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Stabilization of a Class of Fractional-Order Nonlinear Systems Subject to Actuator Saturation and Time Delay

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Abstract: Actuator saturation and time delay are practical issues in practical control systems, significantly affecting their performance and stability. This paper addresses, for the first time, the stabilization problem of fractional-order (FO) nonlinear systems under these two practical constraints. Two primary methodologies are employed: the vector Lyapunov function method, integrated with the M-matrix approach, and the second one is the Lyapunov-like function method, which incorporates diffusive realization and the Lipchitz condition. An optimization framework is proposed to design stabilizing controllers based on the derived stability conditions. The proposed methods are validated numerically through their application to the FO Lorenz and Liu systems, demonstrating their effectiveness in handling actuator saturation and time delay.

Keywords: stability condition; fractional order; time delay; saturation; chaotic system

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1. Introduction

Fractional-order systems (FOSs) have garnered substantial attention in scientific research due to their inherent capability to model memory and hereditary properties. These properties are pivotal in accurately capturing the dynamics of natural and engineered systems, which are often characterized by non-local interactions and long-term dependencies. Unlike integer-order derivatives, fractional derivatives provide a robust mathematical framework to describe such behaviors, enabling the analysis of complex phenomena over multiple time scales. Applications of fractional calculus (FC) have been extensively documented in areas such as anomalous diffusion, viscoelastic materials, control systems, and electrochemical processes, highlighting its interdisciplinary relevance [1–9]. From a control perspective, fractional-order (FO) models excel in capturing systems with inherent memory effects, offering advantages over traditional integer-order approaches. This has been demonstrated in diverse engineering domains, including robotics, signal processing, and materials science [10–16]. Furthermore, fractional derivatives inherently encapsulate the influence of historical states on the current dynamics, providing a more comprehensive representation of system behavior, particularly in systems with hereditary or viscoelastic properties. Stability analysis of FOSs has become a cornerstone in advancing their practical utility. Stability is crucial not only for understanding the inherent dynamics of these systems but also for designing robust control strategies that accommodate uncertainties, time delays, and actuator constraints. This study aims to contribute to the ongoing discourse by addressing stability challenges in FOSs and proposing novel solutions tailored to real-world applications.

1.1. Motivation and Background

The stability and control of FOSs are critical for their effective deployment in practical scenarios, particularly in systems where nonlinearity, input saturation, and time delays play a significant role. Fractional derivatives provide a compelling physical interpretation, wherein the system's present state depends not merely on the instantaneous rate of change but also on the cumulative historical dynamics. This attribute underscores the indispensability of FC in modeling and analyzing systems with long-term dependencies.

A significant challenge in the control of FOSs arises from actuator saturation, a phenomenon that stems from the physical limitations of actuators [17]. If left unaddressed, actuator saturation can lead to degraded performance and, in extreme cases, instability of the closed-loop system. Extensive research has focused on mitigating these effects through advanced control strategies. For instance, stability analysis under saturation has been well studied in linear FOSs, leveraging analytical methods to estimate the domain of attraction while incorporating actuator constraints [18–20]. Nonlinear FOSs introduce additional complexities due to their sensitivity to initial conditions and external perturbations. The authors of [21] addressed this issue by deriving sufficient stabilization conditions using the Gronwall–Bellman lemma and sector-bounded conditions. Other studies have explored uncertain FOSs with saturation constraints, employing the Lipschitz condition to design state feedback controllers that ensure stability [22]. Recent advancements include the development of adaptive backstepping control strategies tailored to incommensurate FOSs under input saturation [23]. These strategies utilize frequency-distributed models and indirect Lyapunov methods to account for control constraints. Moreover, the stochastic stability of FOSs with actuator saturation has been rigorously analyzed, with stability criteria derived using stochastic system theory and FO calculus [24]. Additional research has explored delay-dependent stability in uncertain FOS, particularly those with distributed delays and input saturation. These studies employ Lyapunov–Krasovskii functionals to derive robust stability conditions, expressed as linear matrix inequalities (LMIs), incorporating time delays into their analysis [25]. Despite these advancements, the combined effects of time delays and saturation control in FOSs remain underexplored, underscoring the need for comprehensive stability analyses that integrate these critical factors.

Time delays present another major challenge in the stability analysis of dynamical systems. Considerable research has addressed this issue, often focusing on FOSs with linear components and pure delays. For instance, characteristic polynomial methods have been employed to derive stability conditions for such systems [26]. Additionally, the Mittag–Leffler function has been utilized to develop decentralized FO controllers for nonlinear FOSs, accounting for its effects in stabilizing time-delayed systems [27]. The problem of asymptotic stabilization in nonlinear FOSs with time delays has also been addressed using innovative control strategies. A notable contribution in this area was the development of two novel stabilization methods for FOSs with multiple time delays [28]. This work introduced a stabilization control criterion that integrates a Lyapunov-like function with the M-matrix method, offering a more robust solution for managing the complexities of time delays in nonlinear systems. Another key study extended the application of vector Lyapunov functions to FOSs with time delays, proposing a new stability condition that specifically addressed the challenges posed by time delays in FOSs [29]. A time-delay feedback controller was designed in [30] to suppress the chaos of the FO chaotic jerk system, where delay-independent stability and bifurcation conditions were established. Additionally, a mixed controller, which includes a time-delay feedback controller and an

FO PD controller, was designed to eliminate the chaos of the FO system. The problems of stability and stabilization of the FO power system with time delays were investigated in [31]. Based on the theory of FC and the Lyapunov functional technique, the relevant stability criteria were obtained. Meanwhile, Lyapunov functionals were constructed, and the free-weighing matrix technique was introduced to reduce the conservatism of the criteria. In [32], some novel asymptotic stability criteria were deduced for different forms of multivariable fractional-order systems (MFOSs) with FO parameters between 0 and 1 under time delays based on the M-matrix. Initially, the general asymptotic stability condition of ordinary systems was extended to MFOSs. Subsequently, the stability of linear and nonlinear MFOSs was investigated, and their asymptotic stability criterion was derived using the M-matrix method. Moreover, ref. [33] focused on the stabilization of nonlinear FO time-delay systems in the presence of faults. In this research, a general class of nonlinear FOSs was considered, where faults caused variations in the system dynamics and actuators.

In summary, while significant progress has been made in understanding the stability and control of FOSs under both saturation and time delays, further research is needed to fully comprehend the interplay between these two factors. Developing a more integrated approach could yield valuable insights for creating advanced control strategies that ensure the stability of real-world FOSs operating under practical constraints.

1.2. Contributions

The present study addresses a crucial gap in the research on FOS, where the combined effects of time delay and input saturation have been less explored in prior literature. While these factors are highly relevant in practical applications, previous studies have largely neglected their combined influence on system behavior. This paper introduces several key contributions to overcome these challenges by proposing a novel approach to stability analysis and control design for nonlinear FOSs under both saturation control and time delay constraints.

The key contributions of this paper are as follows:

- Addressing saturation limitations: Two innovative approaches are presented to manage the challenges posed by input saturation, a prevalent constraint in practical control and robotic systems. By incorporating the Lipchitz condition into the saturation model, this paper ensures that the FOSs remain stable even when control inputs are constrained. This method offers a robust solution to the issues related to input saturation.
- Addressing time-delay limitations: Recognizing the shortcomings of traditional Lyapunov-like function methods in analyzing time-delayed FOS, this paper utilizes the M-matrix and vector Lyapunov function techniques as critical tools. These methods create a rigorous framework for overcoming the stability challenges induced by time delays, ensuring system stability under these complex conditions.
- Development of a stable controller: This paper makes a substantial contribution by developing an optimized stable controller that accounts for practical system constraints. The controller is designed to guarantee stability in the presence of both saturation and time-delay conditions. Furthermore, a new stability criterion for delayed nonlinear FOSs with saturation constraints is introduced, which significantly broadens the scope of existing stability analysis techniques.
- Adaptation of the Lyapunov-like function: This research modifies the Lyapunov-like function by incorporating diffusive realization and the Lipchitz condition, providing a rigorous criterion for asymptotic stability. This approach effectively resolves the stability challenges of interconnected FOSs operating under saturation control, ensuring robustness in practical applications.

