Mathematical Methods in the Applied Sciences

WILEY

RESEARCH ARTICLE

The Lifespan of Solutions for a Boussinesq-Type Model

Faramarz Tahamtani[1] | Mohammad Shahrouzi[2]

[1]Department of Mathematics, School of Sciences, Shiraz University, Shiraz, Iran | [2]Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran

Correspondence: Faramarz Tahamtani (tahamtani@shirazu.ac.ir)

Received: 17 February 2025 | **Revised:** 30 April 2025 | **Accepted:** 4 May 2025

Keywords: blow-up | Boussinesq-type equation | decay | existence | logarithmic term

ABSTRACT

This paper investigates a class of Boussinesq-type equations in a bounded domain $\Omega \subset \mathbb{R}^n$ subject to clamped or hinged boundary conditions. The local existence of weak solutions is proven using the classical Faedo-Galerkin method in conjunction with the contraction mapping principle. Within the framework of potential wells, the global existence of solutions and energy decay are established when the solution resides in a subset that is smaller than the stable set. Also, it is shown that solutions exhibit exponential growth either when they belong to the unstable set or when the initial energy is negative. Furthermore, the blow-up of solutions is proven by combining the potential well framework with concavity arguments. For solutions with sufficiently positive initial energy, an upper bound for the blow-up time is derived, while a suitable functional is employed to establish a lower bound for the blow-up time.

MSC2020 Classification: 35A01, 35B35, 35B44

1 | Introduction

We deal with the following Boussinesq-type model

$$(1-\beta\Delta)u_{tt} + (1-\Delta)u + \gamma\Delta^2 u + u_t + \sigma(\nabla u) = 0, \quad x \in \Omega, t > 0, \tag{1}$$

where $\sigma(\nabla u) = div(\nabla u. \ln|\nabla u|^k)$, $\beta > 0, \gamma > 0, k \geq 1$ and $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. We consider two types of boundary conditions, namely, clamped boundary condition

$$u = \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, \tag{2}$$

where ν is the unit exterior normal to $\partial\Omega$, or hinged boundary condition

$$u = \Delta u = 0, \quad x \in \partial\Omega, t > 0. \tag{3}$$

For displacement $u(x,t)$, the initial conditions are given by

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega. \tag{4}$$

Over the last decades, Boussinesq-type equations have enjoyed increasing favor with in the coastal engineering community. The main reasons for this are the optimal blend of physical adequacy (i.e., their ability to represent all main physical phenomena) and computational ease (i.e., their mathematical well-possedness and numerical cheapness). Hence, Boussinesq-type equations beating the computation of nonlinear shallow-water equations have become the most favored approximations of Navier–Stokes equations for coastal-type computations. Beyond the many interesting reviews, see [1–3].

It is well-known that Boussinesq derived some model equations in 1870s, and such equations describe the propagation of small amplitude and long waves on the surface of shallow-water. Boussinesq [4] was the first to give the scientific explanation of the solidarity wave equation as follows:

$$u_{tt} - u_{xx} + \gamma u_{xxxx} = (u^2)_{xx}, \quad x \in \mathbb{R}, t > 0. \tag{5}$$

Equation (5) depends on the sign of γ; indeed, the case $\gamma < 0$ is called the "good" Boussinesq equation because it is linearly stable and governs small nonlinear transverse oscillations of an elastic beam (see [5]), while this equation with $\gamma > 0$ is called the "bad" Boussinesq equation since it possesses the linear instability. There are many researches concerned with the generalized Boussinesq equation: as follows:

$$u_{tt} - u_{xx} + u_{xxxx} = (f(u))_{xx}.$$

For example, Bona and Sachs [6] proved the local well-posedness and global existence of solutions for $f(u) = |u|^{p-1}u$ $(p > 1)$. When $f(u) = u^{p+1}$, Xue [7] investigated the local and global existence of solutions. Wang and Chen [8] studied the existence and uniqueness of the global solutions for the following Cauchy problem of the generalized double dispersion equation

$$u_{tt} - u_{xx} - u_{xxxt} + u_{xxxx} - \alpha u_{xxt} = (f(u))_{xx}, \quad x \in \mathbb{R}, t > 0;$$

also, under some suitable conditions, they established the blow-up result by using the concavity method.

The multidimensional Boussinesq equation is as follows:

$$u_{tt} - a\Delta u_{tt} + \Delta^2 u_{tt} + \Delta^2 u - \Delta u = \Delta f(u),$$

studied by Wang and Mu [9] for Cauchy problem. They proved the global existence and asymptotic behavior of the solutions provided that the initial data is suitably small. As the literature on the Boussinesq-type equation is too extensive, to mention all of them, we only partially refer the reader to [10–18].

In the actual process with so much literature on logarithmic nonlinearity, it has also received great attention of physicists and mathematicians. In fact, this type of nonlinearity can be applied in many branches of physics, such as optics, inflationary cosmology, geophysics, and nuclear physics (see [19–22]).

Liu [23] considered the following plate equation with nonlinear damping and a logarithmic source term:

$$u_{tt} + \Delta^2 u + |u_t|^{m-2}u_t = |u|^{p-2}u \log |u|^k,$$

and by applying the contraction mapping principle, the author established the local existence of solutions. Furthermore, the global existence and decay estimates of the solution were derived under the condition of subcritical initial energy. Additionally, it is proven that solutions with negative initial energy exhibit finite-time blow-up under appropriate conditions. Moreover, for the case of linear damping (i.e., $m = 2$), the finite-time blow-up of solutions is demonstrated for arbitrarily high initial energy levels. In another study, Shao et al. [24] analyzed the following plate equation, which includes weak damping and logarithmic nonlinearity:

$$\phi_{tt} + \Delta^2\phi + \phi + \phi_t = k\phi \ln |\phi|.$$

The study confirmed that the solution experiences blow-up at infinity for subcritical and critical initial energy levels. Additionally, the authors established that solutions with arbitrarily high initial energy exhibit infinite-time blow-up. Al-Gharabli et al. [25] employed the Galerkin method and the multiplier method to establish the existence of solutions and derive an explicit and general decay rate for the energy of solutions to the following plate equation:

$$u_{tt} + \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 u(s)ds = ku \ln |u|.$$

In their study, Pang et al. [26] investigated the well-posedness of solutions for a fourth-order wave equation incorporating strong and weak damping terms, as well as a logarithmic strain term. The equation under consideration is expressed as follows:

$$u_{tt} + \alpha\Delta^2 u - \beta\Delta u + \sum_{i=0}^{n} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) - \Delta u_t + |u_t|^{r-1}u_t = |u|^{q-1}u,$$

where $(x,t) \in \Omega \times [0,T)$ and $\sigma_i(u_{x_i}) = u_{x_i} \ln |u_{x_i}|^p$. Utilizing the Galerkin method in conjunction with the contraction mapping principle, the authors demonstrated the existence of a local solution. Furthermore, they established the existence of global solutions and infinite-time blow-up solutions under subcritical initial energy conditions. These findings were subsequently extended to scenarios involving critical initial energy. Additionally, the study proved the occurrence of infinite-time blow-up for solutions at arbitrary positive initial energy levels. In a recent study, Li and Fang [27] examined a fourth-order damped wave equation featuring logarithmic nonlinearity, which is formulated as follows:

$$u_{tt} + \Delta^2 u + \alpha\Delta u - \omega\Delta u_t + \beta u_t = u \log |u|^\gamma.$$

By employing a combination of the potential well method and a modified differential inequality technique, the authors established the infinite-time blow-up of solutions under arbitrary initial energy conditions. Notably, they demonstrated that the initial velocity and initial displacement need not share the same sign in the context of the L^2-inner product. Specifically, they proved that solutions may still blow-up at infinity even when $\int_\Omega u_0 u_1 dx < 0$.

