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AN OVERVIEW OF BAER'S THEOREM AND ITS EXTENSIONS

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Abstract. Baer's theorem is one of the cornerstone result in group theory, providing critical insights into the relationship between the finiteness of central factor group and that of the commutator subgroup. Building upon Schur's foundational work, Baer's theorem connects the upper and lower central series, establishing constraints on group structure that have far-reaching implications. This paper provides a brief review of Baer's theorem, detailing its historical development, generalizations, and recent extensions. Some key results include exponents, bounds on central series, extensions to locally generalized radical groups, finite rank conditions and applications to automorphism-influenced properties are given. Invoking the notion of variety of groups, we also propound the Baer's (or Schur's) theorem in its most general form as a fundamental question and attempt to identify all classes of groups that are Schur-Baer with respect to some variety as potential answers. Particular attention is also given to some of its applications in diverse areas of mathematics. Furthermore, the paper explores open problems and potential research directions, underscoring the theorem's enduring significance and its role in shaping contemporary mathematical inquiry.

1. **Introduction**

Group theory is a fundamental branch of mathematics, providing deep insights into the structure of algebraic concepts. Among its most significant results is Schur's theorem, which establishes a relationship between the finiteness of central factor group and commutator subgroup. Building on

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Schur's theorem from 1904 [21], Baer extended the understanding of central and commutator subgroups to the roles of the upper and lower central series in group classification [2].

The relevance of Baer's theorem extends far beyond its original formulation. Recent advancements have broadened its applicability to infinite groups [14], groups with finite rank [28], and periodic locally nilpotent groups [30]. The theorem has become a crucial tool in understanding nilpotent, solvable, and residually finite groups, with connections to the other branches of algebra.

This paper reviews the historical development of Baer's theorem, its key generalizations and extensions, and also its applications in diverse areas of mathematics. We aim to provide an overview perspective on this seminal result and its extensions, while identifying open problems and future research directions.

2. Historical Review of Baer's Theorem

One of the foundational results leading to Baer's theorem is Schur's theorem, which links the finiteness of the factor group $G/Z(G)$ to the finiteness of the derived subgroup G':

Schur (1904): If $G/Z(G)$ is finite, then G' is finite [21].

Schur's theorem emerged from studies in linear group theory and primarily dealt with the representation theory of finite groups and their associated matrix groups. Schur's theorem laid the groundwork for understanding the interaction between a group's center and its commutator subgroup. Initially focused on finite groups, the theorem has had profound implications in the study of infinite groups, particularly those that are residually finite or locally finite.

In 1952, Reinhold Baer extended Schur's theorem to higher terms in the central series, providing a framework for analyzing groups with finite upper central factor groups. To fully appreciate Baer's theorem, it is essential to understand the central series:

The *upper central series* of a group G is a family of subgroups $\{Z_\alpha(G)\}_{\alpha\in\text{Ordinals}}$, where Ordinals represents the class of all ordinals and it is defined recursively as follows:
$Z_0(G) = \{e\}$, and for a successor ordinal $\alpha+1$, $Z_{\alpha+1}(G)/Z_\alpha(G) = Z(G/Z_\alpha(G))$, where $Z(G/Z_\alpha(G))$ is the center of the quotient group $G/Z_\alpha(G)$. For a limit ordinal λ, $Z_\lambda(G) = \bigcup_{\alpha<\lambda} Z_\alpha(G)$. The *hypercenter* $Z_\infty(G)$ is defined as the union of all $Z_\alpha(G)$ for all ordinals α:

$$Z_\infty(G) = \bigcup_{\alpha\in\text{Ordinals}} Z_\alpha(G).$$

The *lower central series* of a group G is a family of subgroups $\{\gamma_\alpha(G)\}_{\alpha\in\text{Ordinals}}$, defined as follows: $\gamma_1(G) = G$. For a successor ordinal $\alpha+1$, $\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G]$, where $[\gamma_\alpha(G), G]$ is the subgroup generated by all commutators $[x, y] = x^{-1}y^{-1}xy$ for $x \in \gamma_\alpha(G)$ and $y \in G$. For a limit ordinal λ, $\gamma_\lambda(G) = \bigcap_{\alpha<\lambda} \gamma_\alpha(G)$. The *hypocenter* $\gamma_\infty(G)$ is defined as the intersection of all $\gamma_\alpha(G)$ for all ordinals α:

$$\gamma_\infty(G) = \bigcap_{\alpha\in\text{Ordinals}} \gamma_\alpha(G).$$

The hypercenter $Z_\infty(G)$ of a group G is the terminal subgroup of the upper central series and is the largest normal subgroup of G in which every element is central in some quotient $G/Z_\alpha(G)$.