- Application of vector Lyapunov functions with M-matrix methods: The study applies vector Lyapunov functions in combination with M-matrix methods to establish strong stability conditions. This integration enhances the analytical framework for assessing system stability, particularly in cases involving both time delays and saturation constraints, thereby expanding the potential applicability of the proposed methodologies.
- Application to two significant FO systems: FO Lorenz and FO Liu as illustrative examples: The study of stability and control of the FO Lorenz and FO Liu systems is essential not only for understanding their theoretical behavior but also for investigating their practical relevance in automation and robotics, where stability and precise coordination between system components are critical. By applying the proposed analysis to these FO chaotic systems, we aim to deepen the understanding of their complex dynamics and validate the effectiveness of the advanced stabilization techniques introduced in this work.

These contributions collectively offer a comprehensive stability analysis and control framework for nonlinear FOSs subjected to both time delay and saturation control. The proposed methodologies provide significant insights for future research and practical applications and demonstrate the potential to advance the state-of-the-art in FO control strategies.

This paper is structured into six key sections, each contributing to a rigorous analysis of fractional-order (FO) nonlinear systems under saturation control. Section 2 establishes the theoretical groundwork with essential theorems and definitions. Section 3 presents the primary contributions, including stability conditions and the impact of time delays on FO systems. Section 4 focuses on the optimization and practical implementation of a stable controller. Section 5 validates the proposed approach through numerical simulations, demonstrating its real-world applicability. Finally, Section 6 summarizes key findings, emphasizes the practical significance of the study, and outlines potential future research directions in FO system control.

2. Fundamental Concepts

FC is an extension of classical calculus that allows the differentiation and integration of arbitrary (non-integer) order. The two most commonly used definitions are the Riemann–Liouville and Caputo fractional derivatives. The Riemann–Liouville derivative is defined as [1,3]

$$D_a^v H(t) = \frac{1}{\Gamma(n-v)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-v-1} H(\tau) d\tau, \tag{1}$$

where $n = \lceil v \rceil$, $0 < v < n$, and $\Gamma(.)$ is the Gamma function. This definition is suitable for theoretical analysis but has initial value problems with physical interpretations.

The Caputo fractional derivative, often preferred in practical applications, is defined as [3]

$$^{C}D_a^v H(t) = \frac{1}{\Gamma(n-v)} \int_a^t (t-\tau)^{n-v-1} \frac{d^n H(\tau)}{d\tau^n} d\tau. \tag{2}$$

- v is the order of the derivative, which can be fractional (non-integer). It characterizes the “memory” or “hereditary” properties of the system. The smaller the value of v, the stronger the memory effects.
- $H(t)$ is the function to be differentiated. This could represent a physical quantity such as position, temperature, or voltage, depending on the application.
- a is the lower bound of the fractional derivative and represents the starting point of the integral. Physically, this is often the initial time $t = a$.
- $\Gamma(\cdot)$ is the Gamma function, a generalization of the factorial for non-integer values, ensuring the proper scaling of fractional operations.

It is advantageous because it allows for standard initial conditions, such as $H(0)$ and $H'(0)$, which are consistent with classical systems.

In the entirety of this paper, we adopt the Caputo definition of the fractional derivative, as it is particularly suited for practical applications and allows for the use of standard initial conditions.

If we consider a general non-linear FOS given by

$$^{C}D_t^v x = H(x,t) \tag{3}$$

where $x \in \mathbb{R}^n$ represents the state of Equation (3), v is an FO derivative that includes $v \in (0,1)$ and the function $H : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ holds the condition $H(0) = 0$ and is locally Lipchitz with respect to the states, we can establish the following theorem.

Theorem 1 ([29,34])**.** *If the function $H(x,t)$ is continuous over time, then a unique solution exists for the FOS given by (3). The solution is expressed as*

$$x(t) = x_0 + \int_0^\infty \xi_v(w)\psi(w,t)dw, \tag{4}$$

where $\psi(w,t)$ is the solution of the initial value problem associated with the FOS (3), defined as

$$\frac{\partial\psi(w,t)}{\partial t} = -w\psi(w,t) + H(x(t),t), \quad \psi(0,t) = 0. \tag{5}$$

Here, $\xi_v(w) = w^{-v}\frac{\sin(\pi v)}{\pi}$, where $w \in (0,\infty)$.

Equation (4) represents the solution of the FOS (3) as an integral involving the vector-valued function $\psi(w,t)$. To ensure a precise interpretation, the integral of $\psi(w,t)$ is evaluated component-wise. That is, for $\psi(w,t) = [\psi_1(w,t), \psi_2(w,t), \ldots, \psi_n(w,t)]^\top$, the integral is understood as

$$x(t) = x_0 + \begin{bmatrix} \int_0^\infty \xi_v(w)\psi_1(w,t)\,dw \\ \int_0^\infty \xi_v(w)\psi_2(w,t)\,dw \\ \vdots \\ \int_0^\infty \xi_v(w)\psi_n(w,t)\,dw \end{bmatrix}.$$

This component-wise approach is consistent with the established methods in FC and ensures the solution is well defined in the context of vector-valued systems.

The Lyapunov function is a critical tool for analyzing the stability of FOSs. Consider the system described in Equation (3).

A Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}$ for this system must satisfy

1. Positive Definiteness:

$$V(x) > 0 \quad \text{for } x \neq 0, \quad V(0) = 0. \tag{6}$$

2. Radial Unboundedness:

$$V(x) \to \infty \quad \text{as } \|x\| \to \infty. \tag{7}$$

$\|.\|$ denotes an arbitrary norm.

3. Negative Definiteness of the Fractional Derivative:

$$^{C}D^{v}V(x) < 0 \quad \text{for } x \neq 0. \tag{8}$$

In FOS, the fractional derivative $^{C}D^{v}V(x)$ incorporates memory effects inherent in FC, making stability analysis dependent on the entire history of system states, not just the current state.

Lemma 1 ([35])**.** *For the system (3), if there exists a convex and continuously differentiable Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ and the following inequalities hold, then the nonlinear FOS is asymptotically stable:*

$$\begin{aligned} \alpha_1(\|x\|)| \leq V(x) \leq \alpha_2(\|x\|), \\ ^{C}D^{v}V(x) \leq -\alpha_3(\|x\|), \end{aligned} \tag{9}$$

where $\alpha_i, i = 1,2,3$ are class K functions, $v \in (0,1)$ denotes the FO operator, and $\|.\|$ is arbitrary norm.

Lemma 2 ([36])**.** *Let $x = 0$ be an equilibrium point of the nonlinear system (3). If there exists a convex and continuously differentiable Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, then the following inequality holds for all $t > 0$:*

$$^{C}D^{v}V(x) \leq \nabla_x V(x) \cdot {}^{C}D^{v}x. \tag{10}$$

Lemma 3 ([37])**.** *Let $x(t) \in R^n$ be a differentiable vector function. Then, for any time instant t, the following inequality holds:*

$$^{C}D_t^{v}(x^T(t)x(t)) \leq 2x^T(t)^{C}D_t^{v}x(t), \quad \forall v \in (0,1) \tag{11}$$

Definition 1 ([38,39])**.** *A real $n \times n$ matrix $W = [w_{ij}]$ is an M-matrix if the element $w_{ij} \leq 0$ for $i \neq j$, and all the principal minors of W are positive.*

Lemma 4 ([38])**.** *Let W be an M-matrix, which is a special class of matrices commonly used in stability analysis and control theory. Then, there exists a diagonal matrix $Q = diag(q_1, q_2, \ldots, q_N)$, where $q_i > 0$ for all i, such that the matrix $W^TQ + QW$ is positive-definite.*

This lemma plays a critical role in establishing stability criteria for FOS, as it ensures the existence of a Lyapunov function or matrix that can be used to verify the positive definiteness of the system's dynamics. The diagonal nature of Q simplifies the computational complexity, making it a practical choice for controller design and stability analysis.

3. Main Result

In this section, we introduce the general framework of nonlinear fractional-order systems (FOSs) and explore two distinct stabilization scenarios, addressing practical challenges such as input saturation and time delays. These scenarios provide a comprehensive foundation for advancing the stabilization theory of nonlinear FOSs under realistic constraints.

The first scenario focuses on deriving novel stability theorems to achieve asymptotic stabilization of nonlinear FOSs subjected to input saturation. To accomplish this, two robust methodologies are utilized. In Section 3.1, the first methodology employs a Lyapunov-like function with diffusive realization, originally proposed in [40] and subsequently extended in 2019 for analyzing the external stability of Caputo FOSs [34]. This approach provides a systematic framework for analyzing the stability of nonlinear FOSs with saturation constraints. The second methodology leverages the vector Lyapunov function method, a powerful tool particularly suited for the stability analysis and stabilization of FOSs experiencing time-delay effects.

