rate is not changed as a result of the repair which is equivalent to reduction policy with $\rho = 1$. Maintenance operations are usually time consuming and expensive. We have to turn off the system when we are doing the repair actions. Thus, the reduction method may be cheaper in these situations.

It should be noted that the reduction method may not be used in practice. This method can be considered as a technique for comparing the different redundancy policies. We can find the reduction factor ρ such that two different redundancy methods are equivalent and compare different policies based on their correspond reduction factors. This is one of the most advantage of the reduction policy. This method also has some limitations such as other redundancy methods. Sometimes we can not find the reduction factor ρ such that the system reliability (or mean times to failure) reaches to the desired values. Because, the reduction factor is limited to the interval [0, 1].

The reduction factors are often obtained by numerical methods and mathematical packages in literature. In this paper based on the system reliability function, the equivalence between redundancy and reduction methods in a general coherent system with independent components is investigated. In Sect. 2, we present a closed-formula solution to the problem when the reliability function of one component improved according to the reduction method. In Sect. 3, we introduce a new idea to consider the REF as an importance measure of component. A preliminary comparison indicates that our idea has advantageous in selecting the best component to apply reduction method. We also study the relationship between REF as a measure of importance with some other measures, such as Xie and Shen (1989) measures. Finally in Sect. 4, based on the mean reliability equivalent factor we define a new relative importance measure of components. Some numerical examples are also given to illustrate how the theoretical results obtained in this paper can be applied.

2 Reduction method

The use of redundancy method may not be optimal in systems when the minimum size and weight are overriding considerations. In these cases improving the system reliability through other alternative methods such as reduction method is done. In this section we obtain a closed formula for the REF when the reduction method on a single component of the system is used.

Assume that the reliability of ith component, $p_i(t)$, with failure rate $r_i(t)$, can be increased by reducing its failure rate via $r_i'(t) = \rho_i r_i(t)$ by a factor ρ_i such that $0 \leq \rho_i \leq 1$. Then the reliability function of component i after reducing its failure rate function is given by

$$p_i'(t) = e^{-\int_0^t \rho_i r_i(x)\,\mathrm{d}x} = [p_i(t)]^{\rho_i}.$$

Let

$$(\mathbf{p}(t)\,;\boldsymbol{\rho}) = \mathbf{p}'(t) = (p_1'(t), \ldots, p_n'(t)),$$

denote the improved reliability vector of the system components with corresponded reduction factors $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$. If we reduce the component failure rates of subset K by reduction vector $\boldsymbol{\rho}_K = (\rho_{1_K}, \ldots, \rho_{n_K})$, such that

$$\rho_{i_K} = \begin{cases} \rho_i & i \in K \\ 1 & i \notin K, \end{cases} \tag{2}$$

then the system reliability can be expressed as $R_{\boldsymbol{\rho}_K}(t) = h(\mathbf{p}(t)\,;\boldsymbol{\rho}_K)$; where $R_{\boldsymbol{\rho}_K}(t) \in \left[h(\mathbf{p}(t)), h(\mathbf{1}_K, \mathbf{p}(t))\right]$ and $(\mathbf{1}_K, \mathbf{p}(t))$ denotes an improved reliability vector $\mathbf{p}'(t)$ such that

$$p_i'(t) = \begin{cases} 1 & i \in K \\ p_i(t) & i \notin K. \end{cases}$$

If ω be an arbitrary level of system reliability, then for $\omega \in \left[h(\mathbf{p}(t)), h(\mathbf{1}_K, \mathbf{p}(t))\right]$ there exists a vector of component reduction factors $\boldsymbol{\rho}_K$, such that $R_{\rho_K}(t) = \omega$. Also for some arbitrary sets K_i, $i \in I$ of the system components, if $\omega \in \left[h(\mathbf{p}(t)), \min_i\{h(\mathbf{1}_{K_i}, \mathbf{p}(t))\}\right]$ then there exist some vectors of component reduction factors $\boldsymbol{\rho}_{K_i}$, $i \in I$, where I is a finite index set.

In the next theorem, we present a closed form for the reduction factor of the component i. We suppose that the reliability of the system components are known at a particular instant t, and then reduce the failure rate of the ith component such that the system reliability is increased to ω. Since the time is fixed and the reduction factor depends on i and ω, we denote the reduction factor by $\rho_i(\omega)$.

Theorem 2.1 *Consider a coherent system with n independent component lifetimes. Suppose that the failure rate function of the component i is reduced by reduction factor ρ_i such that the system reliability is increased to specified value ω. Then*

$$\rho_i(\omega) = \begin{cases} 1 & if\ \omega = h(\mathbf{p}) \\ \frac{1}{\ln p_i} \times \ln\left(\dfrac{\omega - h(0_i, \mathbf{p})}{I_B(i; \mathbf{p})}\right) & if\ \omega \in (h(\mathbf{p}), h(1_i, \mathbf{p})) \\ 0 & if\ \omega = h(1_i, \mathbf{p}). \end{cases}$$

In particular, if $p_j = \frac{1}{2}$ for all $j \neq i$, then

$$\rho_i(\omega) = \begin{cases} 1 & if\ \omega = h(p_i, \mathbf{p}^{(i)}_{\frac{1}{2}}) \\ \frac{1}{\ln p_i} \times \ln\left(\dfrac{\omega - h(p_i, \mathbf{p}^{(i)}_{\frac{1}{2}})}{I_\varphi(i)}\right) & if\ \omega \in \left(h(p_i, \mathbf{p}^{(i)}_{\frac{1}{2}}), h(1_i, \mathbf{p}^{(i)}_{\frac{1}{2}})\right) \\ 0 & if\ \omega = h(1_i, \mathbf{p}^{(i)}_{\frac{1}{2}}), \end{cases}$$

where $\mathbf{p}^{(i)}_{\frac{1}{2}} = (\underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{i-1}, \cdot, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{n-i})$.