Also, the hypocenter $\gamma_\infty(G)$ of a group G is the terminal subgroup of the lower central series and is the smallest normal subgroup N such that G/N is nilpotent.

These definitions using ordinal numbers provide a framework for analyzing the hierarchical structure of infinite groups, particularly in the study of nilpotent, solvable, and residually nilpotent groups.

Baer (1952): If $G/Z_n(G)$ is finite, then $\gamma_{n+1}(G)$ is finite for all $n \geq 1$ [2].

Baer's theorem connects the upper central series $Z_n(G)$, which approximates the center, to the lower central series $\gamma_{n+1}(G)$, which measures non-commutativity. This result deepened our understanding of the structure of groups by showing how finiteness in the upper series propagates to the lower series. Baer's work has influenced the classification of nilpotent and solvable groups and has set the stage for numerous generalizations.

In the 1970s, Wiegold introduced quantitative bounds on the size of G' in terms of $|G/Z(G)|$, improving earlier results [42]. Mann extended these results using tools from the solution of the restricted Burnside problem, establishing precise bounds for locally finite groups and groups of finite exponent [35]. Extensions by Ellis [13], Kurdachenko et al. [28], Dixon et al. [9], Wehrfritz [41], and Taghavi, Kayvanfar and Parvizi [23] have been done in order to find some bounds for lower central series of a group. Also efforts by Ellis [14], Mann [35], Dietrich and Moravec [5] and Kurdachenko et al. [30] were done to present bounds for the terms of the lower central series and autocommutator subgroup of a group. Extensions by Dixon, Kurdachenko and others explored the behavior of commutator subgroups under automorphism actions, further broadening the scope of Baer's theorem [6].

Modern research continues to refine Baer's theorem, addressing its applications in infinite group theory, representation theory, and algebraic topology. Relationships between the upper and lower central series, bounds on commutator subgroups, and also verifying some specific groups and properties like locally generalized groups, section rank and priodic groups and so on and also extending the theorem to the notion of variety of groups remain active areas of study, connecting Baer's theorem to new mathematical contexts.

3. Generalizations and Extensions of Baer's Theorem

Since the publication of Baer's original result, there have been numerous extensions and generalizations. These generalizations primarily fall into two categories:

- Generalizations related to the types of groups considered (e.g., solvable groups, polycyclic groups, finite groups, polycyclic by finite groups, Chernikov groups, residually locally finite groups, locally generalized radical groups, etc.).
- Generalizations related to the bounds and conditions imposed (e.g. finite exponent, special rank, section p-rank, rank conditions, etc.).

Nevertheless, depends on the situation, the above specifications and properties are sometimes used simultaneously. For instance, in the analyzing of groups with finite rank by replacing finiteness conditions on central factors with rank constraints, researchers have extended Baer's theorem to locally finite groups, Chernikov groups, and polycyclic-by-finite groups. These results have provided new tools for analyzing the hierarchy and classification of both finite and infinite groups.

Recent studies have also focused on bounding the size and structure of commutator subgroups using advanced algebraic techniques, such as Baer invariants, Schur multipliers, and non-abelian tensor products. These refinements have not only strengthened the applicability of Baer's theorem but also demonstrated its relevance to cohomology of groups, representation theory, and other areas of mathematics.

To state Baer's theorem (or Schur's theorem) in the most general state, we need to know the concept of variety of groups.

A *variety* $\mathfrak{V}$ of groups is defined as a class of groups closed under homomorphisms, subgroups, and direct products [38]. For example, trivial variety $\mathfrak{T}$, abelian variety $\mathfrak{A}$, nilpotent variety $\mathfrak{N}_c$ of class at most c, and solvable variety $\mathfrak{S}_l$ of groups of length l, contains only the trivial group, all abelian groups, groups of nilpotency class c and solvable groups of length l, respectively. The *verbal subgroup* $V(G)$ of a group G with respect to a variety $\mathfrak{V}$ is the subgroup of G generated by all elements that satisfy the defining identities of $\mathfrak{V}$. In other words,

$$V(G) = \langle w(g_1, g_2, \ldots, g_n) \mid w(x_1, x_2, \ldots, x_n) \text{ is a word in } V,\ g_i \in G\rangle,$$

where $w(x_1, \ldots, x_n)$ is any word representing a defining identity of the variety $\mathfrak{V}$, $w(g_1, g_2, \ldots, g_n)$ is the evaluation of that word in G and V is the set according to which the variety $\mathfrak{V}$ is defined. Also, the *marginal subgroup* $V^*(G)$ of a group G with respect to $\mathfrak{V}$ is the set of all elements $g \in G$ such that substituting g into any word $w(x_1, \ldots, x_n)$ defining $\mathfrak{V}$ leaves the word unchanged in all evaluations. that is

$$V^*(G) = \{g \in G \mid w(g, g_2, \ldots, g_n) = 1 \text{ in } G,\ \text{for all } w \in V \text{ and all } g_2, \ldots, g_n \in G\}.$$