For a comprehensive understanding of equations featuring logarithmic nonlinearity, we refer the reader to the studies documented in [28–31] and references therein.

Regarding logarithmic Boussinesq-type wave equations, for Gaussian solitary wave solutions, Wazwaz [32] considered the following:

$$u_{tt} - u_{xx} + u_{xxxx} + (u \log |u|^k)_{xx} = 0, \quad k > 1. \tag{6}$$

By using $u = v_x$, Hu et. al [33] investigated (6) in the following one-dimensional problem:

$$\begin{aligned}
&v_{tt} - v_{xx} + v_{xxxx} + (v_x \log |v_x|^k)_x = 0, && x \in (0,l), t > 0,\\
&v(0,t) = v(l,t) = 0, \quad v_x(0,t) = v_x(l,t) = 0, && t > 0,\\
&v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), && x \in (0,L).
\end{aligned}$$

The authors established the existence of global solutions by using Galerkin method combined with potential-well method and proved that solutions grow up in L^2-norm. In a recent work, Ding and Zhou [34] considered the following problem:

$$\begin{aligned}
&u_{tt} - \beta\Delta u_{tt} - \Delta u + \Delta^2 u + \gamma\Delta^2 u_t - \alpha\Delta u_t + \Delta(u\log|u|) = 0, && x \in \Omega, t > 0,\\
&u = \Delta u = 0, && x \in \partial\Omega, t > 0,\\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), && x \in \Omega.
\end{aligned}$$

By using Galerkin method combined with potential-well method, the authors established the local and global existence of solutions. They also proved infinite time blow-up with subcritical, critical, and high initial energy of solutions.

Motivated by the aforementioned studies, the main purpose of this paper is to deal with the well-posedness and lifespan of solutions for problems (1)–(4). To the best of our knowledge, no existing work has addressed the well-posedness and lifespan for this specific problem. Following the establishment of the local and global existence of solutions, we demonstrate that these solutions exhibit exponential decay. Furthermore, in Theorem 4, we prove that the solutions of problems (1–4) are unstable. Specifically, we show that for small positive initial energy ($E(0) > 0$) or $E(0) < 0$, the solutions grow exponentially in the $H^2(\Omega) \cap H_0^1(\Omega)$-norm as time goes to infinity. While the approach is inspired by the ideas in [26, 27, 33], the proofs presented here are different. Additionally, we establish that the solutions to the problem described by Equations (1.1–1.4) can be extended over the entire time half-line, thereby extending the existence time of the solutions to infinity. Thus, we extend the existence time of solutions to infinity. However, in the problems discussed in [17, 32, 33], the authors did not clarify whether the maximum existence time of solutions can also be extended to infinity. Finally, under appropriate conditions on the initial data, we prove that there exists a finite time (with lower and upper bound) that the solutions blow-up at this time. Hence, our work represents an extension and generalization of many earlier results in this area.

The rest of the paper is structured as follows: Section 2 introduces the necessary notations and preliminaries, including a family of potential wells relevant to our study. In Section 3, we establish the local existence and uniqueness of solutions. Section 4 extends these results to demonstrate the global existence of solutions. Section 5 examines the decay properties of the solutions and establishes their exponential growth for problems (1–4). Finally, Section 6 addresses the blow-up of solutions within the framework of potential wells, complemented by concavity arguments. In this section, upper and lower bounds for the blow-up time are also derived.

2 | Preliminaries and Notations

In this section, we present some notations and preliminary results, which will be used throughout of this work. We use the standard Lebesgue space L^2-inner product $(\cdot,\cdot)$. We denote the Sobolev spaces $H_0^1(\Omega)$, $H_0^2(\Omega)$, and $H^2(\Omega)$ with the following norms:

$$\|(\cdot)\|_{H_0^1(\Omega)} = \sqrt{\|(\cdot)\|_2^2 + \|\nabla(\cdot)\|_2^2},$$
$$\|(\cdot)\|_{H_0^2(\Omega)} = \sqrt{\|(\cdot)\|_2^2 + \|\Delta(\cdot)\|_2^2},$$
$$\|(\cdot)\|_{H^2(\Omega)} = \sqrt{\|(\cdot)\|_2^2 + \|\nabla(\cdot)\|_2^2 + \|\Delta(\cdot)\|_2^2},$$

respectively. By $\langle\cdot,\cdot\rangle$, we represent the duality pairing between $H^2(\Omega)$ and $H^{-2}(\Omega)$.

From problems (1–4), we introduce the following new Sobolev spaces $H^* = H_0^1(\Omega)$ which is given a norm for $\beta > 0$, $\|(\cdot)\|_{H^*} := \sqrt{\|(\cdot)\|_2^2 + \beta\|\nabla(\cdot)\|_2^2}$, $H^{**} = H_0^2(\Omega)$ for boundary condition (2) and $H^{**} = H^2(\Omega) \cap H_0^1(\Omega)$ for boundary condition (3) which is given a norm for $\gamma > 0$, $\|(\cdot)\|_{H^{**}} := \sqrt{\|(\cdot)\|_2^2 + \gamma\|\Delta(\cdot)\|_2^2}$. By Poincaré inequality, it is clear that the norms $\|(\cdot)\|_{H^*}$ and $\|(\cdot)\|_{H^{**}}$ are equivalent to the norms of the spaces $H_0^1(\Omega), H_0^2(\Omega)$ and $H^2(\Omega)$.