In Section 3.2, the second scenario builds on the first by incorporating time-delay effects into the stability analysis. A novel stability theorem is proposed for achieving asymptotic stabilization of delayed nonlinear FOSs, explicitly considering the influence of input saturation. This extension significantly enhances the theoretical relevance and practical applicability of the proposed stabilization strategies, enabling their deployment in real-world control systems with complex dynamic behaviors.

Our study considers the general form of delayed nonlinear FOS, which can be expressed as follows.

$$^{C}D_t^{v}x_i = a_i x_i + f_i(x) + g_i(x_i(t-\tau)) + b_i sat(u_i), i \in \{1,2,\ldots,N\} \quad (12)$$

Let $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ denote the state and input vectors, respectively, where x_i and u_i represent the i^{th} state and input. The functions f_i and g_i correspond to the linear or nonlinear components of the system. Here, N denotes the number of states, τ represents the time delay, and $\text{sat}(\cdot)$ is the saturation function that constrains the input control, defined as $\text{sat}(u) = \text{sign}(u) \cdot \min(|u|, u_0)$, where u_0 is the saturation bound. Additionally, a_i and b_i are specific real constants, and it is assumed that the system is controllable. Figure 1 presents a detailed block diagram of the system under consideration, providing a clear visualization of its structure and functionality.

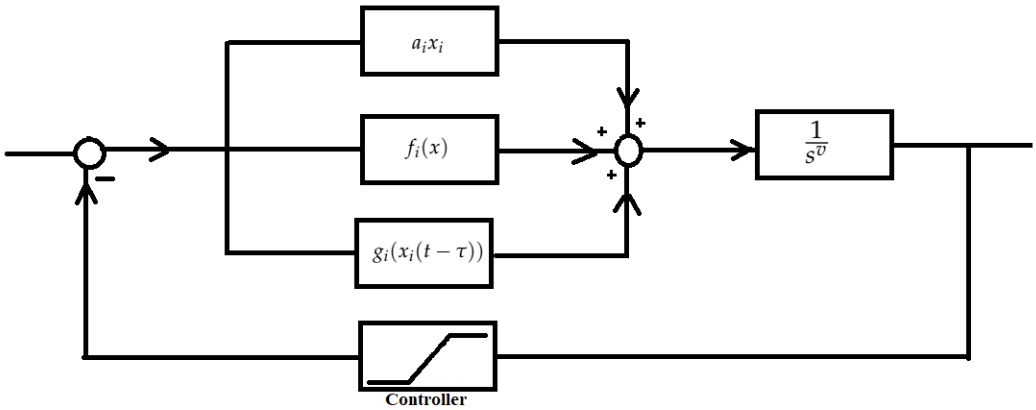

Figure 1. Block diagram of the FO delayed system subject to saturation control.

Assumption 1. *The nonlinear functions f_i and g_i are continuously differentiable in time and satisfy the local Lipchitz condition with Lipchitz parameters l_1 and l_2, respectively.*

Lemma 5 ([41])**.** *The saturation function satisfies the Lipchitz condition with a Lipchitz parameter l_{sat}.*

3.1. Section 1

We consider Equation (12) with $\tau = 0$; then, by considering feedback controller $u_i = k_i x_i$, the closed loop system can be rewritten as

$$^{C}D_t^{v}x_i = a_{cl_i} x_i + f_i(x) + g_i(x_i) + b_i \breve{\psi}_i(k_i x_i), \text{i} \in \{1,2,\ldots,N\}, \quad (13)$$

where $\breve{\psi}_i = sat(k_i x_i) - k_i x_i$ and $a_{cl_i} = a_i + b_i k_i$.

We introduce some assumptions for the present analysis. Some of these relate to the functions $f_i(x)$ and $g(x_i)$, as introduced in [28,29]. Additionally, the following components are defined and utilized in the subsequent discussions:

1. Function $\Phi_{3i}(\|x_i\|)$: Let $\Phi_{3i} : [0,\infty) \rightarrow [0,\infty)$ be a non-negative, continuous, and radially unbounded function that depends on the norm $\|x_i\|$. This function quantifies bounds on the growth of specific terms in the analysis of subsystem i.

2. Lyapunov Function Gradient $\nabla_{x_i} V_i(x_i)$: The term $\nabla_{x_i} V_i(x_i)$ represents the gradient of the Lyapunov function $V_i(x_i)$ with respect to the state x_i. The Lyapunov function $V_i(x_i) : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuously differentiable, positive-definite function used to establish subsystem stability.

3. Upper Bound $\Xi_i^{''}, \Xi_i^{'}$ and Ξ_i: The scalar $\Xi_i^{''}, \Xi_i^{'}$ and $\Xi_i > 0$ serves as an upper bound in inequalities involving $\nabla_{x_i} V_i(x_i)$ and related terms. This constant plays a crucial role in bounding subsystem dynamics and ensuring analytical rigor.

Assumption 2. *Consider a real number $\Xi_i^{''} > 0 (i = 1, \ldots, N)$ satisfying the following inequality:*

$$\nabla_{x_i} V_i(x_i) a_{cli} x_i \leq \sqrt{\Phi_{3i}(\|x_i\|)} \Xi_i^{''} \sqrt{\Phi_{3i}(\|x_i\|)} \tag{14}$$

Assumption 3 ([28,29])**.** *Consider a real number $\Xi_{ij} \geq 0 (i,j = 1, \ldots, N)$ satisfying the following inequality:*

$$\nabla_{x_i} V_i(x_i) f_i(x(t)) \leq \sqrt{\Phi_{3i}(\|x_i\|)} \Xi_{ij} \sqrt{\Phi_{3i}(\|x_j\|)} \tag{15}$$

Assumption 4. *Consider a real number $\Xi_i^{'} > 0 (i = 1, \ldots, N)$ satisfying the following inequality:*

$$\nabla_{x_i} V_i(x_i) \check{\psi}(x_i(t) \leq \sqrt{\Phi_{3i}(\|x_i\|)} \Xi_i^{'} \sqrt{\Phi_{3i}(\|x_i\|)} \tag{16}$$

Assumption 5 ([28,29])**.** *There is some constant $v_i > 0$ for $\tau \geq 0$ such as*

$$\nabla_{x_i} V_i(x) g(x_i(t-\tau)) \leq v_i \Phi_{3i}^{1/2}(\|x_i\|) \Phi_{3i}^{1/2}(\|x_i(t-\tau)\|), i = \{1,2,3,\ldots\} \tag{17}$$

Assumption 6 ([28,29])**.** *There is a continuous non-decreasing function $\zeta_i(\daleth) > \daleth > 0$ for $\tau \geq 0$ such as*

$$\|g_i(x(t-\tau)\| \leq \zeta_i(\|g_i(x(t))\|), i = \{1,2,3,\ldots,N\} \tag{18}$$

According to Theorem 4 in [42] and Remark 2.10 in [43], if we consider a continuous nondecreasing function $\zeta_i(\daleth) > \daleth > 0$ for $\daleth > 0$, then it is clear that

$$\Phi_{3i}^{1/2}(\|x_i(t-\tau)\|) < \zeta_i\left(\Phi_{3i}^{1/2}(\|x(t)\|)\right), i = \{1,2,3,\ldots,N\} \tag{19}$$

Lemma 6. *if each $V_i, i \in \{1,2,\ldots,N\}$ is continuously differentiable and convex, then their sum will also be continuously differentiable and convex.*

Proof. This pertains to properties of convex and continuously differentiable functions. □

Theorem 2. *Considering the closed-loop system described by Equation (13) and assuming that Assumptions 1–5 are satisfied, the asymptotic stabilization of the closed-loop system can be achieved if the following judgement matrix W is an M-matrix:*

$$\text{where } W = \begin{bmatrix} w_{11} & w_{12} & \ldots & w_{1j} \\ w_{21} & w_{21} & \ldots & w_{2j} \\ \vdots & \vdots & \ldots & \vdots \\ w_{i1} & w_{i1} & \ldots & w_{ij} \end{bmatrix}$$

$$w_{ij} = \begin{cases} -\Xi_i^{''} - \zeta_i v_i - \|b_i\| \Xi_i^{'} & if \;\; i = j \\ -\Xi_{ij} & Otherwise \end{cases} \tag{20}$$