In fact, $V(G)$ captures the active "outputs" of the words defined by the variety and $V^*(G)$ identifies the "inactive" elements that do not interfere with the variety's defining laws. For instance, in the variety of abelian groups, $V(G) = G'$, $V^*(G) = Z(G)$, and in the variety of nilpotent groups of class at most c, $V(G) = \gamma_{c+1}(G)$, $V^*(G) = Z_c(G)$.
Now the most general state of Baer's or Schur's theorem can be propounded as a question as follows:

Question: For what variety $\mathfrak{V}$ and class of groups $\mathcal{X}$, the assuption $G/V^*(G) \in \mathcal{X}$ implies $V(G) \in \mathcal{X}$?

If $\mathfrak{V}$ is the variety of abelian groups $\mathfrak{A}$, or the variety of nilpotent groups of class at most c, i.e. $\mathfrak{N}_c$, and $\mathcal{X}$ is the class of all finite groups $\mathcal{F}$, then the answer of the above question is clearly Schur's theorem and Baer's theorem, respectively. Now, one may easily observe that many of the papers which generalize Baer's or Schur's theorem, in fact, answer the above question for different varieties $\mathfrak{V}$ and classes of groups $\mathcal{X}$.

Accordingly, it is appropriate to name such a class of groups in honor of Schur and Baer as follows.

Definition: Let $\mathfrak{V}$ be a variety of groups. A class of groups $\mathcal{X}$ is called a *Schur-Baer class* with respect to the *variety* $\mathfrak{V}$, if for all groups G

$$G/V^*(G) \in \mathcal{X} \quad \text{always implies} \quad V(G) \in \mathcal{X}.$$

The class of all finite groups $\mathcal{F}$ and consequently, the class of all locally finite groups $\mathcal{LF}$ are Schur-Baer with respect to the variety of abelian groups $\mathfrak{A}$. Furthermore, the class of polycyclic-by-finite groups $\mathcal{PF}$, Chernikov groups $\mathcal{C}$ (see [31, Theorem 3.9]) (it was first proved by Polovitskij), and soluble-by-finite minimax groups $\mathcal{SF}_{min}$[26] are Schur-Baer with respect to $\mathfrak{A}$ too. Nevertheless, there are some classes of groups which do not have this property. For instance, Adian [1] shows that the class of periodic groups is not a Schur-Baer with respect to $\mathfrak{A}$.

In this section we intend to have a quick view to some classes of groups $\mathcal{X}$ which are Schur-Baer with respect to some varieties of groups $\mathfrak{V}$.

3.1. Baer's Theorem and Exponents.

The relationship between Baer's theorem and the exponents of groups has been explored in numerous papers. In particular, several results address how the finiteness conditions on the central series influence the exponents of the group. Some of the important results in this area are as follows.

Ellis' Exponent (2001): If $G/Z_n(G)$ is finite group of finite exponent, then exponent of $\gamma_{n+1}(G)$ is bounded by a function [14].

Mann's Exponent (2007): If $G/Z(G)$ is locally finite of finite exponent e, then exponent of $\gamma_2(G)$ is bounded by a function of e [35].

Kurdachenko et. al.'s Exponent (2016): If $G/Z_n(G)$ is locally finite of finite exponent e, then exponent of $\gamma_{n+1}(G)$ is bounded by a function of e [30].

The above statements show that the classes of all finite group of finite exponent $\mathcal{F}_{\mathcal{FE}}$ and locally finite of finite exponent $\mathcal{LF}_{\mathcal{FE}}$ are Schur-Baer with respect to the variety of nilpotent groups $\mathfrak{N}_n$ and therefore with respect to the variety of abelian groups $\mathfrak{A}$.

Dietrich and Moravec's Exponent (2011): If $G/L(G)$ is locally finite of finite exponent e, then exponent of $K(G)$ is bounded by a function of e, where $L(G)$ and $K(G)$ are absolute center and autocommutator subgroup of G [5].

3.2. **Baer's Theorem and Bounds on Central Series.** Another significant area of development has been the quantitative versions of Baer's theorem. Some authors have sought to provide better estimates and bounds for the sizes of $|\gamma_{n+1}(G)|$ in terms of $|G/Z_n(G)|$. These bounds can be used to classify groups more efficiently and to understand the behavior of terms in the central series. Some of these bounds include:

Wiegold's Bound (1965): If $|G/Z(G)| = t$, then $|G'| \leq t^{\frac{1}{2}(\log_2 t - 1)}$ [42].