For the initial boundary value problems (1–4), we define the modified energy, potential energy, and Nehari functionals sequentially

$$E(t) = \frac{1}{2}\|u_t\|_{H^*}^2 + \frac{1}{2}\|u\|_{H^{**}}^2 - \frac{k}{2}\int_\Omega |\nabla u|^2 \ln|\nabla u| dx + \frac{k+2}{4}\|\nabla u\|_2^2, \tag{7}$$

$$J(u(t)) = \frac{1}{2}\|u\|_{H^{**}}^2 - \frac{k}{2}\int_\Omega |\nabla u|^2 \ln|\nabla u| dx + \frac{k+2}{4}\|\nabla u\|_2^2, \tag{8}$$

$$I(u(t)) = \|u\|_{H^{**}}^2 - k\int_\Omega |\nabla u|^2 \ln|\nabla u| dx. \tag{9}$$

From (7) to (9), we can easily see that

$$J(u(t)) = \frac{k+2}{4}\|\nabla u\|_2^2 + \frac{1}{2}I(u(t)), \tag{10}$$

$$E(t) = \frac{1}{2}\|u_t\|_{H^*}^2 + J(u(t)). \tag{11}$$

According to Nehari functional, we introduce the potential-well (stable set)

$$\mathcal{N}^+ = \{u \in H^{**}; I(u(t)) > 0\} \cup \{0\}, \tag{12}$$

the complement of potential-well (unstable set)

$$\mathcal{N}^- = \{u \in H^{**}; I(u(t)) < 0\}, \tag{13}$$

and Nehari manifold

$$\mathcal{N} = \{u \in H^{**}; I(u(t)) = 0\}. \tag{14}$$

The depth of potential-well d (the so called mountain pass level) is defined by the following:

$$d := \inf_{u\in\mathcal{N}} J(u(t)). \tag{15}$$

Meanwhile, we give some properties of the aforementioned manifolds and functionals as follows.

Lemma 1. ([35, 36]). *Let $a > 0$ be a constant and $\Omega \subset \mathbb{R}^n$ be a domain (bounded or unbounded). Then there holds*

$$n(1+\ln a)\|\chi\|_2^2 + 2\int_\Omega |\chi|^2 \ln\left(\frac{|\chi|}{\|\chi\|_2^2}\right)dx \le \frac{a^2}{\pi}\|\nabla\chi\|_2^2, \quad (16)$$

for any $\chi \in H_0^1(\Omega)$.

Lemma 2. *The potential-well depth satisfies the following:*

$$d \ge \frac{k+2}{4}\ell := \frac{k+2}{4}\left(\frac{2\pi\gamma}{k}\right)^{\frac{n}{4}} e^{\frac{n}{2}}. \quad (17)$$

Proof. From (14) and (15), $I(u(t)) = 0$, that is,

$$\|u\|_{H^{**}}^2 - k\int_\Omega |\nabla u|^2 \ln|\nabla u|dx \ge \gamma\|\Delta u\|_2^2 - k\int_\Omega |\nabla u|^2 \ln|\nabla u|dx.$$

By exploiting the logarithmic Sobolev inequality (16) in the above inequality, we get the following:

$$\begin{aligned} 0 = I(u(t)) &\ge \left(\gamma - \frac{a^2}{2\pi}\right)\|\Delta u\|_2^2 \\ &\quad + k\left(\frac{n(1+\ln a)}{2} - \ln\|\nabla u\|_2^2\right)\|\nabla u\|_2^2. \end{aligned}$$

By taking $a = \sqrt{\frac{2\pi\gamma}{k}}$ and $\frac{n(1+\ln\sqrt{\frac{2\pi\gamma}{k}})}{2} - \ln\|\nabla u\|_2^2 < 0$, we get

$$\|\nabla u\|_2^2 > \ell. \quad (18)$$

From (10) and (15), it follows that

$$d \ge \frac{k+2}{4}\|\nabla u\|_2^2 + \frac{1}{2}I(u(t)). \quad (19)$$

Combining (18) and (19) with $I(u(t)) = 0$, prove the result (17). □

Further in (17), we may need to ensure that the equality holds; that is, there exists an extremal of the variational problem (15). For proof and more details, see [37, 38].

Lemma 3. *For $u \in H^{**}$, the following statements hold:*

i. *If $0 \le \|\nabla u\|_2^2 < \ell$, then $I(u(t)) > 0,\ t > 0$,*
ii. *If $I(u(t)) \le 0$, then $\|\nabla u\|_2^2 > \ell,\ t > 0$,*

where ℓ defined in (17).

Proof. (i) From Nehari functional (9) and logarithmic Sobolev inequality (16), we get for any $a > 0$

$$\begin{aligned} &I(u(t)) \\ &= \|u\|_{H^{**}}^2 - k\int_\Omega |\nabla u|^2 \ln\left(\frac{|\nabla u|}{\|\nabla u\|_2^2}\right)dx - k\|\nabla u\|_2^2 \ln\|\nabla u\|_2^2 \\ &\ge \left(1 - \frac{ka^2}{2\pi\gamma}\right)\|u\|_{H^{**}}^2 + k\left(\frac{n(1+\ln a)}{2} - \ln\|\nabla u\|_2^2\right)\|\nabla u\|_2^2. \end{aligned} \quad (20)$$

Taking $a = \sqrt{\frac{2\pi\gamma}{k}}$ in (20), we obtain

$$I(u(t)) \ge k\left(\frac{n(1+\ln a)}{2} - \ln\|\nabla u\|_2^2\right)\|\nabla u\|_2^2.$$

Now, if $0 \le \|\nabla u\|_2^2 < \left(\frac{2\pi\gamma}{k}\right)^{\frac{n}{4}} e^{\frac{n}{2}} = \ell$ implies $I(u(t)) > 0$.

(ii) In (20), if $I(u(t)) < 0$, we find from $n\left(1 + \sqrt{\frac{2\pi\gamma}{k}}\right) - 2\ln\|\nabla u\|_2^2 < 0$ that means $\|\nabla u\|_2^2 > \ell$. □

Lemma 4. ([39]). *Assume that $\psi(t) \in C^2([0,T])$ is a positive function satisfying the following inequality:*

$$\psi\psi'' - \tilde{\alpha}(\psi')^2 + \tilde{\gamma}\psi'\psi + \tilde{\beta}\psi \ge 0, \qquad \tilde{\alpha} > 1, \tilde{\beta} \ge 0, \tilde{\gamma} \ge 0,$$

and $\psi(0) > 0$. If

$$\psi'(0) > \frac{\tilde{\gamma}}{\tilde{\alpha}-1}\psi(0),$$
$$\left(\psi'(0) - \frac{\tilde{\gamma}}{\tilde{\alpha}-1}\psi(0)\right)^2 > \frac{2\tilde{\beta}}{2\tilde{\alpha}-1}\psi(0).$$

Then

$$\psi(t) \to +\infty \qquad \text{as} \qquad t \to T^* \le \psi^{1-\tilde{\alpha}}(0)A^{-1},$$

$$A^2 \equiv (\tilde{\alpha}-1)^2\psi^{-2\tilde{\alpha}}(0)\left[\left(\psi'(0) - \frac{\tilde{\gamma}}{\tilde{\alpha}-1}\psi(0)\right)^2 - \frac{2\tilde{\beta}}{2\tilde{\alpha}-1}\psi(0)\right].$$

Lemma 5. *Let u be a solution of problems (1–4), then the energy functional (7) is nonincreasing with respect to t, that is,*

$$E'(t) = -\|u_t\|_2^2 < 0,\ t > 0. \quad (21)$$

Proof. The functional $E(t)$ comes from multiplying Equation (1) with u_t and integrating on $(0,t)$, that is,

$$E(t) + \int_0^t \|u_\tau\|_2^2 d\tau = E(0),$$

which proves $E(t) \le E(0)$ for $t \ge 0$, where $E(0) = \frac{1}{2}\|u_1\|_2^2 + J(u_0)$. □

Definition 1. For $T > 0$, a function

$$u \in C([0,T]; H^{**}) \cap C^1([0,T]; H^*) \cap C^2([0,T]; H^{-2}(\Omega))$$

is called a weak solution of problems (1–4) if $u(x,0)=u_0(x)$ in H^{**}, $u_t(x,0)=u_1(x)$ in H^* for $t\in[0,T]$; the equality

$$\begin{aligned}\langle u_{tt},\Phi\rangle+\beta(\nabla u_{tt},\nabla\Phi)+(u,\Phi)+(\nabla u,\nabla\Phi)+\gamma(\Delta u,\Delta\Phi)\\ +(u_t,\Phi)=k(\nabla u\ln|\nabla u|,\nabla\Phi)\end{aligned}, \quad (22)$$

holds for any $\Phi\in H^{**}$.