Proof. Let us consider a series of positive-definite convex and continuously differentiable functions $V_i(x_i)$ that satisfy Lemma 1 for the non-linear FOS.

$$\begin{array}{l} \alpha_{1i}(\|x_i\|) \leq V_i \leq \alpha_{2i}(\|x_i\|) \\ {}^{C}D^{v}V(x_i) \leq -\alpha_{3i}(\|x_i\|) \end{array} \tag{21}$$

Then, we chose a vector function, the Lyapunov function of the FOS, with a set of positive-definite constants $q_i > 0$.

$$V(x) = \sum_{i=1}^{N} q_i V_i(x_i) \tag{22}$$

According to Lemma 2, we have

$${}^{C}D^{v}V(x) \leq \sum_{i=1}^{N} q_i \nabla_{x_i} V_i(x_i) {}^{C}D^{v} x_i \tag{23}$$

Substituting the states equation of (13) in (22), we have

$${}^{C}D^{v}V(x) \leq \sum_{i=1}^{N} q_i \nabla_{x_i} V_i(x_i)(a_{cl}{}_i x_i + f_i(x) + g_i(x(t)) + b_i \breve{\psi}_i(k_i x_i)), i \in \{1,2,\ldots,N\}, \tag{24}$$

Simplifying Equation (24), we have

$$\begin{array}{l} {}^{C}D^{v}V(x) \leq \sum_{i=1}^{N} q_i (\nabla_{x_i} V_i(x_i) a_{cl}{}_i x_i + \nabla_{x_i} V_i(x_i) f_i(x) + \\ \nabla_{x_i} V_i(x_i) g_i(x(t)) + \nabla_{x_i} V_i(x_i) b_i \breve{\psi}_i(k_i x_i)), i \in \{1,2,\ldots,N\}), \end{array} \tag{25}$$

Applying Equations (14), (15), (17) and (16),

$$\begin{array}{l} {}^{C}D^{v}V(x) \leq \sum_{i=1}^{N} q_i (\sqrt{\Phi_{3i}(\|x_i\|)} \Xi_i'' \sqrt{\Phi_{3i}(\|x_i\|)} + \sqrt{\Phi_{3i}(\|x_i\|)} \Xi_i \sqrt{\Phi_{3i}(\|x_i\|)} \\ + v_i \sqrt{\Phi_{3i}(\|x_i\|)} \zeta_i \left(\sqrt{\Phi_{3i}(\|x_i(t)\|)} \right) + b_i \sqrt{\Phi_{3i}(\|x_i\|)} \Xi_i' \sqrt{\Phi_{3i}(\|x_i\|)} \\ , i \in \{1,2,\ldots,N\}) \end{array} \tag{26}$$

Simplifying the above equation, we have

$${}^{C}D^{v}V(x) \leq -\frac{1}{2} \Phi_3^T(\|x\|)(WQ + QW)\Phi_3(\|x\|) \tag{27}$$

we are able to find a diagonal matrix $Q = diag\{q_1, q_2, ..., q_N\}$, such that the matrix $WQ + WQ$ is positive-definite. Then, according to Lemma 4, W is an M-matrix. Thus, based on Lemma 3, the closed-loop system is asymptotically stable.

where $\Phi_3(\|x\|) = \left[\begin{array}{cccc} \Phi_{31}^{\frac{1}{2}} & \Phi_{32}^{\frac{1}{2}} & \ldots & \Phi_{3N}^{\frac{1}{2}} \end{array} \right]$. The proof is complete. □

By satisfying the M-matrix properties of W, we can guarantee the stability and asymptotic stabilization of the closed-loop system described by Equation (13). This provides a basis for designing effective stabilization controllers that ensure the convergence of the system's trajectories to the desired equilibrium point.

We could introduce an alternative theorem in order to check a new stability condition by the Lyapunov-like function with the aid of diffusive realization and the Lipchitz condition. The Lyapunov-like function method provides a powerful tool for analyzing the stability of non-linear FOSs. Then, the described nonlinear system with Equation (13) can be rewritten as general form as

$$ {}^{C}D_t^{v}x = A_{cl}x + F(x) + G(x(t)) + B\Psi(x), \tag{28} $$

where $A_{cl} = diag\{a_{cl1}, a_{cl2}, ..., a_{clN}\}$ and $B = diag\{b_1, b_2, \ldots, b_N\}$ are constant matrices, and $F(x)$, $G(x(t)$ are the matrix of the linear or non-linear part of the system with appropriate dimension. Furthermore, $\Psi(x) = Sat(Kx) - Kx$, $K = diag\{k_1, k_2, \ldots, k_N\}$.

Lemma 7 ([19])**.** *By defining function $\Psi(x) = Sat(u) - u$, it satisfies the Lipchitz condition with the Lipchitz parameter of l_s.*

Theorem 3. *Consider the dynamical system described by Equation (28) under the validity of Assumption 1. The system is guaranteed to be asymptotically stable if there exists*

- *A symmetric positive-definite matrix P with all elements satisfying $p_{ij} \geq 0$,*
- *Positive constants k_f, k_g, and k_s,*

such that the following matrix inequality holds:

$$ \begin{bmatrix} \Theta & P & P & P \\ P & -k_f I & 0 & 0 \\ P & 0 & -k_g I & 0 \\ P & 0 & 0 & -k_s I \end{bmatrix} < 0, \tag{29} $$

where

$$ \Theta = PA_{cl} + A_{cl}^T P + 2k_f l_1^2 + 2k_g l_2^2 + 2k_s l_s^2. $$

Proof. In this proof, we use Lyapunov-like function method (often used to establish stability in control theory for FOS), and it contains a dependence on both the state $x(t)$ and the time t because the solution $\psi(w,t)$ depends on t. By introducing the following Lyapunov function based on Theorem 1, we have

$$ V(x,t) = \int_0^\infty \xi_v(w)\psi(w,t)^T P\psi(w,t)dw \tag{30} $$

if we obtain the time derivative from $V(x)$, it is clear that

$$ \dot{V}(x,t) = \int_0^\infty \xi_v(w)\left[\frac{\partial\psi(w,t)}{\partial t}^T P\psi(w,t) + \psi(w,t)^T P\frac{\partial\psi}{\partial t}\right]dw \tag{31} $$

Using Theorem 1, the above equation can be rewritten

$$ \begin{aligned} \dot{V}(x,t) = &\int_0^\infty \xi_v(w)[-w\psi(w,t) + A_{cl}x + F(x) + G(x) + B\Psi(x)]^T P\psi(w,t)dw \\ &+ \int_0^\infty \xi_v(w)\psi^T(w,t)P[-w\psi(w,t) + A_{cl}x + F(x) + G(x) + B\Psi(x)]dw \end{aligned} \tag{32} $$

By simplifying, we have

$$ \begin{aligned} \dot{V}(x,t) = &-\int_0^\infty \xi_v(w)\psi^T(w,t)P\psi(w,t)dw + x^T A_{cl}^T Px + F^T Px + G^T Px + \Psi^T(x)B^T Px \\ &-\int_0^\infty \xi_v(w)\psi^T(w,t)P\psi(w,t)dw + x^T PA_{cl}x + x^T PF + x^T PG(x) + x^T PB\Psi(x) \end{aligned} \tag{33} $$

Then, based on the Lipschitz condition, for any positive constant $k_f, k_g > 0$ and k_s,

$$\begin{aligned} & k_f(F^T(x(t))F(x(t)) - F^T(x(t))F(x(0)) - F(x(t)F^T(x(0)) + F^T(x(0))F(x(0)) \\ & \leq 2k_f l_1^2(x^T(t)x(t) - x^T(t)x(0) - x(t)x^T(0) \\ & + x^T(0)x(0)2k_f l_1^2 x^T(t)x(t) - k_f F^T(x(t))F(x(t)) > 0 \end{aligned} \tag{34}$$

where l_1 is the Lipschitz parameter of $F(.)$[29].

$$\begin{aligned} & k_g(G^T(x(t))G(x(t)) - G^T(x(t))G(x(0)) - G(x(t)G^T(x(0)) + G^T(x(0))G(x(0)) \\ & \leq 2k_g l_2^2(x^T(t)x(t) - x^T(t)x(0) - x(t)x^T(0) \\ & + x^T(0)x(0)2k_g l_2^2 x^T(t)x(t) - k_g G^T(x(t))G(x(t)) > 0 \end{aligned} \tag{35}$$

where l_2 is the Lipchitz parameter of $G(.)$.

$$\begin{aligned} & k_s(\Psi^T(x(t))\Psi(x(t)) - \Psi^T(x(t))\Psi(x(0)) - \Psi(x(t)\Psi^T(x(0)) + \Psi^T(x(0))\Psi(x(0)) \\ & \leq 2k_s l_s^2(x^T(t)x(t) - x^T(t)x(0) - x(t)x^T(0) \\ & + x^T(0)x(0)2k_s l_s^2 x^T(t)x(t) - k_s\Psi^T(x(t))\Psi(x(t)) > 0 \end{aligned} \tag{36}$$

where l_s is the Lipchitz parameter of $\Psi(.)$.