Proof : By using the pivotal decomposition on component i, we have

$$\begin{aligned} h(\mathbf{p}) &= p_i h(1_i, \mathbf{p}) + (1 - p_i) h(0_i, \mathbf{p}) \\ &= h(0_i, \mathbf{p}) + p_i I_B(i; \mathbf{p}). \end{aligned}$$

For reduction set $K_i = \{i\}$; $i = 1, \ldots, n$, we obtain

$$h(\mathbf{p}\,; \boldsymbol{\rho}_{K_i}) = h(0_i, \mathbf{p}) + [p_i]^{\rho_i}\, I_B(i; \mathbf{p}).$$

Note that $h(\mathbf{p}\,; \boldsymbol{\rho}_{K_i}) \in \left[h(\mathbf{p}), h(1_i, \mathbf{p})\right]$. Thus, for any $\omega \in (h(\mathbf{p}), h(1_i, \mathbf{p}))$, we derive $h(0_i, \mathbf{p}) + [p_i]^{\rho_i(\omega)}\, I_B(i; \mathbf{p}) = \omega$ or equivalently $\rho_i(\omega) = \frac{1}{\ln p_i} \times \ln\left(\dfrac{\omega - h(0_i, \mathbf{p})}{I_B(i; \mathbf{p})}\right)$.

Thus,

$$\rho_i(\omega) = \begin{cases} 1 & if\ \omega = h(\mathbf{p}) \\ \frac{1}{\ln p_i} \times \ln\left(\frac{\omega - h(0_i, \mathbf{p})}{I_B(i; \mathbf{p})}\right) & if\ \omega \in (h(\mathbf{p}), h(1_i, \mathbf{p})) \\ 0 & if\ \omega = h(1_i, \mathbf{p}). \end{cases}$$

In particular, if $p_j = \frac{1}{2}$ for all $j \neq i$, then $I_B(i; \mathbf{p}) = I_\varphi(i)$ and the system reliability is given by

$$h(p_i, \mathbf{p}_{\frac{1}{2}}^{(i)}) = h(0_i, \mathbf{p}_{\frac{1}{2}}^{(i)}) + p_i I_\varphi(i),$$

where $\mathbf{p}_{\frac{1}{2}}^{(i)} = (\underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{i-1}, \cdot, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{n-i})$. Since $h(\mathbf{p}_{\frac{1}{2}}^{(i)}; \boldsymbol{\rho}_{K_i}) \in \left[h(p_i, \mathbf{p}_{\frac{1}{2}}^{(i)}), h(1_i, \mathbf{p}_{\frac{1}{2}}^{(i)})\right]$, for any ω belongs to $\left(h(p_i, \mathbf{p}_{\frac{1}{2}}^{(i)}), h(1_i, \mathbf{p}_{\frac{1}{2}}^{(i)})\right)$ we should solve $h(0_i, \mathbf{p}_{\frac{1}{2}}^{(i)}) + [p_i]^{\rho_i(\omega)} I_\varphi(i) = \omega$, that concludes $\rho_i(\omega) = \frac{1}{\ln p_i} \times \ln\left(\frac{\omega - h(0_i, \mathbf{p}_{\frac{1}{2}}^{(i)})}{I_\varphi(i)}\right)$ and this completes the proof. □

In the following, we present some examples to illustrate how the result of Theorem 2.1 can be used to find admissible bounds for a system whose reliability is improved according to the reduction method.

Example 2.1 The reliability function of the bridge system, shown in Fig. 1, is given by

$$\begin{aligned} h(\mathbf{p}) &= p_3 (1 - q_1 q_4)(1 - q_2 q_5) \\ &\quad + q_3 [1 - (1 - p_1 p_2)(1 - p_4 p_5)] \ ; \ q_i = 1 - p_i. \end{aligned}$$

Let $\mathbf{p}_0 = (0.5, 0.5, 0.5, 0.5, 0.5)$ and $\mathbf{p}_1 = (0.8, 0.6, 0.5, 0.3, 0.7)$ be respectively the homogeneous and non-homogeneous vectors of the component reliabilities. Table 1 provides the lower and upper bounds for system reliability when the system is improved by reducing the failure rate of component i. For example, if $\mathbf{p} = \mathbf{p}_0$ and $K_1 = \{1\}$, then there exists the reduction factor ρ_1 such that the system reliability is increased to $\omega \in [0.5, 0.6875]$. Note that by applying the reduction method on component 1, we can not improve the system reliability to $\omega \in [0.6875, 1]$. If $\omega \in \left[h(\mathbf{p}), \min_i\{h(1_i, \mathbf{p})\}\right]$, then all reduction factors ρ_i, $i = 1, \ldots, n$ exist. Thus, for $\mathbf{p} = \mathbf{p}_0$ and by reducing the failure rate of each component of the system, we can improve the system reliability to ω, where $\omega \in [0.5, 0.5625]$. For $\mathbf{p} = \mathbf{p}_1$ this interval becomes $[0.673, 0.748]$. Figure 2 illustrates the reduction factors $\rho_1(\omega) = \frac{1}{\ln p_1} \times \ln\left(\frac{\omega - h(0_1, \mathbf{p}_{\frac{1}{2}}^{(1)})}{I_\varphi(1)}\right)$ and

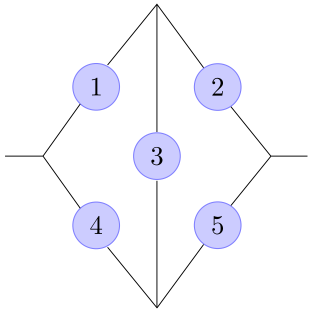

Fig. 1 A bridge system

Table 1 Admissible intervals of ω by reducing the failure rate function of component i in the bridge system

Reduction set	Interval $\left[h(\mathbf{p}), h(1_i, \mathbf{p})\right]$	
	$\mathbf{p} = \mathbf{p}_0$	$\mathbf{p} = \mathbf{p}_1$
$K_1 = \{1\}$	(0.5, 0.6875)	(0.673, 0.7820)
$K_2 = \{2\}$	(0.5, 0.6875)	(0.6730, 0.8510)
$K_3 = \{3\}$	(0.5, 0.5625)	(0.6730, 0.7568)
$K_4 = \{4\}$	(0.5, 0.6875)	(0.6730, 0.8620)
$K_5 = \{5\}$	(0.5, 0.6875)	(0.6730, 0.7480)
$\left[h(\mathbf{p}), \min_i \{h(1_i, \mathbf{p})\}\right]$:	(0.5, 0.5625)	(0.6730, 0.7480)

$\rho_1'(\omega) = \frac{1}{\ln p_1} \times \ln\left(\frac{\omega - h(0_i, \mathbf{p})}{I_B(1; \mathbf{p})}\right)$, respectively for the homogeneous and non-homogeneous vectors.