Furthermore, the solution u can be extended to $[0,T_{max})$ where T_{max} is the maximal existence of $T\in(0,T_{max})$. For $T_{max}=+\infty$, the solution is global and $T_{max}<+\infty$; the solution blows up in a finite time.

For the term $(\nabla u\ln|\nabla u|,\nabla\Phi)$ in (22), it follows from

$$\begin{aligned}\|\nabla u\ln|\nabla u|\|_2^2=&\int_{\{x\in\Omega;|\nabla u|<1\}}(\nabla u\ln|\nabla u|)^2dx\\ &+\int_{\{x\in\Omega;|\nabla u|\geq1\}}(\nabla u\ln|\nabla u|)^2dx.\end{aligned} \quad (23)$$

For the first integral in the right-hand side of (23), we need a minimum of a function $f(s)=s\ln|s|$. According to Fermat's theorem, one has $\min\limits_{0<s<1}f(s)=-\frac{1}{e}$. Therefore, a simple computation and for sufficiently small $\delta>0$, (23) implies

$$\begin{aligned}\|\nabla u\ln|\nabla u|\|_2^2&\leq e^{-2}|\Omega|+\delta^{-2}\int_{\{x\in\Omega;|\nabla u|\geq1\}}|\nabla u|^{2+2\delta}dx\\ &\leq e^{-2}|\Omega|+\delta^{-2}\|\nabla u\|_{2+2\delta}^{2+2\delta}.\end{aligned} \quad (24)$$

Clearly, for $\delta>0$, there holds $H^{**}\hookrightarrow L^{2+2\delta}(\Omega)$, which together with (23) and $u\in H^{**}$ for $t\in[0,T]$ implies that $(\nabla u\ln|\nabla u|,\nabla\Phi)$ is well-defined for all $t\in[0,T]$.

3 | Local Existence

In this section, we use the Faedo–Galerkin approximation method and contraction mapping principle to show the local existence and uniqueness of weak solution for problems (1–4). Here, we will adapt the idea used in [18].

First, we define the space

$$\mathcal{H}=C([0,T];H^{**})\cap C^1([0,T];H^*) \quad (25)$$

equipped with the norm

$$\|v\|_{\mathcal{H}}^2=\max_{0\leq t\leq T}(\|v_t\|_{H^*}^2+\|v\|_{H^{**}}^2) \quad (26)$$

and establish the following Lemma.

Lemma 6. *For $(u_0,u_1)\in H^{**}\times H^*$ and $v\in\mathcal{H}$, there exists a unique solution $u\in\mathcal{H}\cap C^2([0,T];H^{-2}(\Omega))$ and $u_t\in L^2([0,T];H^*)$ that solves*

$$(1-\beta\Delta)u_{tt}+(1-\Delta)u+\gamma\Delta^2u+u_t+\sigma(\nabla v)=0,\quad x\in\Omega,t>0, \quad (27)$$

$$u=\frac{\partial u}{\partial \nu}=0,\ \text{or}\ u=\Delta u=0,\qquad x\in\partial\Omega,t>0, \quad (28)$$

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\qquad x\in\Omega. \quad (29)$$

Proof. **Existence**. For any $m\geq1$, let $W_m=\text{span}\{w_1,w_2,\cdots,w_m\}$ where $\{w_j\}$ is the complete orthogonal system of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$ and $\|w_j\|=1$ for $j=1,2,\cdots,m$. That is, $-\Delta w_j=\lambda_jw_j$ where λ_j is the related eigenvalues in $H_0^1(\Omega)$, respectively.

We search for an approximation solution

$$u^m(x,t)=\sum_{j=1}^m h_j^m(t)w_j(x) \quad (30)$$

of the approximation problem in W_m:

$$\begin{aligned}(1-\beta\Delta)(u_{tt}^m,w_j)+(1-\Delta)(u^m,w_j)+\gamma(\Delta^2u^m,w_j)+(u_t^m,w_j)\\ +(\sigma(\nabla v),w_j)=0,\end{aligned} \quad (31)$$

$$u^m(x,0)=\sum_{j=1}^m a_jw_j\to u_0\ \text{in}\ H^{**} \quad (32)$$

$$u_t^m(x,0)=\sum_{j=1}^m b_jw_j\to u_1\ \text{in}\ H^* \quad (33)$$

as $m\to+\infty$. Inserting (30) into (31–33) gives

$$\begin{aligned}(1+\beta\lambda_j)h_j^{\prime\prime m}(t)+h_j^{\prime m}(t)\\ +(1+\lambda_j+\gamma\lambda_j^2)h_j^m(t)=(\nabla v\ln|\nabla v|^k,\nabla w_j),\end{aligned} \quad (34)$$

$$h_j^m(0)=a_j,\quad h_j^{\prime m}(0)=b_j \quad (35)$$

Based on the theory of ordinary differential equations, Peano's theorem, for each m, there exists $t_m>0$ such that problems (34) and (35) admits a global and unique solution $h_j^m\in C[0,t_m]$. We show that $t_m=T\in(0,T_{max})$ and the local solution is uniformly bounded independent of m and t. Multiplying (31) by $h_j^{\prime m}(t)$ and sum from $j=0$ to m gives

$$\frac{d}{dt}(\|u_t^m\|_{H^*}^2+\|u^m\|_{H^{**}}^2)=-2\|u_t^m\|_2^2+2k\int_\Omega\nabla v\ln|\nabla v|\nabla u_t^m dx,$$

and integrating on $(0,t)$, implies

$$\begin{aligned}\|u_t^m\|_{H^*}^2+\|u^m\|_{H^{**}}^2\leq\|u_1^m\|_{H^*}^2+\|u_0^m\|_{H^{**}}^2\\ +2k\int_0^t\int_\Omega\nabla v\ln|\nabla v|\nabla u_\tau^m dxd\tau.\end{aligned} \quad (36)$$