Then, we can continue as follows

$$\begin{aligned} & \dot{V}(x,t) = -2\int_0^\infty \xi_v(w)\psi^T(w,t)P\psi(w,t)dw + x^T(PA_{cl} + A_{cl}^T P)x + x^T Pf(x) \\ & + x^T Pg(x) + x^T P\Psi(x) + f^T(x)Px + g^T Px + \Psi^T(x)B^T Px + 2k_f l_1^2 x^T x \\ & - k_f f^T(x)f(x) + 2k_g l_2^2 x^T x - k_g g^T(x)g(x) + 2k_s l_s^2 x^T x - k_s\Psi^T(x)\Psi(x) \end{aligned} \tag{37}$$

By definition of a new vector as $\Xi = \begin{bmatrix} x^T & f^T & g^T & \psi^T \end{bmatrix}^T$, we have

$$\Xi^T \begin{bmatrix} \Theta & P & P & P \\ P & -k_f I & 0 & 0 \\ P & 0 & -k_g I & 0 \\ P & 0 & 0 & -k_s I \end{bmatrix} \Xi < 0 \tag{38}$$

Then, the FO non-linear system subject to saturation is asymptotically stable, which completes the proof. □

The stability conditions provided by Theorem 3 offer valuable insights into the design of controllers and stability analysis for the considered system. By ensuring that the matrix inequality (29) is satisfied, we can guarantee the stability of the system, enabling effective control strategies.

3.2. Section 2

The second analysis considers the presence of a time delay, indicated by $\tau \neq 0$ in Equation (12). Then, Equation (39) presents the closed-loop system:

$${}^C D_t^v x_i = a_{cli}x_i + f_i(x) + g_i(x(t-\tau)) + b_i\breve{\psi}_i(k_i x_i), \mathrm{i} \in \{1, 2, \ldots, N\}, \tag{39}$$

Theorem 4. *Consider the closed-loop system described by Equation (39), which satisfies Assumptions 2–6. If the judgment matrix W is an M-matrix, then the closed-loop system (39) is asymptotically stable.*

The judgment matrix W is defined as

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1j} \\ w_{21} & w_{22} & \cdots & w_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i1} & w_{i2} & \cdots & w_{ij} \end{bmatrix},$$

where the elements w_{ij} of W are given by

$$w_{ij} = \begin{cases} -\Xi_i'' - \zeta_i v_i - \|b_i\|\Xi_i' & \text{if } i = j, \\ -\Xi_{ij} & \text{otherwise.} \end{cases} \tag{40}$$

This theorem establishes that if the matrix W satisfies the properties of an M-matrix, the stability of the closed-loop system is guaranteed. The conditions on the elements w_{ij} ensure that the system's stability criteria are met even in the presence of nonlinearities and external constraints.

Proof. For a closed-loop system (39), similar to the previous proof, let us consider a series of positive-definite functions $V_i(x_i)$ as the Lyapanov function that satisfies Lemma 1 such as

$$V(x) = \sum_{i=1}^{N} q_i V_i(x_i) \tag{41}$$

where q_i is a set of positive-definite constants.

Then, according to Lemma 2, the following inequality satisfies

$$^C D^v V(x) \leq \sum_{i=1}^{N} q_i \nabla_{x_i} V_i(x_i)^C D^v x_i \tag{42}$$

Substituting the states equation of (39) in (42), we have

$$^C D^v V(x) \leq \sum_{i=1}^{N} q_i \nabla_{x_i} V_i(x_i)(a_{cl}{}_i x_i + f_i(x) + g_i(x(t-\tau)) + b_i \breve{\psi}_i(k_i x_i)), i \in \{1,2,\ldots,N\}, \tag{43}$$

Simplifying Equation (43), we have

$$\begin{aligned} ^C D^v V(x) \leq \sum_{i=1}^{N} q_i(\nabla_{x_i} V_i(x_i) a_{cl}{}_i x_i + \nabla_{x_i} V_i(x_i) f_i(x) + \\ \nabla_{x_i} V_i(x_i) g_i(x(t-\tau)) + \nabla_{x_i} V_i(x_i) b_i \breve{\psi}_i(k_i x_i)), i \in \{1,2,\ldots,N\}, \end{aligned} \tag{44}$$

Using (15) and (17)

$$\begin{aligned} ^C D^v V(x) \leq \sum_{i=1}^{N} q_i(\nabla_{x_i} V_i(x_i) a_{cl}{}_i x_i + \sqrt{\Phi_{3i}(\|x_i\|)}\Xi_i \sqrt{\Phi_{3i}(\|x_j\|)} + \\ v_i \Phi_{3i}^{1/2}(\|x_i\|)\Phi_{3i}^{1/2}(\|x_i(t-\tau)\|) + \nabla_{x_i} V_i(x_i) b_i \breve{\psi}_i(k_i x_i)), i \in \{1,2,\ldots,N\}, \end{aligned} \tag{45}$$

Applying Assumption 6 to Equation (45),

$$\begin{aligned} ^C D^v V(x) \leq \sum_{i=1}^{N} q_i \sqrt{\Phi_{3i}(\|x_i\|)}\Xi_i'' \sqrt{\Phi_{3i}(\|x_i\|)} + \sqrt{\Phi_{3i}(\|x_i\|)}\Xi_i \sqrt{\Phi_{3i}(\|x_i\|)} \\ + v_i \sqrt{\Phi_{3i}(\|x_i\|)}\zeta_i \left(\sqrt{\Phi_{3i}(\|x(t)\|)} \right) + b_i \sqrt{\Phi_{3i}(\|x_i\|)}\Xi_i' \sqrt{\Phi_{3i}(\|x_i\|)} \\ , i \in \{1,2,\ldots,N\}, \end{aligned} \tag{46}$$

Simplifying the above equation, we have

$$^{C}D^{v}V(x) \leq -\frac{1}{2}\Phi_3^T(\|x\|)(WQ + QW)\Phi_3(\|x\|) \tag{47}$$

we are able to always find a diagonal matrix $Q = diag\{q_1, q_2, \ldots, q_N\}$, such that the matrix $WQ + WQ$ is positive-definite. Then, according to Lemma 4, W is M-matrix. Thus, based on Lemma 3, the closed-loop system is asymptotically stable.

where $\Phi_3(\|x\|) = \left[\begin{array}{cccc} \Phi_{31}^{\frac{1}{2}} & \Phi_{32}^{\frac{1}{2}} & \ldots & \Phi_{3N}^{\frac{1}{2}} \end{array}\right]$.

Hence, the proof is complete. □

Note 1. *The selection of an appropriate vector Lyapunov function is of paramount importance in the stability analysis of the system. It serves as the foundation for ensuring that all the required assumptions are satisfied. By carefully choosing a vector Lyapunov function that aligns with the system's dynamics and constraints, the stability conditions can be rigorously verified, thereby guaranteeing the desired asymptotic behavior of the system.*

4. Controller Design

Designing a stable controller is a critical challenge in the practical implementation of control systems. While the previous section established sufficient stability conditions for ensuring the stability of the closed-loop system, the task of designing an effective and implementable controller remains a significant objective. In this section, we address this challenge by formulating the controller design problem as an optimization problem, which is a well-established methodology in control theory. This approach provides both flexibility and robustness, allowing for the systematic incorporation of various performance criteria into the design process. Moreover, the optimization framework ensures the derivation of a smooth and computationally efficient controller that meets the desired stability and performance requirements, making it practical for real-world implementation.