In the next example, we are interested in finding the most appropriate component for reducing failure rate when the importance measures are given.

Example 2.2 Consider the bridge system in Fig. 1. The Birnbaum structural importance measures of the system components are given by $I_\varphi(1) = I_\varphi(2) = I_\varphi(4) = I_\varphi(5) = 0.375$ and $I_\varphi(3) = 0.125$. Thus, the components 1,2,4 and 5 have the same and the highest structural importance. If $\mathbf{p} = \mathbf{p}_1 = (0.8, 0.6, 0.5, 0.3, 0.7)$, then the most important component based on the Birnbaum reliability importance measure is component 1. It is known that for $\mathbf{p} = \mathbf{p}_0 = (0.5, 0.5, 0.5, 0.5, 0.5)$ the Birnbaum reliability importance reduces to Birnbaum structural importance. If we compute the general importance measure introduced by Shen and Xie (1989), we find that the most important components for $\mathbf{p} = \mathbf{p}_0$ are $\{1, 2, 4, 5\}$ and for $\mathbf{p} = \mathbf{p}_1$ is component 4. Now, we consider the reliability vectors $\mathbf{p}_0$ and $\mathbf{p}_1$ and find the best component for reducing the failure rate. From Table 1 we know that for $\mathbf{p} = \mathbf{p}_0$ and $\omega \in [0.5, 0.5625]$ all the reduction factors ρ_i; $i = 1, \ldots, 5$ exist. So let us to consider $\omega = 0.5625$ and calculate the reduction factors. Table 2 shows that for $\mathbf{p} = \mathbf{p}_0$ we have $\rho_1 = \rho_2 = \rho_4 = \rho_5 = 0.5850 \geq \rho_3 = 0$. Thus, reducing the failure rate of component i; $i \in \{1, 2, 4, 5\}$ concludes more improving than component 3. For $\mathbf{p} = \mathbf{p}_1$ we select $\omega = 0.748$ because there exist all of the reduction factors for $\omega \in [0.5, 0.748]$. From

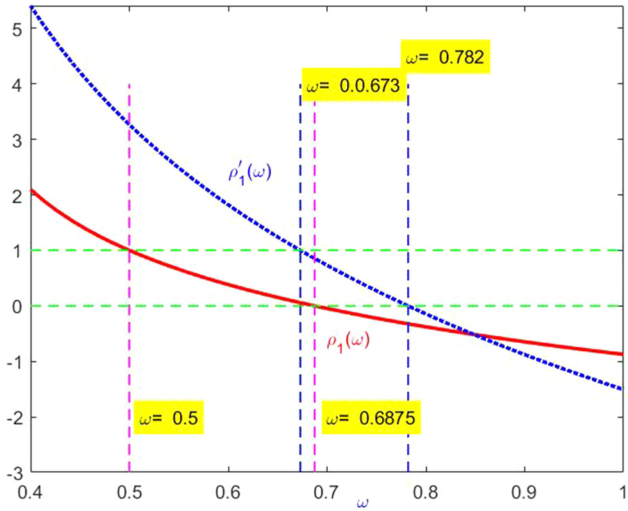

Fig. 2 The values of $\rho_1(\omega)$ (solid line) and $\rho_1'(\omega)$ (dotted line) in Example 2.1

Table 2, we find that component 2 which is the most important component has the highest reduction factor and therefore is the best component for reducting the failure rate.

Example 2.2 motivates us to a new light of the reduction factor as a new importance measure. We follow this idea in the next section.

3 Relative importance of components based on reliability equivalence factor

As mentioned in Sect. 1, in redundancy method, the system reliability can be improved using active or standby redundancy in three types of hot, warm, and cold standby. We suppose that the system reliability can be improved by a set of different methods of redundancy that we call them redundancy mechanisms. We denote by $\mathcal{Q}$ the set of all redundancy mechanisms, and an arbitrary element of $\mathcal{Q}$ is denoted by Q. For example, $Q_1 = \{H = \{i\}\}$ means that we improve the system reliability by adding a hot standby component to the component i or $Q_2 = \{H = \{i\}, C = \{j\}\}$ means that the system reliability is improved by adding a hot standby component to the component i and a cold standby component to the component j of the system. We will denote the reliability of improved system via an arbitrary redundancy mechanism Q, by $h(\mathbf{p}; Q)$.

The survival reliability equivalence factor (SREF) for a redundancy mechanism Q is defined in the following.

Definition 3.1 Let $R_{\boldsymbol{\rho}_{0_K}}(t) = h(\mathbf{p}(t); \boldsymbol{\rho}_{0_K})$ denote the reliability function of the improved system by reducing the component failure rates of the subset K by the

Table 2 Reduction factors and component importance measures in Example 2.1

Reduction set	$I_\varphi(K_i)$	$I_B(K_i; \mathbf{p})$		$h(\mathbf{p}; \boldsymbol{\rho}_{K_i}) - h(p)$		$\rho_{K_i}(\omega = 0.5625)$	$\rho_{K_i}(\omega = 0.748)$
		$p = p_0$	$p = p_1$	$p = p_0$	$p = p_1$	$p = p_0$	$p = p_1$
$K_1 = \{1\}$	0.375	0.375	0.545	0.1875	0.1090	0.5850	0.2887
$K_2 = \{2\}$	0.375	0.375	0.445	0.1875	0.1780	0.5850	0.5154
$K_3 = \{3\}$	0.125	0.125	0.167	0.0625	0.0838	0	0.0778
$K_4 = \{4\}$	0.375	0.375	0.270	0.1875	0.1890	0.5850	0.4556
$K_5 = \{5\}$	0.375	0.375	0.250	0.1875	0.0750	0.5850	0
Best component	{1, 2, 4, 5}	{1, 2, 4, 5}	{1}	{1, 2, 4, 5}	{4}	{1, 2, 4, 5}	{2}

reduction vector $\boldsymbol{\rho}_{0_K} = (\rho_{1_K}, \ldots, \rho_{n_K})$, where

$$\rho_{i_K} = \begin{cases} \rho_0 & i \in K \\ 1 & i \notin K, \end{cases}$$

and $h(\mathbf{p}(t)\,;\,Q)$ denote the reliability function of the improved system by using an arbitrary redundancy mechanism Q. Then, a solution ρ_0 of equation

$$h(\mathbf{p}(t)\,;\,Q) = h(\mathbf{p}(t)\,;\,\rho_K), \tag{3}$$

is said the SREF of mechanism Q in redundancy method.