To estimate the integral term on the right-hand side of (36), it follows from (24) and Hölder's and Young's inequalities for sufficiently small $\delta>0$,

$$\int_0^t\int_\Omega \nabla v \ln|\nabla v| \nabla u_\tau^m \, dx d\tau$$

$$\leq \int_0^t \|\nabla v \ln|\nabla v|\|_2^2 \|\nabla u_\tau^m\|_2^2 d\tau$$

$$\leq \int_0^t \left(\|\nabla v \ln|\nabla v|\|_2^{\frac{2}{2+\delta}} \|\nabla u_\tau^m\|_2^{\frac{2}{2+\delta}} \right)^{\frac{2+\delta}{2}} d\tau$$

$$\leq \int_0^t \left(\frac{1+\delta}{2+\delta} \|\nabla v \ln|\nabla v|\|_2^{\frac{2}{1+\delta}} + \frac{1}{2+\delta} \|\nabla u_\tau^m\|_2^2 \right)^{\frac{2+\delta}{2}} d\tau$$

$$\leq \int_0^t \left[\frac{1+\delta}{2+\delta} e^{-\frac{2}{1+\delta}} \left(|\Omega|^{\frac{1}{1+\delta}} + \|\nabla v\|_{2+2\delta}^{2+2\delta} \right) + \frac{1}{2+\delta} \|\nabla u_\tau^m\|_2^2 \right]^{\frac{2+\delta}{2}} d\tau$$

$$\leq \int_0^t \left[\frac{1+\delta}{2+\delta} e^{-\frac{2}{1+\delta}} \left(|\Omega|^{\frac{1}{1+\delta}} + C_* \|v\|_{\mathcal{H}}^2 \right) + \frac{1}{2+\delta} \|\nabla u_\tau^m\|_2^2 \right]^{\frac{2+\delta}{2}} d\tau$$

$$\leq C_{**} t + \frac{2^{\frac{\delta}{2}}}{2+\delta} \int_0^t \left(\|u_\tau^m\|_{H^*}^2 + \|u^m\|_{H^{**}}^2 \right)^{\frac{2+\delta}{2}} d\tau, \tag{37}$$

where

$$C_{**} = \frac{2^{\frac{\delta}{2}}}{2+\delta} \left[\frac{1+\delta}{2+\delta} e^{-\frac{2}{1+\delta}} \left(|\Omega|^{\frac{1}{1+\delta}} + C_* \|v\|_{\mathcal{H}}^2 \right) \right]^{\frac{2+\delta}{2+2\delta}}.$$

Here, we used the fact that $v \in \mathcal{H}$ and C_* is the optimal embedding constant of $\mathcal{H} \to L^{2+2\delta}(\Omega)$.

Substituting (37) in (36) gives

$$\mathcal{K}_1(t) \leq \mathcal{K}_1(0) + C_{**} t + \frac{2^{\frac{\delta}{2}}}{2+\delta} \int_0^t \mathcal{K}_1^{\frac{2+\delta}{2}}(\tau) d\tau \tag{38}$$

which we let

$$\mathcal{K}_1(t) = \frac{1}{2k} \left(\|u_t^m\|_{H^*}^2 + \|u^m\|_{H^{**}}^2 \right),$$
$$\mathcal{K}_1(0) = \frac{1}{2k} \left(\|u_1^m\|_{H^*}^2 + \|u_0^m\|_{H^{**}}^2 \right).$$

At this point, we are going to obtain an upper bound for $\mathcal{K}_1(t)$ that shows the behavior of $t \in [0, T]$. Define

$$\mathcal{K}_2(t) = \mathcal{K}_1(0) + C_{**} t + \frac{2^{\frac{\delta}{2}}}{2+\delta} \int_0^t \mathcal{K}_1^{\frac{2+\delta}{2}}(\tau) d\tau. \tag{39}$$

Thus, $\mathcal{K}_1(t) = \left[\frac{2+\delta}{2^{\frac{\delta}{2}}} (\mathcal{K}_2'(t) - C_{**}) \right]^{\frac{2}{2+\delta}}$ and $\mathcal{K}_1(t) \leq \mathcal{K}_2(t)$ implies

$$\mathcal{K}_2'(t) \leq \frac{2^{\frac{\delta}{2}}}{2+\delta} \mathcal{K}_2^{\frac{2+\delta}{2}}(t) + C_{**}$$
$$\leq \frac{2^{\frac{\delta}{2}}}{2+\delta} \left[\mathcal{K}_2(t) + \left(\frac{2+\delta}{2^{\frac{\delta}{2}}} C_{**} \right)^{\frac{2}{2+\delta}} \right]^{\frac{2+\delta}{2}},$$

$$\mathcal{K}_1(0) = \mathcal{K}_2(0).$$

Letting, $\mathcal{K}_3(t) = \mathcal{K}_2(t) + \left(\frac{2+\delta}{2^{\frac{\delta}{2}}} C_{**} \right)^{\frac{2}{2+\delta}}$ in the above inequality and by simple calculation, that is, it is easy to get

$$\mathcal{K}_1(t) \leq \mathcal{K}_2(t) \leq \mathcal{K}_3(t) \leq \left(\mathcal{K}_3^{-\frac{\delta}{2}}(0) - \frac{\delta 2^{\frac{\delta}{2}-1}}{2+\delta} t \right)^{-\frac{2}{\delta}}, \tag{40}$$

where $\mathcal{K}_3(0) = \mathcal{K}_1(0) + \left(\frac{2+\delta}{2^{\frac{\delta}{2}}} C_{**} \right)^{\frac{2}{2+\delta}}$. Clearly, the right-hand side of (40) blow-up when $t \to \frac{2+\delta}{\delta 2^{\frac{\delta}{2}-1}} \mathcal{K}_3^{-\frac{\delta}{2}}(0)$, but for $T = \frac{2+\delta}{\delta 2^{\frac{\delta}{2}}} \mathcal{K}_3^{-\frac{\delta}{2}}(0)$, we have

$$\|u_t^m\|_{H^*}^2 + \|u^m\|_{H^{**}}^2 \leq k e^{\frac{2+\delta}{\delta}} \mathcal{K}_3(0).$$

Consequently, for large m, this implies

$$\sup_{t \in [0, t_m]} \left(\|u_t^m\|_{H^*}^2 + \|u^m\|_{H^{**}}^2 \right) \leq k e^{\frac{2+\delta}{\delta}} \mathcal{K}_3(0). \tag{41}$$

Hence, the approximation solution (30) is uniformly bounded in m on the interval $[0, T]$ and $h_j^m(t)$ can be extended in $[0, T]$. Combining this fact and problems (34) and (35) admits a unique global solution $h_j^m \in C^2[0, T]$. Also, in view of (41), it is easy to see that $\|u_{tt}^m\|_{H^{-2}(\Omega)}$ be finite for all $t \in [0, T]$. Therefore,

$$\{u^m\} \text{ is bounded in } L^\infty([0, T]; H^{**}),$$
$$\{u_t^m\} \text{ is bounded in } L^\infty([0, T]; H^*),$$
$$\{u_{tt}^m\} \text{ is bounded in } L^2([0, T]; H^{-2}(\Omega)).$$

Uniqueness. Let u_1 and u_2 be strong solutions of problems (27–29). The equation

$$\begin{aligned} &(1 - \beta\Delta)(u_1 - u_2)_{tt} + (1 - \Delta)(u_1 - u_2) + \gamma \Delta^2 (u_1 - u_2) \\ &\quad + (u_1 - u_2)_t = 0, \quad x \in \Omega, t > 0 \end{aligned},$$

shares homogenous initial and boundary conditions. By (36), we get

$$\|(u_1 - u_2)_t\|_{H^*}^2 + \|u_1 - u_2\|_{H^{**}}^2 \leq 0,$$

that implies $u_1 = u_2$ in $\mathcal{H}$. □

Next, based on Lemma 6 and contraction mapping principle, we are going to prove that problems (1–4) exist uniquely local weak solution.