We propose the following optimization problem to achieve the optimal output controller:

$$\begin{aligned} &\text{Minimize } \ u \\ &\text{subject to: Stability Condition} \end{aligned} \tag{48}$$

In the optimization problem (48), the control input u is defined as $u = Kx$, where K represents the controller gain matrix. This optimization problem leverages the stability condition derived from Theorem 2, specifically using Equation (29), which converts the optimization into a nonlinear problem. The formulation ensures that the controller adheres to stability constraints while optimizing system performance. Furthermore, the controller gain matrix K is selected to ensure the asymptotic stabilization of the system.

The utilization of optimization in the design process not only addresses the inherent complexities in FO nonlinear systems but also ensures that the controller remains stable in the presence of input saturation and time delays. The method we present provides a systematic approach to balancing the trade-off between stability and performance, thereby offering a comprehensive solution to controller design in FOSs subject to nonlinear dynamics and practical constraints.

5. Simulation Results

In this section, we investigate the effectiveness of the proposed analysis using two well-known FOSs: the FO Lorenz system and the FO Liu system. Both systems serve as paradigms in the study of chaotic dynamics, providing valuable insights into complex behaviors that are pivotal in numerous applications, including robotics and automation, where precise control and adaptation to uncertain environments are critical.

The FO Lorenz system extends the classical Lorenz system [44] into the FO domain, thereby exhibiting intricate chaotic behavior that differs from its integer-order counterpart. A recent survey [45] provides a new classification of FO Lorenz-type systems, detailing their equilibria, eigenvalues, and attractors within the three-dimensional state space. These characteristics make the FO Lorenz system particularly suitable for modeling and analyzing systems that are sensitive to initial conditions and exhibit long-term unpredictability.

In the field of robotics and automation, chaotic systems like the FO Lorenz can be employed for secure communication between robotic units, as the chaotic nature of the system provides an inherent level of security in signal transmission. Additionally, the complex dynamics of the FO Lorenz system may be leveraged for trajectory planning and optimization algorithms, which are essential in scenarios where robots must navigate unpredictable or dynamic environments. The practical applications of the FO Lorenz system are extensive, encompassing fields such as weather forecasting, cryptography, and fluid dynamics. These applications underscore its significance in understanding and predicting complex phenomena, including those encountered in robotic control systems and autonomous navigation.

Similarly, the FO Liu system, a variant of the classical Liu system, incorporates FO derivatives that introduce non-locality and nonlinearity, resulting in highly complex and chaotic dynamics. This system has attracted considerable attention in the field of nonlinear dynamics due to its potential applications in secure communications and cryptography. In robotics and automation, the FO Liu system could be applied to design robust controllers capable of handling nonlinearities and uncertainties in robotic systems, particularly in environments where precise synchronization and adaptability are required. Its chaotic behavior also makes it suitable for applications in swarm robotics, where chaotic motion patterns may enhance exploration and task distribution among robots.

Thus, studying the stability, control, and synchronization of the FO Liu system is essential not only for comprehending its theoretical behavior but also for exploring its practical implications in automation and robotic systems where stability and precise coordination between components are critical. By applying the proposed analysis to these FO chaotic systems, we aim to deepen the understanding of their complex dynamics and demonstrate the efficacy of the advanced stabilization techniques introduced in this work.

5.1. Example 1

For the first example, we consider the FO Lorenz system to demonstrate the advantage of the proposed desired controller by utilizing the Lyapunov-like function. The system can be described by the following equations, which capture the FO dynamics:

$$\begin{aligned} D^v x &= \sigma(y - x) + \mathrm{sat}(k_1 x) \\ D^v y &= x(\rho - z) - y + \mathrm{sat}(k_2 y) \\ D^v z &= xy - \beta z + \mathrm{sat}(k_3 z) \end{aligned} \tag{49}$$

where x, y, and z represent the state variables, and σ, ρ, and β are system parameters. By designing a suitable controller based on the Lyapunov-like function and applying it to the FO Lorenz system, we aim to demonstrate the ability of the proposed control approach to stabilize and control the chaotic behavior of the system. The analysis and simulation results will provide insights into the effectiveness and performance of the controller in achieving stabilization and desired system behavior.

The model parameters are given by $\sigma = 10$, $\rho = 28$, $\beta = \frac{8}{3}$, $v = 0.993$, and $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. The open-loop system is depicted in Figure 2.

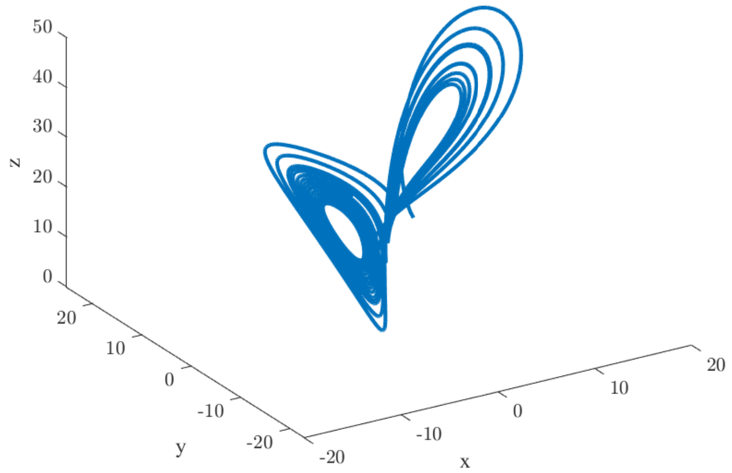

Figure 2. Phase plot of the open-loop FO Lorenz system for $v = 0.993$.

Next, we consider the condition of Theorem 2 as follows:

$$A_{cl} = \begin{bmatrix} k_1 - \sigma & 0 & 0 \\ 0 & k_2 - 1 & 0 \\ 0 & 0 & k_3 - \beta \end{bmatrix}, \quad F(x) = \begin{bmatrix} 0 \\ -xz \\ xy \end{bmatrix}, \quad G(x) = \begin{bmatrix} \sigma \\ \rho \\ 0 \end{bmatrix}.$$

We can now design the controller using the proposed method outlined in Section 4 and solve the optimization problem given by (48). We then consider $\|u\|$:

$$\begin{aligned} &\text{Minimize} \quad \|u\| \\ &\text{subject to:} \\ &\left\{ \begin{bmatrix} \Theta & P & P & P \\ P & -k_f I & 0 & 0 \\ P & 0 & -k_g I & 0 \\ P & 0 & 0 & -k_s I \end{bmatrix} < 0 \right. \end{aligned} \tag{50}$$

We achieve the following results:

$$\begin{aligned} P &= \begin{bmatrix} 1 & 0.001 & 0.001 \\ 0.001 & 1 & 0.001 \\ 0.001 & 0.001 & 1 \end{bmatrix} \\ K &= \text{diag}\begin{bmatrix} -68.8939 & -77.8701 & -57.2020 \end{bmatrix} = \text{diag}\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \\ k_f &= 0.8132, \quad k_g = 0.0295, \quad k_s = 0.0825 \end{aligned} \tag{51}$$

Figures 3 and 4 exhibit the phase plots of the system and demonstrate the behavior of the closed-loop system. These plots provide valuable insights into the dynamics and convergence characteristics of the system. In Figure 3, the phase plot illustrates the trajectory of the system's state variables over time, allowing us to visualize the system's attractor and observe its overall behavior. A well-defined and structured attractor indicates the stability and regular behavior in the system. The plot reveals the desirable convergence speed of the system, with the trajectory approaching a steady state or periodic orbit. Similarly, in Figure 4, the behavior of the closed-loop system with initial states $x(0) = [10, 14, 7]$ is depicted. The closed-loop response reflects the impact of the control input on the system's dynamics. The figure demonstrates the effectiveness of the control strategy in guiding the system toward a desired state or trajectory. The convergence speed, as evident from the plot, indicates that the closed-loop system can reach the desired equilibrium or follow a desired trajectory in a relatively short time. The rapid convergence speed observed in Figures 3 and 4 is a positive outcome, signifying the effectiveness and efficiency of the control approach. A fast convergence speed implies that the system can quickly adjust and stabilize itself in response to perturbations or changes in initial conditions.

