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A note on the stress-strength reliability of a coherent system based on signature

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Abstract

This article considers the stress-strength reliability of a coherent system and discusses its computation based on the concept of system signature. The system components may experience the same or different stress levels. We have found some mistake results given by Eryilmaz (2008). Some other mistake results of Bhattacharya and Roychowdhury (2013) was pointed out by Sadegh (2021). All these mistake results are due to misapplication of the system reliability in case of the system components are subjected to a common stress level. It is shown that the system signature can be used for calculating of the stress-strength reliability of a coherent system with different stress levels whereas when the system components are subjected to a common stress level, the use of system signature may leads to a mistake result. Regarding this, some mistake results given in Eryilmaz (2008) are pointed out.

Keywords: Coherent systems, stress-strength reliability, system signature.

1 Introduction

Stress-strength models are important in reliability literature and engineering applications. A system or unit may be subjected to randomly occurring environmental stress such as pressure, temperature and humidity and survival of the system depends on its resistance. In the simplest setup of stress-strength models, a unit functions if its strength is greater than the stress imposed on it. The reliability of the unit is then defined as $R = P(Y > X)$, where Y and X represent the random values

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of strength of the unit and stress placed on the unit, respectively. The estimation of R has been widely studied under various distributional assumptions on X and Y(see e.g., Kotz et al.(2003)). These models have also been studied for the systems consist of more than one component. Eryilmaz (2008) considered a multivariate stress-strength model for a coherent system. He assumed that the components are subjected to a common random stress and then used the system signature to obtain the stress-strength system reliability. We show in the next section, that his results given in Theorem 3 and Corollary 1 are not correct. Eryilmaz (2010) expressed the system stress-strength reliability in terms of those of series systems and presented some approximations for system reliability. Eryilmaz (2013) studied the stress-strength reliability of a system with a time dependent(dynamic) strength and a static and common random value of the stress. Bhattacharya and Roychowdhury (2013) studied the stress-strength reliability of a system with different stress levels and claimed their results include the case when the system components are subjected to a common stress level as a special case. Sadegh (2021) showed that their claim is not correct and gave the correct argument.

2 Main result

For the sake of completeness, in this section we first give some results of Sadegh (2021) and then give our main results about the use of system signature in computing of the stress-strength system reliability.

Let ϕ be the structure function of a coherent system with n components whose random strengths are $Y_1, \ldots, Y_n$ and suppose the components are subjected to the stress levels $X_1, \ldots, X_n$, respectively. The ith component fails if the imposed stress exceeds its strength at any time, i.e. if $X_i \geq Y_i$. Thus $p_i = P(Y_i > X_i)$ gives the stress-strength reliability of the ith component. We define the status of components as follow:

$$Z_i = \begin{cases} 1 & \text{if} \quad Y_i > X_i \\ 0 & \text{if} \quad Y_i \leq X_i \quad i = 1, 2, \ldots, n \end{cases} \tag{2.1}$$

where we assume that $X_1, \ldots, X_n$ are independent and have a common continuous distribution function G. Also assume that $Y_1, \ldots, Y_n$ are independent random variables and have a common continuous distribution function F. We also assume that F and G are independent distributions. Then the reliability of the coherent system ϕ under the above mentioned stress-strength setup is given by

$$R_\phi = Pr\{\phi(Z_1, \ldots, Z_n) = 1\}$$

where $\phi(\mathbf{z})$ indicates the state of the system. Note that the binary random variables defined by (2.1) are independent and have a common distribuon $Binomial(1, p)$, where

$$p = P(Y > X) = \int \bar{G}(x) dF(x), \tag{2.2}$$

and $\bar{G} = 1 - G$.