Theorem 1. *For $(u_0, u_1) \in H^{**} \times H^*$, problems (1–4) admit a unique solution u in $[0, T]$ for some $T > 0$.*

Proof. For any $T > 0$, we consider the space

$$\mathcal{M}_T = \{u \in \mathcal{H}; \|u\|_{\mathcal{H}} \leq M\}.$$

Define a map $\mathcal{S} : \mathcal{M}_T \to \mathcal{M}_T$. Lemma 6 shows that problems (27–29) exist a unique solution $u = \mathcal{S}(v)$ for $v \in \mathcal{M}_T$. We show that $\mathcal{S}(\mathcal{M}_T) \subseteq \mathcal{M}_T$, while $T > 0$ small enough; that is, if $\|u\|_{\mathcal{H}} \leq M$, then $\|\mathcal{S}(v)\|_{\mathcal{H}} \leq M$.

Multiplying (27) by u_t and integrating over $\Omega \times (0,t)$, we obtain

$$\|u_t\|_{H^*}^2 + \|u\|_{H^{**}}^2 \le \|u_1\|_{H^*}^2 + \|u_0\|_{H^{**}}^2 + 2k\int_0^t\int_\Omega \nabla v \ln|\nabla v| \nabla u_\tau dx d\tau, \tag{42}$$

where $u = \mathcal{S}(v)$ is the corresponding solution to problems (27–29) for fix $v \in \mathcal{M}_T$. By similar arguments in (37), it follows from (42)

$$\|u_t\|_{H^*}^2 + \|u\|_{H^{**}}^2 \le \|u_1\|_{H^*}^2 + \|u_0\|_{H^{**}}^2 + C_M T + \frac{2^{\frac{\delta}{2}}}{2+\delta}\int_0^t \left(\|u_\tau\|_{H^*}^2 + \|u\|_{H^{**}}^2\right)^{\frac{2+\delta}{2}} d\tau, \tag{43}$$

where $C_M = \frac{2^{\frac{\delta}{2}}}{2+\delta}\left[\frac{1+\delta}{2+\delta}e^{-\frac{2}{1+\delta}}\left(|\Omega|^{\frac{1}{1+\delta}} + C_* M^2\right)\right]^{\frac{2+\delta}{2+2\delta}}$. According to (38–40), it follows from (43)

$$\|u_t\|_{H^*}^2 + \|u\|_{H^{**}}^2 \le \|u_1\|_{H^*}^2 + \|u_0\|_{H^{**}}^2 + \left[C_M + \frac{2^{\frac{\delta}{2}}}{2+\delta}\left(k2^{\frac{2+\delta}{2}}\mathcal{K}_3(0)\right)^{\frac{2+\delta}{2}}\right]T.$$

Choose $T > 0$ sufficiently small, $M > 0$ large enough so that

$$\|u_1\|_{H^*}^2 + \|u_0\|_{H^{**}}^2 \le \frac{M^2}{2},$$

$$\left[C_M + \frac{2^{\frac{\delta}{2}}}{2+\delta}\left(k2^{\frac{2+\delta}{2}}\mathcal{K}_3(0)\right)^{\frac{2+\delta}{2}}\right]T \le \frac{M^2}{2},$$

we get

$$\|u_t\|_{H^*}^2 + \|u\|_{H^{**}}^2 \le M^2,$$

that implies $\|u\|_{\mathcal{H}} \le M$, and so $\mathcal{S}(\mathcal{M}_T) \subseteq \mathcal{M}_T$.

Now, we are in a position to show for $v_1, v_2 \in M_T$ and $u_1 = \mathcal{S}(v_1), u_2 = \mathcal{S}(v_2)$, $\|u_1 - u_2\|_{\mathcal{H}} \le \bar{k}\|v_1 - v_2\|_{\mathcal{H}}$ for $\bar{k} \in (0,1)$. To do this, taking $z = u_1 - u_2$ solves

$$(1-\beta\Delta)z_{tt} + (1-\Delta)z + \gamma\Delta^2 z + z_t + (\sigma(\nabla v_1) - \sigma(\nabla v_2)) = 0, \ x \in \Omega, t > 0, \tag{44}$$

$$z = \frac{\partial z}{\partial \nu} = 0 \quad \text{or} \quad z = \Delta z = 0, \ x \in \partial\Omega, t > 0, \tag{45}$$

$$z(x,0) = z_t(x,0) = 0, \ x \in \Omega. \tag{46}$$

Testing both sides of (44) by z_t over $\Omega \times (0,t)$ gives

$$\begin{aligned} &\|z_t\|_{H^*}^2 + \|z\|_{H^{**}}^2 \\ &= -2k\int_0^t \|z_\tau\|_2^2 d\tau \\ &\quad + 2k\int_0^t\int_\Omega div(\nabla v_1 \ln|\nabla v_1| - \nabla v_2 \ln|\nabla v_2|) z_\tau dx d\tau \\ &\le 2k\int_0^t\int_\Omega |\nabla v_1 \ln|\nabla v_1| - \nabla v_2 \ln|\nabla v_2|| \nabla z_\tau dx d\tau. \end{aligned} \tag{47}$$

We make use of the Lagrange mean value theorem $\frac{\nabla v_1 \ln|\nabla v_1| - \nabla v_2 \ln|\nabla v_2|}{\nabla v_1 - \nabla v_2} = \ln e|\xi|$, where for any $\theta \in (0,1)$

$$|\xi| = |(\theta\nabla v_1 + (1-\theta)\nabla v_2)| \le |\nabla v_1| + |\nabla v_2|.$$

Thus, from (47), we have

$$\|z_t\|_{H^*}^2 + \|z\|_{H^{**}}^2 \le Q_1 + Q_2, \tag{48}$$

where

$$Q_1 := 2k\int_0^t\int_\Omega |\nabla v_1 - \nabla v_2| \nabla z_\tau dx d\tau,$$

$$Q_2 := 2k\int_0^t\int_\Omega \ln(|\nabla v_1| + |\nabla v_2|)|\nabla v_1 - \nabla v_2| \nabla z_\tau dx d\tau.$$