Since $\max_{\rho_K} h(\mathbf{p}(t)\,;\,\rho_K) = h(\mathbf{1}_K, \mathbf{p}(t))$ and $\min_{\rho_K} h(\mathbf{p}(t)\,;\,\rho_K) = h(\mathbf{p}(t))$, so for any Q such that $h(\mathbf{p}(t)\,;\,Q) \in \left[h(\mathbf{p}(t)), h(\mathbf{1}_K, \mathbf{p}(t))\right]$ there exists a ρ_0 that satisfies in (3).

Now, we can introduce a new measure of importance based on the equivalence factors in reduction method.

Definition 3.2 Consider an arbitrary redundancy mechanism Q such that $h(\mathbf{p}(t)\,;\,Q) = \omega$. Suppose that ρ_C and ρ_D are the SREFs for the reduction subsets C and D of the components, respectively, such that

$$h(\mathbf{p}(t)\,;\,\rho_C) = h(\mathbf{p}(t)\,;\,\rho_D) = \omega.$$

We say that the subset C is more important than the subset D in the sense of SREF, written as $C \geq_{(\rho^S,\omega)} D$, if and only if $\rho_C \geq \rho_D$.

In Definition 3.2, if $C = \{i\}$ and $D = \{j\}$, then we say that component i is more important than component j in the sense of SREF, denoted by $i \geq_{(\rho^S,\omega)} j$, if $\rho_i \geq \rho_j$.

Example 3.1 Assume that the component lifetimes of bridge system in Fig. 1 are independent and distributed exponentially with parameters $\lambda_i = i, i = 1, \ldots, 5$. Consider a redundancy mechanism Q, as $Q_1 = \{H = \{1\}\}$, such that a hot standby component from exponential distribution with $\lambda = 2$ is added to component 1. The reliability function of the bridge system whose reliability is improved according to the redundancy mechanism Q_1 is given by

$$\begin{aligned} h(\mathbf{p}(t)\,;\,Q_1) = &\left(\mathrm{e}^{-3t} - 1\right)\left(\left(\mathrm{e}^{-3t} - 1\right)\left(\mathrm{e}^{-9t} - 1\right) - 1\right) \\ &+ \mathrm{e}^{-3t}\left(\left(\mathrm{e}^{-t} - 1\right)\left(\mathrm{e}^{-4t} - 1\right) - 1\right)\left(\left(\mathrm{e}^{-2t} - 1\right)\left(\mathrm{e}^{-5t} - 1\right) - 1\right). \end{aligned}$$

The SREFs can be obtained by solving the following set of equations

$$h(\mathbf{p}(t)\,;\,Q_1) = h(\mathbf{p}(t)\,;\,\rho_{K_i}) = \omega, i = 1, \ldots, 5,$$

for the appropriate reduction factor ρ_{K_i} and time fractile $t = t_0$ corresponding to specified reliability ω. Note that we have $2^5 - 1 = 31$ possible reduction sets and we only present some of them in Table 3. The following results can be observed from Table 3.

(1) The values ρ_{K_i}; $i = 1, \ldots, 10$ in this table show that reducing the failure rate of each component belonging to the reduction set K_i by reduction factor ρ_{K_i} improves the system reliability like adding a hot redundant component from exponential distribution with $\lambda = 2$ to component 1 where the system reliability is chosen to be specified value ω. For example, for $\omega = 0.5$ after solving equation $h(\mathbf{p}(t_0)\,;\,Q_1) = 0.5$, we obtain $t_0 = 0.3316$, i.e., the system reliability of the bridge system with a hot duplication on component 1 at time $t_0 = 0.3316$ is equal to 0.5. Now if we want to have the same reliability by reducing the failure rate of only one component of the system, component 2 is the best possible for improvement.
(2) Missing values of ρ_{K_i} for $i = 3$ mean that it is not possible to reduce the failure rate for the set K_3 in order to improve the system reliability to be equivalent with the system reliability that is obtained by hot duplication on component 1. Figure 3 clearly presents $h(\mathbf{p}(t)\,;\,Q_1)$, $h(\mathbf{p}(t)\,;\,\rho_{K_3} = 1)$ and $h(\mathbf{p}(t)\,;\,\rho_{K_3} = 0)$. We see that $h(\mathbf{p}(t)\,;\,\rho_{K_3} = 0)$ has no intersection with $h(\mathbf{p}(t)\,;\,Q_1)$ except at the start and end points. We also have the same result for $\omega = 0.01$ and K_4.
(3) If $K_i \subseteq K_j$, then $\rho_{K_i} \le \rho_{K_j}$.
(4) Based on the concept of SREF, the subsets of original components that their reduction factors exist (belong to [0, 1]) are comparable . For example when $\omega = 0.01$ we have

$$\rho_{K_{10}} \ge \rho_{K_8} \ge \rho_{K_7} \ge \rho_{K_2} \ge \rho_{K_6} \ge \rho_{K_1} \ge \rho_{K_9} \ge \rho_{K_5}.$$

It can be written equivalently as

$$\begin{aligned} K_{10} &\ge_{(\rho^S,\omega=0.01)} K_8 \ge_{(\rho^S,\omega=0.01)} K_7 \ge_{(\rho^S,\omega=0.01)} K_2 \\ K_2 &\ge_{(\rho^S,\omega=0.01)} K_6 \ge_{(\rho^S,\omega=0.01)} K_1 \ge_{(\rho^S,\omega=0.01)} K_9 \\ &\ge_{(\rho^S,\omega=0.01)} K_5. \end{aligned}$$

The following theorem pertains to the comparison between original components of the system based on the SREF which is used for improving survival function in reduction method.

Theorem 3.3 *Let $h(\mathbf{p}, \rho_i)$ denote the reliability function of a coherent system improved by reducing the failure rate function of the component i. If $h(\mathbf{p}, \rho_i) = h(\mathbf{p}, \rho_j)$, then we have the following results.*

1) The reduction factors ρ_i and ρ_j satisfy in

$$(p_i^{\rho_i} - p_i) I_B(i; \mathbf{p}) = (p_j^{\rho_j} - p_j) I_B(j; \mathbf{p}). \tag{4}$$

2) If $\frac{p_j I_B(j;\mathbf{p})}{p_i I_B(i;\mathbf{p})} \ge 1$ and $p_i \ge p_j$ then $\rho_i \le \rho_j$.