In the following, using minimal path(cut) sets of the system, a general expression for R_ϕ is given (for a details on the coherent structures, minimal path(cut) sets etc. see e.g. Barlow and Proschan (1975)). Suppose now that the coherent system has p minimal path sets given by $P_1, \ldots, P_p$ and c minimal cut sets $C_1, \ldots, C_c$. It is known that

$$\phi(\mathbf{z}) = \max_{1\le i\le p} \min_{j\in P_i} z_j = \min_{1\le i\le c} \max_{j\in C_i} z_j$$

$$= 1 - \prod_{i=1}^{p}(1 - \prod_{j\in P_i} z_j) = \prod_{i=1}^{c}\left[1 - \prod_{j\in C_i}(1 - z_j).\right]$$

Lemma 1. We have

$$R_\phi = Pr\left\{\cap_{i=1}^{c}[\cup_{j\in C_i}(X_j > Y_j)]\right\} = Pr\left\{\cup_{i=1}^{p}[\cap_{j\in P_i}(X_j > Y_j)]\right\} \tag{2.3}$$

Proof. The first equality was proved by Bhattacharya and Roychowdhury (2013). The second equality can be similarly proved.

Remark 1. The Equation (2.3) holds true in general even if the independence assumption does not hold. Under independence assumption and according to the form of the minimal cut(path) sets of the system, the first or the second equality in (2.3) may be easier to use than the other. For example in a consecutive-k-out-of-n:F system, in which the system fails if at least k out of its n components are consecutively failed, minimal cut sets are $C_i = \{i, i+1, \ldots, i+k-1\}$, $i = 1, \ldots, c = n-k+1$ which are of simple form and easy to use whereas the minimal path sets of this system do not have such a simple form and also determining of $p(> c)$ for this system is usually complicated. Hence the first equality in (2.3) is easier to use than the second one.

Remark 2. Note that when $X_i = X$, the binary random variables $Z_1, \ldots, Z_n$ (or equivalently the events $(Y_i > X)$) are not independent. For example in a series system we now have $R_\phi = Pr(minZ_i = 1) = Pr(Z_1 = 1, \ldots, Z_n = 1) = Pr(Y_1 > X, \ldots, Y_n > X) = Pr(minY_i > X) \neq \prod_1^n Pr(Y_i > X)$. Hence those expressions given by Bhattacharya and Roychowdhury (2013) for the reliability of k-out-of-n systems, a series-parallel system(including Examples 1, 2 and 4), a hi-fi system(Example 5) and a bridge system(Example 6) are not correct. To see the correct values of those examples see Sadegh (2021).

We now give our main results related to the use of system signature in computing of the stress-strength system reliability. Let $T_1, \ldots, T_n$ and $T = \phi(T_1, \ldots, T_n)$ be the component lifetimes and the system lifetime, respectively. When T_is are continuous and are independent and identically distributed (IID), the following well known and important result is obtained by Samaniego (1985):

$$P(T > t) = \sum_{i=1}^{n} s_i P(T_{i:n} > t), \tag{2.4}$$

where $T_{i:n}$ is the ith ordered lifetime of components and $s_i = P(T = T_{i:n})$. The probability vector $\mathbf{s} = (s_1, \ldots, s_n)$ is called the signature of the system.

In the following lemma we use the system signature to obtain R_ϕ.

Lemma 2. Let X_is denote the stresses at component level and Y_is are component strengths of the coherent system ϕ, $i = 1, \dots, n$. Suppose X_is are IID and have a common distribution F and Y_is are IID with a common distribution G which is independent of F. We have

$$R_\phi = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (1-p)^j p^{n-j}, \tag{2.5}$$

where $\mathbf{s} = (s_1, \dots, s_n)$ is the system signature and p is given by (2.2).
Proof. By using of Equation (2.4) and in view of the distribution function of order statistic $T_{i:n}$, if we replace $P(T > t)$ and $P(T_i > t)$ by R_ϕ and p, respectively then the proof of the lemma follows. Note that under the assumptions of the lemma, the component lifetimes T_is or equivalently the binary random variables Z_is are IID.

Remark 3. Although $P(T_{1:n} < \cdots < T_{n:n}) = 1$, but under the conditions of Lemma 2, it does not necessarily imply that a component with small strength fails before a component with large strength. For example when $n = 2$ we have

$$P(X_{R_1} < Y_{1:2}, X_{R_2} > Y_{2:2}) > 0,$$

where $1 \leq R_1, R_2 \leq 2$ and $Y_{R_1} = Y_{1:2}$ and $Y_{R_2} = Y_{2:2}$, as the event $X_{R_1} < Y_{1:2} < Y_{2:2} < X_{R_2}$ has a positive probability.