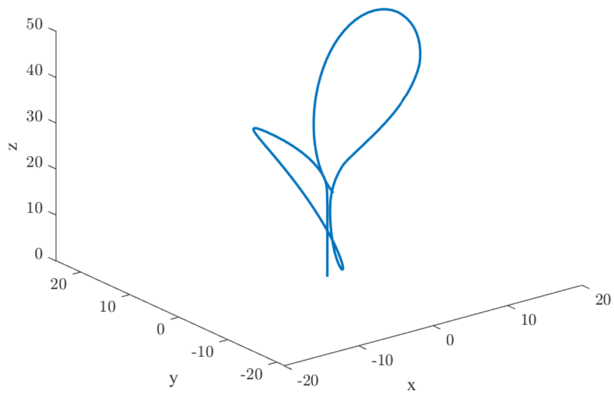

Figure 3. Phase plot of the closed-loop FO Lorenz system for $v = 0.993$.

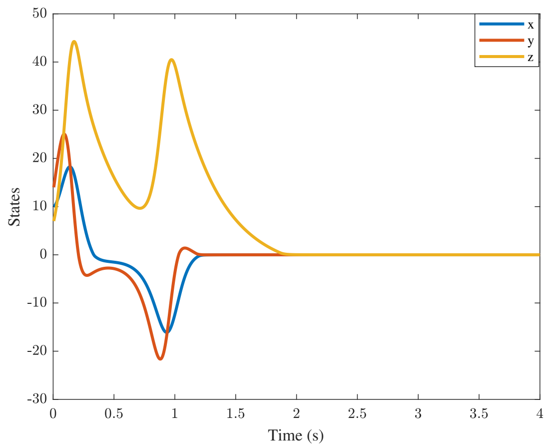

Figure 4. The behavior of the closed-loop FO Lorenz system for $v = 0.993$.

In Figure 5, we present a comparison of the states between FO and integer-order system modes. The figure clearly demonstrates that the states of the FO system converge faster than those of the integer-order system in [46]. This observation can be attributed to two possible reasons. First, the stability region of the FO system, when $v \in (0, 1)$, is inherently broader than that of the integer-order system. Second, different controllers were employed in the FO system and the integer-order system, which may contribute to the observed performance differences.

Furthermore, we were unable to extend the presented analysis to the integer-order system due to the specific constraints of Lemma 1 and Theorem 1, which apply only when $v \in (0, 1)$. Conducting a similar analysis for integer-order systems could represent a valuable direction for future research.

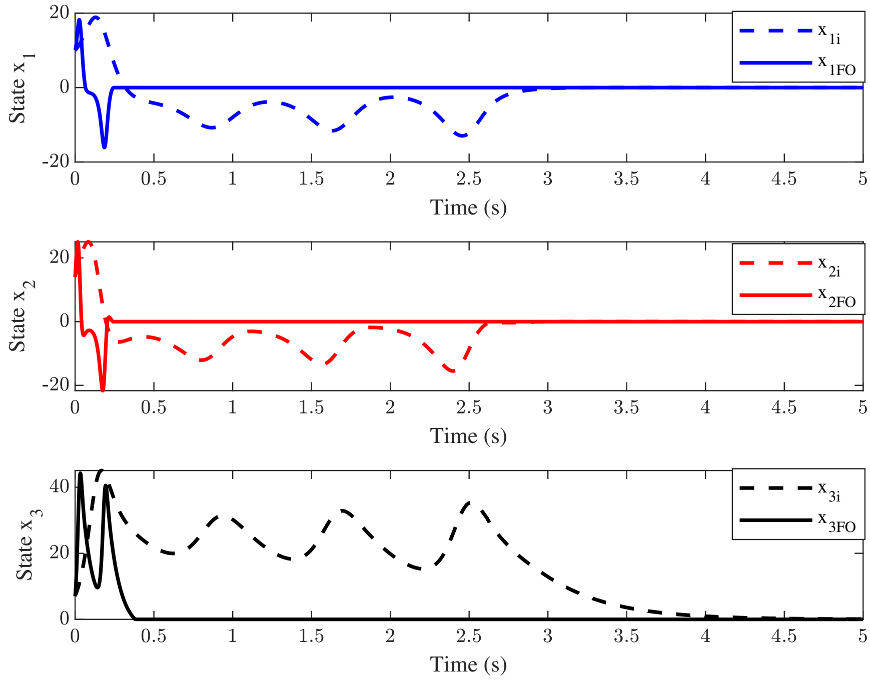

Figure 5. Comparison of states of FO and integer-order system modes presented in [46].

5.2. Example 2

In this section, we extend the application of our proposed control approach to the FO Liu system. By incorporating time delays, we aim to further validate the effectiveness and applicability of our control methods across different chaotic systems. We begin by transforming the traditional FO Liu system into a form that incorporates multiple time delays [29], as represented in Equation (51). The system dynamics are governed by three state variables x_1, x_2, and x_3, with corresponding fractional derivatives in the Caputo sense denoted by D_t^v. The system coefficients a, b, c, and d are predefined constants, while $\tau = 1$ represents the time delay. We select specific coefficient values ($a = 2.5$, $b = -4$, $c = -5$, $d = 4$) and an initial condition of $\begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}$ to simulate the FO Liu system without control. The phase plot of the system is presented in Figure 6.

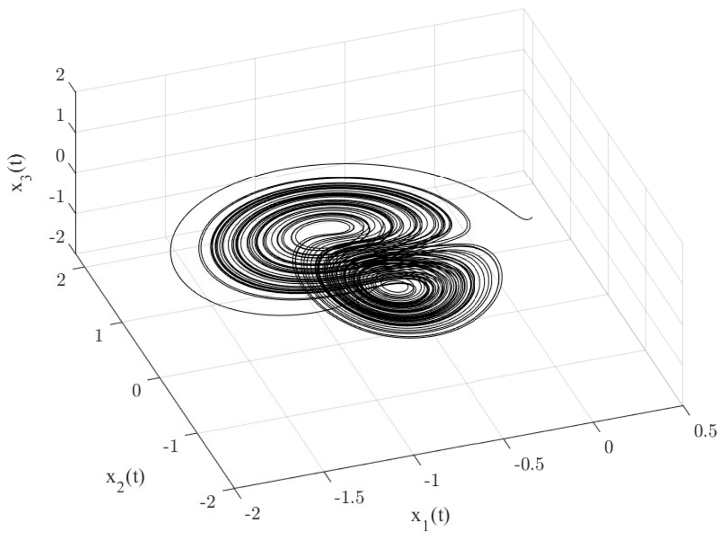

Figure 6. Phase plot of the open-loop FO Liu system for $v = 0.95$.

$$\begin{cases} D_t^v x_1 = -x_1(t-\tau) + x_2^2(t) \\ D_t^v x_2 = a x_2(t-\tau) + b x_1(t) x_3(t) \\ D_t^v x_3 = c x_3(t-\tau) + d x_1(t) x_2(t) \end{cases} \tag{52}$$

It presents the phase plot of the FO Liu system without control, showcasing the relationships between the state variables and visually representing the system's attractor. This allows us to identify any attractor structure and assess the system's stability and dynamic behavior. In [29], the system was studied without any limitations on controller.

By contrasting the results obtained with and without control, we can evaluate the effectiveness of our methods in stabilizing the system and regulating its behavior.