Table 3 SREFs for $Q_1 = \{H = \{1\}\}$ in bridge system

Some reduction set	Reduction factor	ω				
		0.01	0.1	0.5	0.9	0.99
		t_0				
		1.5854	0.8445	0.3316	0.1083	0.0320
$K_1 = \{1\}$	ρ_{K_1}	0.9048	0.7405	0.4437	0.1865	0.0611
$K_2 = \{2\}$	ρ_{K_2}	0.9524	0.8702	0.7247	0.6205	0.5963
$K_3 = \{3\}$	ρ_{K_3}	–	–	–	–	–
$K_4 = \{4\}$	ρ_{K_4}	−0.1227 $\otimes$	0.0158	0.2066	0.1898	0.0962
$K_5 = \{5\}$	ρ_{K_5}	0.1129	0.2481	0.4572	0.5367	0.5725
$K_6 = \{1, 5\}$	ρ_{K_6}	0.9049	0.7601	0.6904	0.6905	0.7011
$K_7 = \{2, 3\}$	ρ_{K_7}	0.9524	0.8719	0.7536	0.6683	0.6236
$K_8 = \{1, 2, 3\}$	ρ_{K_8}	0.9683	0.9142	0.8278	0.7617	0.7295
$K_9 = \{3, 4, 5\}$	ρ_{K_9}	0.4914	0.5778	0.7009	0.7249	0.7193
$K_{10} = \{1, 2, 4, 5\}$	$\rho_{K_{10}}$	0.9683	0.9161	0.8628	0.8450	0.8425

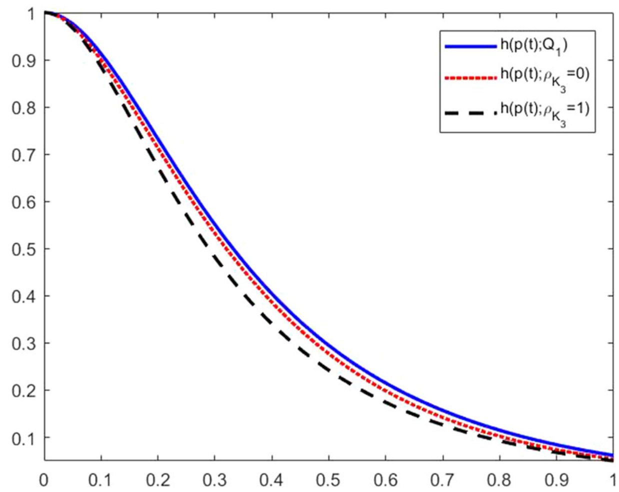

Fig. 3 Plots of $h(\mathbf{p}(t)\,;\,Q_1)$(solid line), $h(\mathbf{p}(t)\,;\,\rho_{K_3}=1)$ (dashed line) and $h(\mathbf{p}(t)\,;\,\rho_{K_3}=0)$ (dotted line) versus time

Table 4 The Birnbaum reliability importance measure of component i

ω	t_0	$I_B(1;\mathbf{p}(t_0))$	$I_B(2;\mathbf{p}(t_0))$	$I_B(3;\mathbf{p}(t_0))$	$I_B(4;\mathbf{p}(t_0))$	$I_B(5;\mathbf{p}(t_0))$
0.01	2.6921	0.0711	0.0711	0.0080	0.0711	0.0711
0.1	1.5866	0.2157	0.2157	0.0530	0.2157	0.2157
0.5	0.7489	0.3735	0.3735	0.1243	0.3735	0.3735

Table 5 The importance measure based on SREF

Reduction set	Reduction factor	ω		
		0.01	0.1	0.5
		t_0		
		2.6921	1.5866	0.7489
$K_1=\{1\}$	ρ_{K_1}	0.4886	0.4525	0.3514
$K_2=\{2\}$	ρ_{K_2}	0.4886	0.4525	0.3514
$K_3=\{3\}$	ρ_{K_3}	0.4170	0.3398	0.1267
$K_4=\{4\}$	ρ_{K_4}	0.4886	0.4525	0.3514
$K_5=\{5\}$	ρ_{K_5}	0.4886	0.4525	0.3514

3) If the system components are i.i.d and $I_B(i;\mathbf{p}) \geq I_B(j;\mathbf{p})$ then $\rho_i \geq \rho_j$.

Proof Part 1): From $h(\mathbf{p},\rho_i)=h(\mathbf{p},\rho_j)$ we obtain

$$h(\mathbf{p},\rho_i)-h(\mathbf{p})=h(\mathbf{p},\rho_j)-h(\mathbf{p}),$$

and this means that $I_{RIP}(i;\mathbf{p}) = I_{RIP}(j;\mathbf{p})$. Then from Eq. (1) we have

$$(p_i' - p_i)I_B(i;\mathbf{p}) = (p_j' - p_j)I_B(j;\mathbf{p}), \tag{5}$$

and since $p_i' = R_i'(t) = e^{-\int_0^t \rho_i r_i(x)\,\mathrm{d}x} = \left[e^{-\int_0^t r_i(x)\,\mathrm{d}x}\right]^{\rho_i} = p_i^{\rho_i}$, then

$$(p_i^{\rho_i} - p_i)I_B(i;\mathbf{p}) = (p_j^{\rho_j} - p_j)I_B(j;\mathbf{p}).$$

Part 2): From Eq. (4), we obtain

$$(p_i^{\rho_i - 1} - 1) = (p_j^{\rho_j - 1} - 1)\frac{p_j I_B(j;\mathbf{p})}{p_i I_B(i;\mathbf{p})}.$$

Now as $\frac{p_j I_B(j;\mathbf{p})}{p_i I_B(i;\mathbf{p})} \geq 1$, we get that $(p_i^{\rho_i - 1} - 1) \geq (p_j^{\rho_j - 1} - 1)$, so

$$p_i^{\rho_i - 1} \geq p_j^{\rho_j - 1},$$

or equivalently

$$\frac{(\rho_i - 1)}{(\rho_j - 1)} \geq \frac{\ln p_j}{\ln p_i} \geq 1,$$

as $p_i \geq p_j$. Therefore $\rho_i \leq \rho_j$.

Part 3): The result follows immediately from $p_i = p_j = p$ and Eq. 5). □

Example 3.2 Consider the bridge system in Fig. 1 with i.i.d components whose lifetimes are distributed exponentially with $\lambda = 1$ and a redundancy mechanism $Q_1 = \{H = \{1\}\}$, such that a hot standby component from exponential lifetime distribution with $\lambda = 2$ is added to component 1.