Ellis' Bound (1998): Let G be a nilpotent group whose n-th upper central quotient is a direct product of finite prime-power group, i.e., $G/Z_n(G) \cong S_1 \times \cdots \times S_l$.

Suppose that each S_i is a d_i-generator group whose order is a power of some prime p_i say, where $p_i \neq p_j$ for $i \neq j$. Suppose that the Frattini subgroup ϕ_i of S_i is such that $|\gamma_j(S_i, \phi_i)| = p_i^{m_i j}$. Then

$$| \gamma_{n+1}(G) | \leq \prod_{i=1}^{l} p_i^{\chi_{n+1}(d_i) + m_{in} d_i + m_{i(n-1)} d_i^2 + \cdots + m_{i1} d_i^n}$$

[13].

Kurdachenko et al.'s Bound (2013): If $G/Z_\infty(G)$ is finite group of order t, then G has a finite normal subgroup L of order at most $|L| \leq t^{\frac{1}{2}(\log_2 t + 1)}$such that G/L is hypercentral, where $Z_\infty(G)$ is hypercenter of G [28].

Dixon et al.'s Bound (2017): Let G be a group and suppose that there is a natural number n such that $G/Z_n(G)$ is finite of order t. Then there is a function β_1 of n, t only such that $\gamma_{n+1}(G)$ is finite of order at most $\beta_1(t, n)$ is defined inductively by

$$\beta_1(t,1) = t^{\frac{1}{2}(\log_p t - 1)}, \quad \beta_1(t,n) = \beta_1(t, n-1)^{\frac{1}{2}(\log_p \beta_1(t,n-1) - 1)} t^{\beta_1(t,n-1)}$$

[7].

The above bound improved in 2021 by Taghavi, Kayvanfar and Parvizi [23] and it will be presented in the following.

Wehrfritz's Bound (2018): Let G be a group such that $|G/Z(G)| = t$. Then $|\gamma_2(G)| \leq t^{\frac{1}{2}(t'-1)}$, where t' is

$$t' = \begin{cases} \log_p t, & \text{if } r' = 1; \\ [\log_p t], & \text{if } r' \neq 1; \end{cases}$$

[41].

Taghavi et al.'s Bound (2021): Let G be a group such that $G/Z_n(G)$ is generated by d elements and is finite of order $t = p_1^{r_1} \cdots p_s^{r_s}$. Then

$$|\gamma_{n+1}(G)| \leq \min\{t^{\frac{1}{2}(t'-1)(\frac{d^n - 1}{d-1})}, t^{\frac{1}{2}(t'+1)} \textstyle\prod_{i=1}^{s} p_i^{\chi_{n+1}(r_i)}\}$$

[23].

The significance of investigating on bounds for the lower central series are particularly useful when working with large groups or groups where direct computation is not feasible.

Remark: Some of the above results have been extended to A-central series by many authors, where A is a subgroup of $Aut(G)$ such that $Inn(G) \leq A$.

For instance:

Hegarty's Bound (1994): Let $|G/C_G(Aut(G))| = t$. Then

$$|\gamma_2(G, Aut(G))| \leq t^{t((t-1)^2+[\frac{t}{2}])[\log_2 t]}$$

[18].

Dixon et al.'s Bound (2014): Let G be a group and let A be a subgroup of $Aut(G)$ such that $Inn(G) \leq A$ and $|A/Inn(G)| = k$ is finite. Let Z be the upper A-hypercenter of G. Suppose that $zl(G, A) = m$ is finite and G/Z is finite of order t. Then $\gamma_{m+1}(G, A)$ is finite, and there is a function β such that

$$|\gamma_{m+1}(G, A)| \leq \beta(k, m, t),$$

where $\beta(k, m, t)$ is defined inductively by

$$\beta(k, 1, t) = t^{k+\frac{1}{2}(\log_p t-1)}, \quad \beta(k, m+1, t) = (\beta(k, m, t))^{k+d(m+1)},$$

$$d(m+1) = \frac{1}{2}(\log_p \beta(k, m, t) - 1)$$

[7].

Dixon et al.'s Bound (2014): Let G be a group and let A be a subgroup of $Aut(G)$ such that $Inn(G) \leq A$ and $|A/Inn(G)| = k$ is finite. Let Z be the upper A-hypercenter of G. Suppose that $zl(G, A) = m$ is finite and G/Z is finite of order t. Then there exists a function β_1 such that $|\gamma_\infty(G, A)| \leq t^{k+\frac{1}{2}(\log_p t+1)} = \beta_1(k, t)$ [7].

Kurdachenko et al.'s Bound (2015): Let G be a group and let A be a subgroup of $Aut(G)$ such that $Inn(G) \leq A$ and $G/C_G(A)$ is finite of order t. If $A/Inn(G)$ has finite special rank r then $[G, A]$ is finite and there exists a function δ_3 such that $|[G, A]| \leq \delta_3(t, r)$ [29].