Without control, the time-delayed FO Liu system exhibits unstable states for every $\tau \geq 0$. Thus, we introduce controllers as follows:

$$\begin{cases} D_t^v x_1 = -x_1(t-\tau) + x_2^2(t) + b_1^{'}\mathrm{sat}(u_1(t)) \\ D_t^v x_2 = a x_2(t-\tau) + b x_1(t) x_3(t) + b_2^{'}\mathrm{sat}(u_2(t)) \\ D_t^v x_3 = c x_3(t-\tau) + d x_1(t) x_2(t) + b_3^{'}\mathrm{sat}(u_3(t)) \end{cases} \tag{53}$$

We set the input coefficients $b_1^{'} = b_2^{'} = b_3^{'} = 1$, and the vector Lyapunov function is defined as follows:

$$V(x_i(t)) = \Sigma_{i=0}^{3} {x_i}^2 = V_1 + V_2 + V_3 \tag{54}$$

Selecting $\zeta = 3/2$ satisfies the requirements of Theorem 4. We can derive the following inequality:

$$\nabla_{x_i} V_i(x_i) x_i(t-1) \leq 2\|x_i(t)\|\|x_i(t-1)\| \leq 3\|x_i(t)\|^2, \tag{55}$$

$$\begin{cases} \nabla_{x_1} V_1(x_1) g_1(x(t)) \leq 2\|x_1(t)\|\|x_2(t)\|^2 \\ \nabla_{x_2} V_2(x_2) g_2(x(t)) \leq 8\|x_1(t)\|\|x_2(t)\|\|x_3(t)\| \\ \nabla_{x_3} V_3(x_3) g_3(x(t)) \leq 8\|x_1(t)\|\|x_2(t)\|\|x_3(t)\| \end{cases} \tag{56}$$

Choosing $u_1(x(t)) = 9x_1(t)$, $u_2(x(t)) = 15x_2(t)$, and $u_3(x(t)) = 13x_3(t)$, and based on Theorem 4, the matrix W is defined as follows:

$$W = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 28.5 & 0 \\ 0 & 0 & 24 \end{bmatrix}.$$

It is evident that the matrix W is an M-matrix. We now proceed to analyze the closed-loop system described by (53) with the proposed controller. The time responses of the controlled FO Liu system are illustrated in Figures 7 and 8, showcasing the system's behavior under the influence of the designed controller. These figures provide insights into the convergence speed and stability of the controlled system. Figure 9 depicts the output of the saturation control.

To further demonstrate the versatility of the proposed controller, we consider different FO parameters and initial values for the FO Liu system. Figures 10 and 11 depict the time responses of the system for randomly selected FO parameters and initial values. These results confirm that the proposed controller, based on the vector Lyapunov function, effectively satisfies the conditions outlined in Theorem 4, ensuring stability and convergence of the system across various parameter settings $v = [0.78, 0.78, 0.78]$, with initial values as $x_0 = [2.21, 3.1, 4]$).

In conclusion, the application of the proposed controller to the FO Liu system yields promising results. The controlled system exhibits stable behavior and convergence toward desired states. These outcomes affirm the viability of the vector Lyapunov function-based control approach and its ability to meet the requirements specified in Theorem 4.

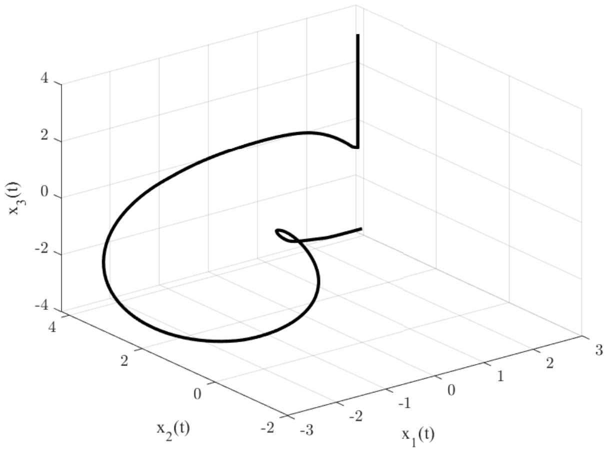

Figure 7. Phase plot of the closed-loop FO Liu system for $v = 0.95$.

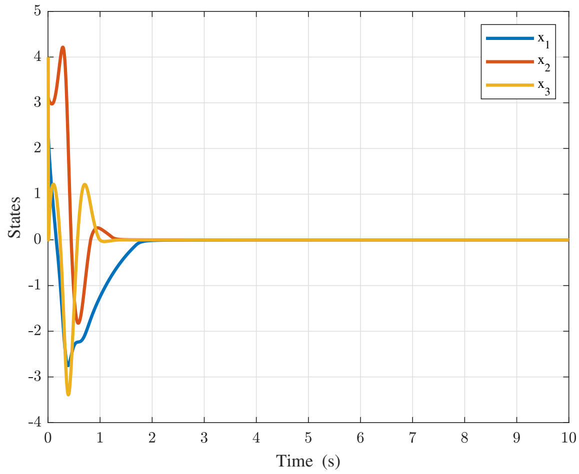

Figure 8. Time evolution of state variables of the closed-loop FO Liu system for $v = 0.95$.

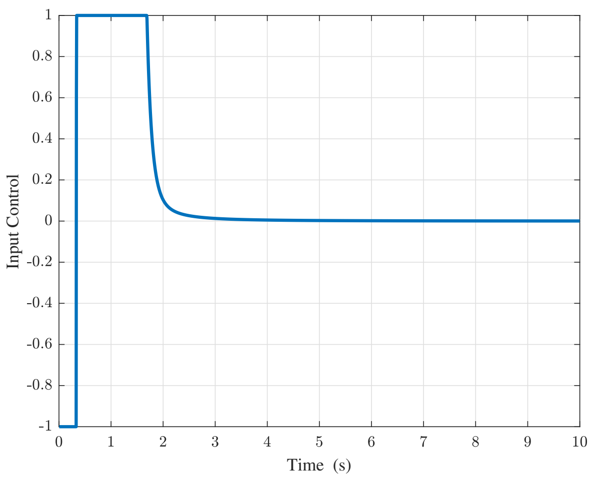

Figure 9. Input saturation control.

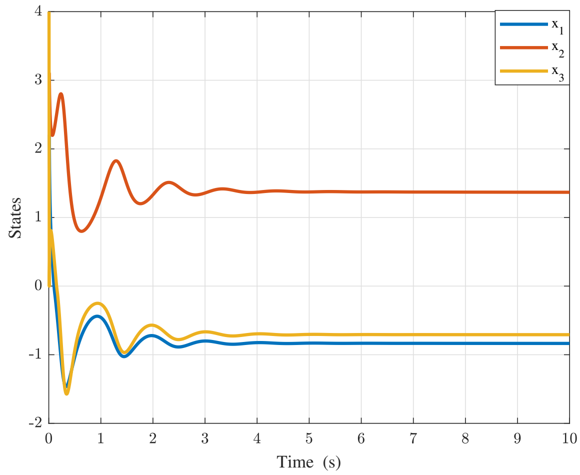

Figure 10. Time evolution of state variables of the closed-loop FO Liu system for order $v = 0.78$.

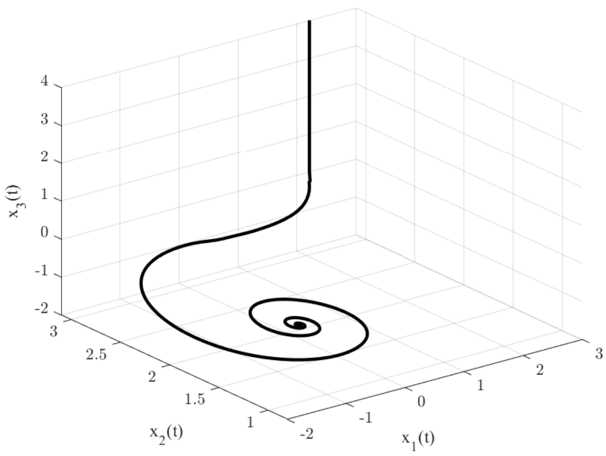

Figure 11. Phase plot of the closed-loop Liu system for order $v = 0.78$.

6. Conclusions

This research introduces novel stabilization criteria for a class of fractional-order (FO) nonlinear systems characterized by saturation control and time delays. The stability analysis hinges on two primary methodologies: the Lyapunov-like function and the vector Lyapunov function tailored for FO nonlinear systems with control constraints. The former technique leverages Lyapunov-like functions, diffusive realization, and Lipschitz conditions, while the latter employs vector Lyapunov functions, the M-matrix method, and specific assumptions. Notably, this latter approach effectively addresses the asymptotic stability problem of nonlinear systems with time delays.

A pivotal aspect of this work is the practical application of the derived stability conditions. A dedicated section outlines the design of a stabilized controller within the context of an optimization problem, considering constraints. To validate the efficacy of the proposed control strategies, two well-established chaotic systems, the FO Lorenz and Liu systems, were employed as case studies. The simulation results unequivocally demonstrate the successful stabilization of these systems, underscoring the practical significance and applicability of the developed techniques. Future research directions include extending these control methods to encompass uncertain FO nonlinear systems.

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Abbreviations

The following abbreviations are used in this manuscript:

FOSs	Fractional-Order Systems
FO	Fractional Order
FC	Fractional Calculus
MFOS	Multivariable Fractional-Order Systems

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