Table 5 lists the SREFs which are calculated for $\omega = 0.01, 0.1, 0.5$ and all singletone reduction sets. In this table we first find t_0 such that $h(\mathbf{p}(t_0)\,;\,Q_1) = \omega$ for a specified reliability requirement ω, and then obtain ρ_{K_i}; $i = 1, 2, 3$ such that satisfies in $h(\mathbf{p}(t_0)\,;\,\rho_{K_i}) = \omega$. The Birnbaum reliability importance measures at time t_0 are presented in Table 4. The results show that for all values of ω, $I_B(3;\mathbf{p}(t_0)) < I_B(i;\mathbf{p}(t_0))$ and also $\rho_i \geq \rho_3$; $i = 1, 2, 4, 5$ which support part 3 of Theorem 3.3.

The next example is given to clarify the result of part 2) in Theorem 3.3.

Example 3.3 Reonsider the system in Example 3.1. It is not difficult to verify that for $\omega = 0.1$ and $t_0 = 0.8445$, we have $p_4(t_0) = 0.0341$, $p_5(t_0) = 0.0147$, $I_B(4,\mathbf{p}(t_0)) = 0.0213$, $I_B(5,\mathbf{p}(t_0)) = 0.0580$ and then $\frac{p_5 I_B(5;\mathbf{p})}{p_4 I_B(4;\mathbf{p})} = \frac{0.0009}{0.0007} \geq 1$. Since $p_4 = 0.0341 \geq p_5 = .0147$ and $\rho_4 = 0.0158 \leq \rho_5 = 0.2481$, the results support part 2 of Theorem 3.3.

A random vector $\mathbf{T} = (T_1, \ldots, T_n)$ with mutually s-independent random variables is said to follow the proportional hazard rates (PHR) model (denoted as $\mathbf{T} \sim PHR(\bar{F}, \boldsymbol{\lambda})$), if

$$P(T_i > t) = \bar{F}^{\lambda_i}(t), \ \ for \ \lambda_i > 0, \ \ i = 1, \ldots, n, \tag{6}$$

where $\bar{F}$ is the baseline survival function and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is the proportional hazard vector. For more details on PHR models refer to Kumar and Klefsjö (1994) and references therein.

In the following theorem, we present a result about relative importance of survival equivalence factor in a series system when the components having lifetimes following the proportional hazard rates model.

Theorem 3.4 *In a series system with independent components, suppose there exist ρ_i and ρ_j such that $h(\mathbf{p}, \rho_i) = h(\mathbf{p}, \rho_j)$, and $\mathbf{p} \sim PHR(p, \boldsymbol{\lambda})$ where p is the baseline survival function and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is the proportional hazard vector. Then $\lambda_i > \lambda_j$ if and only if $\rho_i > \rho_j$.*

Proof For simplicity we ignored the time t in $\mathbf{p}(\mathbf{t})$ and $p(t)$. In a series system $h(\mathbf{p}) = \prod_{i=1}^{n} p_i$ implies that $I_B(i; \mathbf{p}) = \prod_{k \neq i}^{n} p_k$. So from Eq. 4 we have

$$(p^{\rho_i \lambda_i} - p^{\lambda_i}) \prod_{k \neq i}^{n} p_k = (p^{\rho_j \lambda_j} - p^{\lambda_j}) \prod_{k \neq j}^{n} p_k.$$

According to the PHR model property we have

$$p^{\lambda_i}(p^{\lambda_i(\rho_i - 1)} - 1) p^{\sum_{k \neq i} \lambda_k} = p^{\lambda_j}(p^{\lambda_j(\rho_j - 1)} - 1) p^{\sum_{k \neq j} \lambda_k},$$

which means that $\lambda_i(1 - \rho_i) = \lambda_j(1 - \rho_j)$. So $\lambda_i > \lambda_j$ if and only if $\rho_i > \rho_j$. □

4 Relative importance of components based on the mean reliability equivalence factor

The mean time to failure (MTTF) is a measure that indicates how long a device can last on average when no repairs are allowed. The MTTF of a system can be derived from its reliability function as follows:

$$MTTF = \int_0^\infty h(\mathbf{p}(t)) dt.$$

So we will denote by $MTTF_Q = \int_0^\infty h(\mathbf{p}(t)\,;\,Q) dt$, the MTTF of an improved system via an arbitrary redundancy mechanism Q. Similarly the MTTF of an improved system by reducing the component failure rates of the subset K is given by $MTTF_{\boldsymbol{\rho}_K} = \int_0^\infty h(\mathbf{p}(t)\,;\boldsymbol{\rho}_K)dt$, where $\boldsymbol{\rho}_K = (\rho_{1_K}, \ldots, \rho_{n_K})$ is defined in (2).

The mean reliability equivalence factor, (MREF), is defined as a factor ρ such that the failure rates of some system components should be reduced to obtain a MTTF equals to that of a system improved by a redundancy method, see Sarhan (2000) and Sarhan (2009).

In the following, we extend the definition of the MREF for a redundancy mechanism Q.

Definition 4.1 Let $R_{\boldsymbol{\rho}_{0_K}}(t) = h(\mathbf{p}(t)\,;\boldsymbol{\rho}_{0_K})$ and $h(\mathbf{p}(t)\,;Q)$ be as defined in Definition 3.1. A solution ρ_0 of the equation

$$\int_0^\infty h(\mathbf{p}(t)\,;Q)dt = \int_0^\infty h(\mathbf{p}(t)\,;\boldsymbol{\rho}_{0_K})dt,$$

is said the mean reliability equivalence factor.

Now similar to the previous section, we suggest a new light to the MREF as an importance measure in reduction method.

Definition 4.2 Consider a coherent system improved by redundancy mechanism Q such that $\int_0^\infty h(\mathbf{p}(t)\,;Q)dt = \mu$. Suppose ρ_C and ρ_D are the MREFs of the reduction subsets C and D of the original components, respectively, that satisfy in

$$\int_0^\infty h(\mathbf{p}(t)\,;\rho_C)dt = \int_0^\infty h(\mathbf{p}(t)\,;\rho_D)dt = \mu.$$

We say that the subset C is more important than the subset D in the sense of MREF, written as $C \geq_{(\rho^M,\mu)} D$, if and only if $\rho_C \geq \rho_D$.