Taghavi et al.'s Bound (2021): Let G be a group such that $|G/Z_n(G, A)| = t$ for some n, where $Inn(G) \leq A \leq Aut(G)$. Suppose that $G/Z_n(G, A)$ and $A/Inn(G)$ are finitely generated and d and k are the minimal numbers of generators of them, respectively. Then $|\gamma_{n+1}(G, A)| \leq t^{k(k+d)^{n-1}+\frac{1}{2}(t'-1)(\frac{(k+d)^n-1}{d+k-1})}$ [23].

Taghavi et al.'s Bound (2021): Let G be a group with $zl(G, A) < \infty$, where $Inn(G) \leq A \leq Aut(G)$. Assume $A/Inn(G)$ has finite special rank k and $|G/Z| = t$ where Z is the A-hypercenter of G. Then $|\gamma_\infty(G, A)| \leq t^{k+\frac{1}{2}(t'+1)} = \beta_1(k, t)$ [23].

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Taghavi et al.'s Bound (2021): Let G be a group such that $G/Z_n(G,A)$, for some n, is generated by d elements and is finite of order $t = p_1^{r_1} \cdots p_s^{r_s}$. If $A/Inn(G)$ has finite special rank k, then

$$|\gamma_{n+1}(G,A)| \leq \min\{t^{k(k+d)^{n-1}+\frac{1}{2}(t'-1)(\frac{(k+d)^n-1}{d+k-1})}, t^{k+\frac{1}{2}(t'+1)} \prod_{i=1}^{s} p_i^{\chi_{n+1}(r_i+k)-\chi_{n+1}(k)}\}$$

[23].

Taghavi and Kayvanfar's Bound (2024): Let G be an A-nilpotent group such that $G/Z_n(G,A)$ is finite of order $p_1^{n_1} \cdots p_s^{n_s}$ and for $1 \leq i \leq s$ its Sylow p_i-subgroup is generated by r_i elements, where n_i's and r_i's are positive integers. Suppose that $A/Inn(G)$ is generated by d elements. If the Frattini subgroup $\Phi(G/Z_n(G,A))$ has order $p_1^{m_1} \cdots p_s^{m_s}$, m_i's are non-negative integers, then

$$|\gamma_{n+1}(G,A)| \leq \prod_{i=1}^{s} p_i^{\chi_{n+1}(r_i+d)-\chi_{n+1}(d)+m_i(r_i+d)^n},$$

where χ_{n+1} is introduced in [22].

Taghavi and Kayvanfar's Bound (2024): Let $|G/Z_n(G,A)| = t = {p_1}^{n_1} \cdots {p_s}^{n_s}$. Suppose that each Sylow p_i-subgroup of $G/Z_n(G,A)$ is generated by r_i elements and $A/Inn(G)$ has finite special rank d. If the Frattini subgroup of each Sylow p_i-subgroup of $G/Z_n(G,A)$ is of order $p_i^{m_i}$, then

$$|\gamma_{n+1}(G,A)| \leq \prod_{i=1}^{s} p_i^{\chi_{n+1}(r_i+d)-\chi_{n+1}(k)+m_i(r_i+d)^n} \beta_1(d,t)$$

[22].

3.3. **Baer's theorem and Extensions to Specific Classes of Groups.** Baer's theorem has been extended to various specific classes of groups, such as *solvable groups*, *polycyclic groups*, and *locally generalized radical groups* [17, 22, 23, 36]. These extensions help classify certain kinds of groups that do not necessarily fit the traditional constraints of Baer's original result. For instance, in the case of *solvable groups*, some authors have shown that if $G/Z_n(G)$ is finite or has finite rank, the derived length of G is bounded by a function of the parameters of $G/Z_n(G)$. Similar results hold for *polycyclic groups* and some other classes of groups where finiteness of terms in the central series provides important structural information (for instance see [28]).

These extensions refine the theorem's scope and applicability.

- **Finite Rank Conditions and Locally Generalized Radical Groups**

In this subsection some important results are given which show the connection between Baer‘s theorem and "finite rank conditions" and "locally generalized radical groups".

Makarenko (2000). If G is finite group and $G/Z_n(G)$ has finite special rank, then $\gamma_{n+1}(G)$ also has finite rank [36].

Kurdachenko and Shumyatsky (2013). Let G be a locally generalized radical group such that

$G/Z(G)$ has finite special rank r. Then the special rank of $\gamma_2(G)$ is bounded by a function depending on r [28].