In view of (24) and according to Cauchy's, Hölder's, and Sobolev's inequalities, we have

$$\begin{aligned} Q_1 &\le 2k\int_0^t \|\nabla v_1 - \nabla v_2\|_2 \|\nabla z_\tau\|_2 d\tau \\ &\le k\int_0^t \|\nabla v_1 - \nabla v_2\|_2^2 d\tau + k\int_0^t \|\nabla z_\tau\|_2^2 d\tau \\ &\le k\lambda\gamma^{-1}\int_0^t \gamma\|\Delta v_1 - \Delta v_2\|_2^2 d\tau + k\int_0^t (\|z_\tau\|_2^2 + \|\nabla z_\tau\|_2^2) d\tau \\ &\le k\lambda\gamma^{-1}\int_0^t \|v_1 - v_2\|_{H^{**}}^2 d\tau + k\int_0^t \|z_\tau\|_{H^*}^2 d\tau \\ &\le k\lambda\gamma^{-1}T\|v_1 - v_2\|_{\mathcal{H}}^2 + k\int_0^t (\|z_\tau\|_{H^*}^2 + \|z\|_{H^{**}}^2) d\tau, \end{aligned} \tag{49}$$

$$\begin{aligned} Q_2 &\le 2k\int_0^t \left(\|\ln(|\nabla v_1| + |\nabla v_2|)\|_n \|\nabla v_1 - \nabla v_2\|_{\frac{2n}{n-2}} \|\nabla z_\tau\|_2\right) d\tau \\ &\le k\int_0^t \left(\|\ln(|\nabla v_1| + |\nabla v_2|)\|_n \|\nabla v_1 - \nabla v_2\|_{\frac{2n}{n-2}}\right)^2 d\tau \\ &\quad + k\int_0^t \|\nabla z_\tau\|_2^2 d\tau \\ &\le kC(\lambda,\gamma)\tilde{C}\int_0^t \|\ln(|\Delta v_1| + |\Delta v_2|)\|_n^2 \|\gamma(\Delta v_1 - \Delta v_2)\|_{\frac{2n}{n-2}}^2 d\tau \\ &\quad + k\int_0^t (\|z_\tau\|_2^2 + \|\nabla z_\tau\|_2^2) d\tau \\ &\le kC(\lambda,\gamma)\tilde{C}\tilde{\tilde{C}}T\|v_1 - v_2\|_{\mathcal{H}}^2 + k\int_0^t (\|z_\tau\|_{H^*}^2 + \|z\|_{H^{**}}^2) d\tau, \end{aligned} \tag{50}$$

where $C(\lambda, \gamma)$ depends on $\gamma > 0$ and Poincaré constant λ, $\tilde{C}$ is embedding constant and for small $\delta > 0$

$$\tilde{\tilde{C}} = \left(\frac{|\Omega|}{e^n} + 2C_{n\delta}^{n\delta}(e\delta)^{-n}M^{n\delta} \right)^{\frac{2}{n}}.$$

Combining (48–50), we arrive at

$$\|z_t\|_{H^*}^2 + \|z\|_{H^{**}}^2 \leq k(\lambda\gamma^{-1} + C(\lambda,\gamma)\tilde{C}\tilde{\tilde{C}})T\|v_1 - v_2\|_{\mathcal{H}}^2 + k\int_0^t (\|z_\tau\|_{H^*}^2 + \|z\|_{H^{**}}^2)d\tau,$$

then the Gronwall inequality yields

$$\|z_t\|_{H^*}^2 + \|z\|_{H^{**}}^2 \leq k(\lambda\gamma^{-1} + C(\lambda,\gamma)\tilde{C}\tilde{\tilde{C}})Te^{kT}\|v_1 - v_2\|_{\mathcal{H}}^2.$$

Choose T sufficiently small and $\overline{k} = k(\lambda\gamma^{-1} + C(\lambda,\gamma)\tilde{C}\tilde{\tilde{C}})Te^{kT} < 1$ implies that $\mathcal{S}$ is a contractive map in $\mathcal{M}_T$. Thus, by contraction mapping principle, we conclude that problems (1–4) admit uniquely solution. □

4 | Global Existence

In this section, we are concerned with the existence of global weak solution to problems (1–4). In fact,

$$u \in C([0, T_{max}); H^{**}) \cap C^1([0, T_{max}); H^*) \cap C^2([0, T_{max}); H^{-2}(\Omega))$$

holds for any $T \in (0, T_{max})$ and $T_{max} = +\infty$ gives the existence of global weak solution. That is, the local solution of problems (1–4) can be continued in time and the lifespan of solution will be $[0, +\infty)$.

Lemma 7. *Let $u = u(t)$ be a weak solution of problems (1–4).*

i. *For $(u_0, u_1) \in \mathcal{N}^+ \times H^*$, then $u = u(t) \in \mathcal{N}^+$ for all $t \in [0, T_{max})$, where $\mathcal{N}^+$ is defined in (12).*
ii. *For $(u_0, u_1) \in \mathcal{N}^- \times H^*$, then $u = u(t) \in \mathcal{N}^-$ for all $t \in [0, T_{max})$, where $\mathcal{N}^-$ is defined in (13).*

Proof. Because the proofs of (i) and (ii) are similar, we only give the proof of (i). Since $u = u(t)$ is a weak solution of problems (1–4), so $u \in C([0, T_{max}); H^{**})$ implies $I(u(t)) \in C[0, T_{max})$. Arguing by contradiction, suppose there exists $t_0 \in [0, T_{max})$ such that $u(t_0) \in \mathcal{N}$ (see (14)) where t_0 is the first time. For sufficiently $\varepsilon_0 > 0$, there exists $t_{\varepsilon_0} > 0$ such that $u(t_{\varepsilon_0}) \in \mathcal{N}^-$ for $0 < t_0 < t_{\varepsilon_0}$ and

$$I(u(t_{\varepsilon_0})) = -2\varepsilon_0 < 0. \tag{51}$$

From $J(u(t))$ (see (10)), we have

$$J(u(t_{\varepsilon_0})) = \frac{k+2}{4}\|\nabla u(t_{\varepsilon_0})\|_2^2 + \frac{1}{2}I(u(t_{\varepsilon_0})). \tag{52}$$

By recalling Lemma 5 together with (11) and (51), it follows from (52) that

$$\begin{aligned} \|\nabla u(t_{\varepsilon_0})\|_2^2 &\leq \frac{4}{k+2}\left(J(u(t_{\varepsilon_0})) + \varepsilon_0\right) \\ &\leq \frac{4}{k+2}\left(E(t_{\varepsilon_0}) + \varepsilon_0\right) \\ &\leq \frac{4}{k+2}(E(0) + \varepsilon_0). \end{aligned} \tag{53}$$

By $I(u(t))$ (see (9)) and using logarithmic Sobolev inequality (16), we obtain for any $a > 0$

$$\begin{aligned} I(u(t_{\varepsilon_0})) &= \|u(t_{\varepsilon_0})\|_{H^{**}}^2 - k\int_\Omega |\nabla u(t_{\varepsilon_0})|^2 \ln|\nabla u(t_{\varepsilon_0})|dx \\ &\geq k\left(\frac{n(1+\ln a)}{2} - \ln\|\nabla u(t_{\varepsilon_0})\|_2^2\right)\|\nabla u(t_{\varepsilon_0})\|_2^2 \\ &\quad + \left(1 - \frac{ka^2}{2\pi\gamma}\right)\|u(t_{\varepsilon_0})\|_{H^{**}}^2. \end{aligned}$$