Remark 4. In case of stress at system level, that is $X_i = X$, obviously a component with low strength fails before another component with high strength. For example when $n = 2$ we have

$$P(X < Y_{1:2}, X > Y_{2:2}) = 0.$$

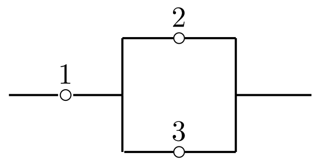
Lemma 3. The Equation (2.5) does not hold true if $X_i = X$, $i = 1, \dots, n$.
Proof. In view of Remark 2, when $X_i = X$, the binary random variables Z_is and therefore the component lifetimes T_is are not IID. Hence the Equation (2.4) and therefore the Equation (2.5) does not hold.

Remark 5. Based on the Lemma 3, the results given in Theorem 3 and Corollary 1 of Eryilmaz (2008) are not correct. See the following examples.

Example 1. Consider the following series-parallel system.

It is known that the signature of this system is $\mathbf{s} = (1/3, 2/3, 0)$. Also its minimal path sets are $\{1, 2\}$ and $\{1, 3\}$.
(a). If Y_is are distributed as $\text{Exp}(\lambda)$, $i = 1, 2, 3$ and X_is are distributed as $\text{Exp}(\mu)$, and X_is and Y_is are independent, then

$$p = P(Y > X) = \frac{\mu}{\mu + \lambda}.$$



Therefore we have the following expression for R_ϕ

$$Pr\left[(Y_1 > X_1) \cap \{(Y_2 > X_2) \cup (Y_3 > X_3)\}\right] = 2p^2 - p^3 = 2(\frac{\mu}{\mu+\lambda})^2 - (\frac{\mu}{\mu+\lambda})^3.$$

Equivalently using Equation (2.5) we have

$$R_\phi = \sum_{i=1}^{3} s_i \sum_{j=0}^{i-1} \binom{3}{j} (1-p)^j p^{3-j} = 2p^2 - p^3.$$

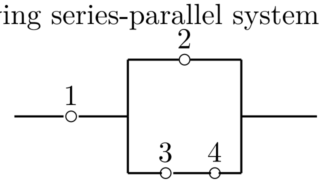
(b). If $X_i = X$ then

$$R_\phi = Pr\left[(Y_1 > X) \cap \{(Y_2 > X) \cup (Y_3 > X)\}\right],$$

$$= Pr(min(Y_1, Y_2) > X) + Pr(min(Y_1, Y_3) > X) - Pr(Y_{1:3} > X)$$

$$= \frac{2\mu}{\mu + 2\lambda} - \frac{\mu}{\mu + 3\lambda} \neq 2p^2 - p^3.$$

Example 2. Consider the following series-parallel system in 4 components.



The minimal path sets of this system are $\{1, 2\}$ and $\{1, 3, 4\}$ and system signature is $\mathbf{s} = (6/24, 14/24, 4/24, 0)$. Under the assumption of exponential distributions for strength and stress variables in Example 1, the stress-strength reliability of the system is given as follows:
(a). In case of stress at component level we have

$$R_\phi = Pr\left[(Y_1 > X_1) \cap \{(Y_2 > X_2) \cup (Y_3 > X_3) \cap (Y_4 > X_3)\}\right] = p^2 + p^3 - p^4,$$

where $p = P(Y > X) = \frac{\mu}{\mu+\lambda}$. Equivalently using Equation (2.5) we have

$$R_\phi = \sum_{i=1}^{4} s_i \sum_{j=0}^{i-1} \binom{4}{j} (1-p)^j p^{3-j} = p^2 + p^3 - p^4.$$

(b). In case of stress at system level, that is $X_i = X$, we have

$$R_\phi = Pr\left[(Y_1 > X) \cap \{(Y_2 > X) \cup (Y_3 > X) \cap (Y_4 > X)\}\right],$$

which is equal to

$$Pr(min(Y_1, Y_2) > X) + Pr(min(Y_1, Y_3, Y_4) > X) - Pr(\min\{Y_1, Y_2, Y_3, Y_4\} > X)$$

$$= \frac{\mu}{\mu + 2\lambda} + \frac{\mu}{\mu + 3\lambda} - \frac{\mu}{\mu + 4\lambda}, \text{ which is quite different with} p^2 + p^3 - p^4.$$

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