Kurdachenko and Otal (2013). Let G be a locally generalized radical group and Z be the upper hypercenter of G. Suppose that $zl(G) = k$ is finite and G/Z has finite special rank r. Then $\gamma_{k+1}(G)$ has finite special rank and there exists a function $\tau_1(r,k)$ such that $r(\gamma_{k+1}(G)) \leq \tau_1(r,k)$, in which

$$\tau_1(r,1) = \kappa(r) = \frac{r(r+1)}{2} + r^2\iota(\log_2 r) + r^2, \quad \tau_1(r,2) = \kappa(\kappa(r)) + r\kappa(r),$$
$$\tau_1(r,k) = \kappa(\tau_1(r,k-1)) + r\tau_1(r,k-1),$$

where $\iota(\alpha)$ denotes the smallest integer not less than the real number α [32].

The above theorems illustrate that for the class of groups of finite special rank $\mathcal{FSR}$ with respect to $\mathfrak{N}_n$, we took a step forward for being Schur-Baer class, as the condition holds for finite groups and locally generalized radical groups. These also can be seen in theorems of Taghavi and Kayvanfar (2024) in this section. The same is somehow true for finite section rank, as it can be seen by Dixon et al.'s theroem as follows.
The next two statements will illustrate how the condition “section rank” provides Baer‘s theorem.

Ballester et al. (2013). There exists a function $\lambda_2(G)$ such that if G is a locally generalized radical group and C is a central subgroup such that G/C has section p-rank r, then

$$r_p(\gamma_2(G)) \leq \lambda_2(r) = 3r\rho(r)^{2r+2}(\rho(r)!) + \frac{5r^3 + 12r^2 + r}{2} + 2r^2\iota(\log_2 r)$$

[3].

Dixon et al. (2015). Let G be a locally generalized radical group and let p be a prime. Suppose that $G/Z_n(G)$ has finite section p-rank at most r. Then $\gamma_{n+1}(G)$ has finite section p-rank. Moreover, there exists a function $\tau(r,n)$ such that

$$r_p(\gamma_{n+1}(G)) \leq \tau(r,n) = \lambda_2(\tau(r,n-1)) + \theta_4(r,\tau(r,n-1))$$

[6].

Finally, in the following two propositions which connect “special rank” and Baer's theorem, the author in his joint paper [22] could improve the bound obtained by Kurdachenko et al. in [30].

Taghavi and Kayvanfar (2024). Let G be a locally generalized radical group such that $G/Z_n(G)$ has finite special rank r.

(1) If $n = 1$, then $r(\gamma_2(G)) \leq \kappa(r)$, where $\kappa(r)$ is introduced in the previous theorem of Kurdachenko and Otal (2013).
(2) Otherwise

$$r(\gamma_{n+1}(G)) \leq \min\{\chi_{n+1}(r) + ([\log_2(n-1)] + 1)r^{n+1} + \kappa(r), (\tfrac{r^n-1}{r-1})\kappa(r)\},$$

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where

$$\chi_{n+1}(r) = \frac{1}{n+1}\sum_{l|n+1}\mu(l)r^{\frac{n+1}{l}},$$

and $\mu(l)$ is the Mobious function and is defined as

$$\mu(l) = \begin{cases} 1 & \textit{if}\ l = 1, \\ 0 & \textit{if}\ l = p_1^{r_1}\cdots p_s^{r_s},\ \exists r_i > 1, \\ (-1)^s & \textit{if}\ l = p_1\cdots p_s, \end{cases}$$

[22].

The above theorem also can be generalized as follows.

Taghavi and Kayvanfar (2024). Let G be a locally generalized radical group such that $G/Z_n(G,A)$ and $A/Inn(G)$ have finite special ranks r and d, respectively.

(1) If $n = 1$, then $r(\gamma_2(G,A)) \le \kappa(r) + dr$, where $\kappa(r)$ is introduced in the previous theorem of Kurdachenko and Otal (2013).

(2) Otherwise

$$r(\gamma_{n+1}(G,A)) \le \min\{(\frac{(r+d)^n - 1}{(r+d)-1})\kappa(r) + (r+d)^{n-1}dr,$$
$$\chi_{n+1}(r+d) - \chi_{n+1}(d) + ([\log_2(n-1)] + 1)(r+d)^n r + \kappa(r)\},$$

where χ_{n+1} is introduced in the previous theorem of Taghavi and Kayvanfar (2024) [22]. Finitely generated groups

- **Finitely generated groups**

Donadze et al. illustrated a criterion in the class of finitely generated groups $\mathcal{FG}$ under which they are Schur-Baer with respect to the variety $\mathcal{N}_n$.

Donadze et al. (2021). If $G/Z_n(G)$ is finitely generated, then $\gamma_{n+1}(G)$ is finitely generated if and only if $\gamma_{n+1}(G/Z_n(G))$ is finitely generated [11].