In particular, if $C = \{i\}$ and $D = \{j\}$ we say that component i is more important than component j in the sense of MREF, written as $i \geq_{(\rho^M,\mu)} j$, if and only if $\rho_i \geq \rho_j$.

Theorem 4.3 *In a coherent system with i.i.d components and MREFs ρ_i and ρ_j satisfying in $MTTF_{\boldsymbol{\rho}_{K_i}} = MTTF_{\boldsymbol{\rho}_{K_j}}$,.*
$I_B(i;\mathbf{p}(t)) \geq I_B(j;\mathbf{p}(t))$, if and only if $i \geq_{(\rho^M,\mu)} j$.

Proof Since $h(\mathbf{p}\,;\boldsymbol{\rho}_{K_i}) = h(\mathbf{p}) + (p_i^{\rho_i} - p_i)I_B(i;\mathbf{p})$, we can write

$$MTTF_{\boldsymbol{\rho}_{K_i}} = MTTF + \int_0^\infty (p_i^{\rho_i}(t) - p_i(t))I_B(i;\mathbf{p}(t))dt. \tag{7}$$

Table 6 The Barlow–Proschan importance measure of system components in Example 4.2

	Reduction set				
	$K_1 = \{1\}$	$K_2 = \{2\}$	$K_3 = \{3\}$	$K_4 = \{4\}$	$K_5 = \{5\}$
$I_{BP}(K_i; \boldsymbol{F})$	0.2266	0.4642	0.0489	0.0955	0.1648

If $MTTF_{\boldsymbol{\rho}_{K_i}} = MTTF_{\boldsymbol{\rho}_{K_j}} = \mu$, then

$$\int_0^\infty (p_i^{\rho_i}(t) - p_i(t)) I_B(i; \mathbf{p}(t)) dt = \int_0^\infty (p_j^{\rho_j}(t) - p_j(t)) I_B(j; \mathbf{p}(t)) dt.$$

Since $p_i(t) = p_j(t) = p(t)$ and $I_B(i; \mathbf{p}(t)) \geq I_B(j; \mathbf{p}(t))$, then $\rho_i \geq \rho_j$. □

Example 4.1 Consider the bridge system with i.i.d component lifetimes from exponential distribution with parameter λ. Since $I_B(3; \mathbf{p}(t)) \leq I_B(i; \mathbf{p}(t))$; $i = 1, 2, 4, 5$, Theorem 4.3 concludes $i \geq_{(\rho^M, \mu)} 3$; $i = 1, 2, 4, 5$.

Barlow and Proschan (1975b) introduced the importance measure of component i as

$$I_{BP}(i; \bar{\mathbf{F}}) = \int_0^\infty \left[h(1_i, \bar{\mathbf{F}}(t)) - h(0_i, \bar{\mathbf{F}}(t))\right] \mathrm{d}F_i(t),$$

where $(0_i, \bar{\mathbf{F}}(t)) = (\bar{F}_1(t), \ldots, \bar{F}_{i-1}(t), 0, \bar{F}_{i+1}(t), \ldots, \bar{F}_n(t))$ and $(1_i, \bar{\mathbf{F}}(t)) = (\bar{F}_1(t), \ldots, \bar{F}_{i-1}(t), 1, \bar{F}_{i+1}(t), \ldots, \bar{F}_n(t))$. The Barlow–Proschan importance measure can be interpreted as the probability that component i is critical to the system functioning on infinite interval. Since the Barlow–Proschan and the MREF, as importance measures, do not dependent on time t, we compare them in the next example.

Example 4.2 Reconsider bridge system in Example 3.1. The Barlow-Proschan importance measure of the system components are presented in Table 6. The most important component is component 2. The MREFs are obtained by Matlab software and presented in Table 7 for all possible reduction sets. Based on MREF, component 2 is the best component in reduction method. In view of all possible reduction sets, we find that $K_{30} = \{1, 2, 3, 4, 5\}$ is the most appropriate subset for reducing the failure rates based on the MREF measure. The values of MREFs outside of [0, 1] are not acceptable and marked with the symbol $\bigotimes$ in Table 7.

In the next theorem we present a result similar to Theorem 4.3 based on MREF measures.

Theorem 4.4 *Let T denote the lifetime of a series system with independent and non-identical component lifetimes $T_1, \ldots, T_n$.*

Table 7 The MREFs in Example 4.2

Reduction set	$K_1 = \{1\}$	$K_2 = \{2\}$	$K_3 = \{3\}$	$K_4 = \{4\}$	$K_5 = \{5\}$
ρ_{K_i}	0.6369	0.8196	−0.2978 ⊗	0.1195	0.3816
Reduction set	$K_6 = \{1, 2\}$	$K_7 = \{1, 3\}$	$K_8 = \{1, 4\}$	$K_9 = \{1, 5\}$	$K_{10} = \{2, 3\}$
ρ_{K_i}	0.8793	0.6741	0.6923	0.7346	0.8289
Reduction set	$K_{11} = \{2, 4\}$	$K_{12} = \{2, 5\}$	$K_{13} = \{3, 4\}$	$K_{14} = \{3, 5\}$	$K_{15} = \{4, 5\}$
ρ_{K_i}	0.5937	0.8423	0.7726	0.5335	0.6032
Reduction set	$K_{16} = \{1, 2, 3\}$	$K_{17} = \{1, 2, 4\}$	$K_{18} = \{1, 2, 5\}$	$K_{19} = \{1, 3, 4\}$	$K_{20} = \{1, 3, 5\}$
ρ_{K_i}	0.8832	0.88591	0.8898	0.7213	0.7609
Reduction set	$K_{21} = \{1, 4, 5\}$	$K_{22} = \{2, 3, 4\}$	$K_{23} = \{2, 3, 5\}$	$K_{24} = \{3, 4, 5\}$	$K_{25} = \{1, 2, 3, 4\}$
ρ_{K_i}	0.7739	0.8434	0.8503	0.6537	0.8896
Reduction set	$K_{26} = \{1, 2, 3, 5\}$	$K_{27} = \{1, 2, 4, 5\}$	$K_{28} = \{1, 3, 4, 5\}$	$K_{29} = \{2, 3, 4, 5\}$	$K_{30} = \{1, 2, 3, 4, 5\}$
ρ_{K_i}	0.8935	0.8961	0.7919	0.8633	0.8994