- **Hypercenter**

Falco et al. (2011). If $G/Z_\infty(G)$ is finite group, then G has a finite normal subgroup L such that G/L is hypercentral, where $Z_\infty(G)$ is hypercenter of G [15].

As stated in Subsection 3.2, two years later, in 2013 Kurdachenko et al. [32] could find a bound for L.

3.4. **Baer's Theorem and connections to Varieties, Isoclinism, and Isologism.** According to the results obtained so far, Baer's theorem has been generalized to some classes of groups $\mathcal{X}$. In other words, we know that for the variety of nilpotent groups of class at most n, $\mathfrak{N}_n$, the following classes of groups are Schur-Baer:

- $\mathcal{T}$: the class of trivial groups,
- $\mathcal{F}$: the class of finite groups,
- $\mathcal{LF}$: the class of locally finite groups,
- $\mathcal{LF}_\pi$: the class of locally finite π-groups,
- $\mathcal{FE}$: the class of finite groups of finite exponent,
- $\mathcal{LFE}$: the class of locally finite groups of finite exponent,
- $\mathcal{LGR}_{FSR}$: the class of locally generalized radical groups of finite special rank,
- $\mathcal{LGR}_{fsr}$: the class of locally generalized radical groups of finite section rank,

and for the variety of abelian groups $\mathfrak{A}$, beside the above cases, the following classes of groups are Schur-Baer too:

- $\mathcal{PF}$: the class of polycyclic-by-finite groups,
- $\mathcal{C}$: the class of Chernikov groups,
- $\mathcal{SF}$: the class of solvable-by-finite groups,
- $\mathcal{SF}_{min}$: the class of solvable-by-finite minimax groups,
- $\mathcal{RLF}$: the class of residually locally finite groups,
- $\mathcal{FE}$: the class of finite exponent groups,

In addition to the above items, Hekster in 1989 proved that for a finitely based and locally residually finite variety $\mathfrak{V}$, the class of finite π-groups is Schur-Baer.

Hekster (1989). Let $\mathfrak{V}$ be a finitely based and locally residually finite variety. If $G/V^*(G) \in \mathcal{X}_\pi$, then $V(G) \in \mathcal{X}_\pi$, where $\mathcal{X}_\pi$ will denote a class of finite π-groups [19].

In the recent paper by Taghavi and Kayvanfar, Baer's theorem has been generalized for any arbitrary variety of groups, but for some specific classes. In other words, we could find more classes of groups which are Schur-Baer in the most general case, i.e. with respect to any variety of groups.

Taghavi and Kayvanfar (2024). Let $\mathfrak{V}$ be a variety of groups. If $G/V^*(G) \in \mathcal{X}$, then $V(G) \in \mathcal{X}$, whenever $\mathcal{X}$ is one of the following classes:

(i) $\mathcal{N}(\mathcal{S})$: the class of nilpotent (solvable) groups,

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(ii) $\mathcal{FN}(\mathcal{FS})$: the class of finite-by-nilpotent (finite-by-solvable) groups,

(iii) $\mathcal{NF}(\mathcal{NS})$: the class of nilpotent-by-finite (solvable-by-finite) groups,

(iv) $\mathcal{LN}(\mathcal{LS})$: the class of locally nilpotent (locally solvable) groups,

(v) $\mathcal{LFN}(\mathcal{LFS})$: the class of locally finite-by-nilpotent (locally finite-by-solvable) groups,

(vi) $\mathcal{LNF}(\mathcal{LSF})$: the class of locally nilpotent-by-finite (locally solvable-by-finite) groups, [24].

Two groups G and H are isoclinic if there exist isomorphisms between their central factors and commutator subgroups which are compatible together (for more details see [19]). Baer's theorem has been extended in this context to study isoclinic groups:

Theorem (Baer and Isoclinism): If G and H are isoclinic groups and $G/Z_n(G)$ is finite, then $\gamma_{n+1}(H)$ is finite [19].

4. Applications of Baer's Theorem

Baer's theorem has profound implications even beyond group theory. Its generalizations and refinements have influenced various areas of mathematics, including algebra, topology, Galois theory, and representation theory. Below, we outline some of its applications.

- **Algebra: Nilpotent and Solvable Groups**

Classification of Groups: Baer's theorem provides tools for analyzing the structure of nilpotent and solvable groups by relating their upper and lower central series. These results are critical in understanding the structure of groups and their role in the classification of finite and infinite groups [2, 28, 36].
Rank-Based Results: Extensions of Baer's theorem to groups with finite rank central factors have been used to classify groups such as Chernikov groups, polycyclic-by-finite groups, and groups of bounded rank [3, 34].

- **Representation Theory**

Schur Multipliers and cohomology: Baer's theorem is closely tied to Schur's theorem, which plays a significant role in the study of Schur multipliers. The Schur multiplier $M(G)$ of a group G is the second cohomology group $H^2(G, \mathbb{C}^\times)$. In representation theory, we know that projective representations correspond to central extensions of groups. In fact, central extensions influence the lifting of projective

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representations to linear representations. Schur's theorem ensures that the commutator subgroup G', which plays a role in these extensions, has constrained size when $G/Z(G)$ is finite [4, 25, 42].

Homology: It is also known that the Schur multiplier $M(G)$ of a group G is isomorphic to the second integral homology group $H_2(G, \mathbb{Z})$ when G is a finite group. Baer's theorem relates to $H_2(G, \mathbb{Z})$ through its impact on the structure of G and its commutator subgroups, which influence the second homology group. In fact, a smaller G' often implies a simpler or smaller $H_2(G, \mathbb{Z})$.

Extensions of Baer's theorem to groups with finite rank $G/Z_n(G)$ provide bounds on the rank of $H_2(G, \mathbb{Z})$, ensuring that the second homology group retains controlled structure [25, 35, 41]. **Extension Problems:** By bounding commutator subgroup sizes, Baer's theorem contributes to the study of group extensions, particularly in determining the structure of central extensions and their representations [6].

- **Topology and Homotopy**

Fundamental Groups: Baer's theorem has significant implications in topology and homotopy theory, particularly through its connections to fundamental groups, covering spaces, and homological invariants.

The fundamental group $\pi_1(X)$ of a topological space X is a primary tool in algebraic topology and Baer's theorem constrains the structure of $\pi_1(X)$. The second homology group of a space X relates directly to the group-theoretic second integral homology $H_2(\pi_1(X), \mathbb{Z})$. Baer's theorem impacts $H_2(X, \mathbb{Z})$ in the following ways. When $\pi_1(X)$ satisfies the conditions of Baer's theorem (e.g., $\pi_1(X)/Z_n(\pi_1(X))$ is finite), then $H_2(X, \mathbb{Z})$ is constrained. This affects the classification of spaces through their homology groups and influences the possible central extensions of $\pi_1(X)$. It is also applies to homotopy theory, since homotopy theory often involves central extensions of fundamental groups, which are classified by the second homology group $H_2(\pi_1(X), \mathbb{Z})$. [4, 14, 40].

- **Galois Theory**

Galois Groups: Extensions of Baer's theorem are used in analyzing the structure of Galois groups, particularly those arising in infinite Galois extensions. The theorem provides finiteness conditions for commutator subgroups, simplifying the study of Galois cohomology. Also, many Galois groups are residually finite. Baer's theorem can help analyze the interaction between residual finiteness and central series properties in such groups. [2, 37].

Profinite Groups: Baer's theorem applies to profinite groups, helping to study their central series and aiding in the classification of infinite Galois groups with finite generator properties [13, 37]. (A profinite group is a topological group, that is compact, totally disconnected and inverse limit of finite groups)

5. Open Problems and New Research Directions

Baer's theorem, which establishes the generalization of the finiteness of commutator subgroups based on the finiteness of central factor groups, continues to inspire research in group theory and related fields. Below, we outline some important open problems and potential research directions based on Baer's Theorem.

- **Finiteness Conditions Beyond Central Factors**

Problem: Can Baer's theorem be extended to groups where $G/Z_n(G)$ has bounded exponent but infinite rank?
Extending Baer's theorem to these cases could provide insights into the structure of infinite groups, particularly profinite or residually finite groups.

- **Infinite Groups with Finiteness Constraints**

Problem: Baer's theorem applies to locally finite and residually finite groups. What are the corresponding results for groups with torsion-free conditions?

Advances in this direction can help classify broader classes of infinite groups.

- **More Accurate Bounds for Verbal Subgroups**

As it is explained in Section 3, for the varieties of abelian groups and nilpotent groups, some authors obtained bounds for the verbal subgroups.

Problem: Can one obtain more accurate bounds for verbal subgroups?

- **New Varieties and Classes of Groups**

So far, we have known some varities of groups $\mathfrak{V}$ and some classes of groups $\mathcal{X}$ for which the generalization of Baer's theorem is true. Finally, again we are achieving to the main question that we have stated in this article:

Problem: For what variety $\mathfrak{V}$ and class of groups $\mathcal{X}$ Baer's theorem can be extended?

Advances in this area could refine classification techniques for groups.

These open problems and research directions illustrate the versatility of Baer's theorem and its potential for further exploration in group theory, algebra, topology and representation theory. With its foundational role and ongoing generalizations, Baer's theorem continues to inspire new lines of inquiry across mathematics.

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