(1) Let $\bar{F}_i(t)$ is the survival function of component $i\,;i = 1,\ldots,n$, and $\bar{\mathbf{F}}(t) = (\bar{F}_1(t),\ldots,\ldots,\bar{F}_n(t))$. The MTTF of an improved series system by reducing the failure rate function of component i ,$MTTF_{\rho_i}$, is given by

$$MTTF_{\rho_i} = \int_0^{\infty} \bar{F}_i^{\rho_i-1}(t)h\left(\bar{\mathbf{F}}(t)\right)\,\mathrm{d}t. \tag{8}$$

(2) Under the PHR model (6) and baseline distribution $\bar{F}(t) = e^{-x};\ x > 0$, the MREF for the reduction set $K_i = \{i\}$ and an arbitrary redundancy mechanism Q which $MTTF_Q = \mu$ is given by

$$\rho_i = 1 + \frac{1}{\lambda_i}\left(\frac{1}{\mu} - \sum_{i=1}^{n}\lambda_i\right)$$

when $\sum_{j\neq i}\lambda_j \leq \frac{1}{\mu} \leq \sum_{i=1}^{n}\lambda_i$. So in this condition, if there exist MREFs ρ_i and ρ_j satisfying in $MTTF_{\rho_i} = MTTF_{\rho_j}$, then $i \geq_{(\rho^M,\mu)} j$ if and only if $\lambda_i \geq \lambda_j$.

Proof Part 1): Since $h(\bar{\mathbf{F}}(t)\,;\boldsymbol{\rho}_i) = h(\bar{\mathbf{F}}(t)) + (\bar{F}_i^{\rho_i}(t) - \bar{F}_i(t))I_B(i;\bar{\mathbf{F}}(t))$

$$MTTF_{\boldsymbol{\rho}_i} = MTTF + \int_0^{\infty}(\bar{F}_i^{\rho_i}(t) - \bar{F}_i(t))I_B(i;\bar{\mathbf{F}}(t))dt. \tag{9}$$

In series system $I_B(i;\bar{\mathbf{F}}(t)) = \prod_{k\neq i}\bar{F}_k(t)$, so

$$MTTF_{\rho_i} = \int_0^{\infty} \bar{F}_i^{\rho_i-1}(t)h\left(\bar{\mathbf{F}}(t)\right)\,\mathrm{d}t.$$

Part 2): From Eq. 8, $MTTF_{\rho_i}$ in a series system with component lifetimes $T_1,\ldots,T_n$ which are under the PHR model with a baseline distribution $\bar{F}(t) = e^{-x};\ x > 0$, is given by

$$MTTF_{\rho_i} = \int_0^{\infty} \bar{F}_i^{\rho_i-1}(t)h\left(\bar{\mathbf{F}}(t)\right)\,\mathrm{d}t = \int_0^{\infty}\left(e^{-\lambda_i t}\right)^{\rho_i-1} e^{-\sum_{i=1}^{n}\lambda_i t}\,\mathrm{d}t$$
$$= \frac{1}{\sum_{i=1}^{n}\lambda_i + \lambda_i(\rho_i-1)}.$$

So the MREF for the reduction set $K_i = \{i\}$ is given by

$$\rho_i = 1 + \frac{1}{\lambda_i}\left(\frac{1}{\mu} - \sum_{i=1}^{n}\lambda_i\right)$$

when $\sum_{j\neq i}\lambda_j \leq \frac{1}{\mu} \leq \sum_{i=1}^{n}\lambda_i$. □

5 Conclusion

There are different techniques for improving the system reliability. It is important for the reliability engineers to find the most appropriate technique. We investigated the analysis of improving system reliability via two types of reliability equivalence factors; respectively, SREF and MREF. These factors help us to equivalent and compare various improving methods. In this paper, we could offer a closed form of SREF in general, based on the improvement level of the original system reliability. We proposed a new light on REFs as importance measures in the analysis of improving system reliability. We also obtained some basic properties of the concepts of survival and mean reliability equivalence factors as importance measures. Some examples with numerical calculations are given to illustrate the results.

Acknowledgements We would like to thank the anonymous reviewers for several helpful suggestions that allow us to improve the paper.

References

Barlow RE, Proschan F (1975a) Statistical theory of reliability and life testing: probability models, Florida State University Tallahassee

Barlow RE, Proschan F (1975b) Importance of system components and fault tree events. Stoch Process Appl 3(2):153–173

Bayramoglu Kavlak K (2017) Reliability and mean residual life functions of coherent systems in an active redundancy. Naval Res Logist 64(1):19–28

Billinton R, Allan RN (1992) Reliability evaluation of engineering systems. Springer, Berlin

Kumar D, Klefsjö B (1994) Proportional hazards model: a review. Reliab Eng Syst Saf 44(2):177–188

Kuo W, Zuo MJ (2003) Optimal reliability modeling: principles and applications. Wiley, New York

Råde L (1989) Reliability equivalence, studies in statistical quality control and reliability Mathematical Statistics. Chalmers University of Technology, Goteborg

Råde L (1990) Reliability systems of 3-state components. Avdelningen för matematisk statistik, Chalmers tekniska högskola

Råde L (1991) Performance measures for reliability systems with a cold standby with a random switch. Avdelningen för matematisk statistik, Chalmers tekniska högskola

Sarhan AM (2000) Reliability equivalence of independent and non-identical components series systems. Reliab Eng Syst Saf 67(3):293–300

Sarhan AM (2002) Reliability equivalence with a basic series/parallel system. Appl Math Comput 132(1):115–133

Sarhan AM (2005) Reliability equivalence factors of a parallel system. Reliab Eng Syst Saf 87(3):405–411

Sarhan AM (2009) Reliability equivalence factors of a general series-parallel system. Reliab Eng Syst Saf 94(2):229–236

Sarhan AM, Tadj L, Al-Khedhairi A, Mustafa A (2008) Equivalence factors of a parallel-series system. Appl Sci 10:219–230

Shaked M, Shanthikumar JG (1990) Reliability and maintainability. Handb Oper Res Manag Sci 2:653–713

Shen K, Xie M (1989) The increase of reliability of k-out-of-n systems through improving a component. Reliab Eng Syst Saf 26(3):189–195

Xie M, Shen K (1989) On ranking of system components with respect to different improvement actions. Microelectron Reliab 29(2):159–164

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.