Taking $a = \sqrt{\frac{2\pi\gamma}{k}}$ and (53) in the above inequality, we get

$$I(u(t_{\varepsilon_0})) \geq k\left[\frac{n\left(1 + \ln\sqrt{\frac{2\pi\gamma}{k}}\right)}{2} - \ln\frac{4}{k+2}(E(0) + \varepsilon_0)\right]\times \|\nabla u(t_{\varepsilon_0})\|_2^2.$$

For $(u_0, u_1) \in \mathcal{N}^+ \times H^*$, satisfying $\frac{4}{k+2}\left(\frac{2\pi\gamma}{k}\right)^{-\frac{n}{4}}e^{-\frac{n}{2}}(E(0) + \varepsilon_0) < 1$ gives $I(u(t_{\varepsilon_0})) > 0$ contradict with (51). Therefore, $u = u(t) \in \mathcal{N}^+$ for all $t \in [0, T_{max})$. Further,

$$E(0) < E(0) + \varepsilon_0 < \frac{k+2}{4}\left(\frac{2\pi\gamma}{k}\right)^{\frac{n}{4}}e^{\frac{n}{2}} = \frac{k+2}{4}\ell < d,$$

where d is defined in (15). □

Theorem 2. *For $(u_0, u_1) \in \mathcal{N}^+ \times H^*$, the local weak solution u got in Theorem 1 exists globally.*

Proof. Let $u = u(t)$ for $t \in [0, T_{max})$ be the local weak solution of problems (1–4). To show $T_{max} = +\infty$, it is enough to find a constant $C > 0$ such that

$$\sup_{t\in[0,T_{max})}\left(\|u_t\|_{H^*}^2 + \|u\|_{H^{**}}^2\right) \leq C. \tag{54}$$

By virtue of Lemma 5, (10), (11), and Lemma 7, we obtain the following:

$$\begin{aligned} E(0) &\geq \frac{1}{2}\|u_t\|_{H^*}^2 + \frac{k+2}{4}\|\nabla u\|_2^2 + \frac{1}{2}I(u(t)) \\ &\geq \frac{1}{2}\|u_t\|_{H^*}^2 + \frac{k+2}{4}\lambda^{-1}\|u\|_2^2 \\ &\geq \frac{1}{2}\|u_t\|_{H^*}^2 + \frac{k+2}{4}\lambda^{-1}\|u\|_{H^{**}}^2 \\ &\geq \min\left\{\frac{1}{2}, \frac{k+2}{4}\lambda^{-1}\right\}\left(\|u_t\|_{H^*}^2 + \|u\|_{H^{**}}^2\right). \end{aligned}$$

This shows that (54) holds, where λ is the Poincaré constant and

$$C = \left(\min\{\frac{1}{2}, \frac{k+2}{4}\lambda^{-1}\}\right)^{-1}E(0).$$

□

5 | Decay and Growth Estimates of Solutions

5.1 | Decay of Solution

This subsection is devoted to the study of the stability of solution to problems (1–4) emanating from initial data satisfying the conditions required by that for the global solutions in Theorem 2. So to prove this result, we consider $L(t) : [0, T_{max}) \to \mathbb{R}^+$ defined by

$$L(t) = E(t) + \varepsilon \int_\Omega uu_t dx + \varepsilon\beta \int_\Omega \nabla u \nabla u_t dx + \frac{\varepsilon}{2}\|u\|_2^2, \ \varepsilon > 0. \tag{55}$$

It is easy to see that $L(t)$ and $E(t)$ are equivalent in the sense that there exist two positive constants α_1 and α_2 such that

$$\alpha_1 E(t) \le L(t) \le \alpha_2 E(t), \ t \ge 0. \tag{56}$$

Theorem 3. *Let $(u_0, u_1) \in \mathcal{N}^+ \times H^*$. Further assume*

$$0 < E(0) < \alpha\ell < d, \tag{57}$$

where α is a positive constant satisfying $0 < \left(\frac{4}{k+2}\right)^{\frac{2}{n}} \alpha^{\frac{2}{n}} \frac{2}{\sqrt{k(k+2)}} < 1$. Then, there exist two positive constants c_1 and c_2 such that

$$0 < E(t) \le c_1 e^{-c_2 t}, \quad \forall t \ge 0. \tag{58}$$

Proof. Taking the derivative of (55) with respect to t together with Equation (1), we get the following:

$$\begin{aligned} L'(t) = {} & E'(t) + \varepsilon\|u_t\|_2^2 + \varepsilon\beta\|\nabla u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 - \varepsilon\|u\|_{H^{**}}^2 \\ & + \varepsilon k \int_\Omega |\nabla u|^2 \ln|\nabla u| dx. \end{aligned} \tag{59}$$

Utilizing the energy functional $E(t)$ (see (7)), (21), and for any $N > 0$, it follows from (59)

$$\begin{aligned} L'(t) = {} & -\varepsilon N E(t) - \left(1 - \varepsilon - \varepsilon\frac{N}{2}\right)\|u_t\|_2^2 + \varepsilon\beta\left(1 + \frac{N}{2}\right)\|\nabla u_t\|_2^2 \\ & - \varepsilon\left(1 - \frac{N}{2}\right)\|u\|_{H^{**}}^2 - \varepsilon\left(1 - \frac{k+2}{2}\frac{N}{2}\right)\|\nabla u\|_2^2 \\ & + \varepsilon k\left(1 - \frac{N}{2}\right)\int_\Omega |\nabla u|^2 \ln|\nabla u| dx. \end{aligned}$$

According to logarithmic Sobolev inequality (16), the above equality implies

$$\begin{aligned} L'(t) \le {} & -\varepsilon N E(t) - \left(1 - \varepsilon - \varepsilon\frac{N}{2}\right)\|u_t\|_2^2 + \varepsilon\beta\left(1 + \frac{N}{2}\right)\|\nabla u_t\|_2^2 \\ & - \varepsilon\left(1 - \frac{N}{2}\right)\|u\|_{H^{**}}^2 - \varepsilon\left(1 - \frac{k+2}{2}\frac{N}{2}\right)\|\nabla u\|_2^2 \\ & + \varepsilon N\left(1 - \frac{N}{2}\right)\frac{a^2}{2\pi}\|\Delta u\|_2^2 \\ & + \varepsilon k\left(1 - \frac{N}{2}\right)\left(\ln\|\nabla u\|_2^2 - \frac{n(1+\ln a)}{2}\right)\|\nabla u\|_2^2. \end{aligned} \tag{60}$$

Choose the appropriate $N = \frac{4}{k+2}$ and picking $\beta \le \frac{N}{2(N+2)}$, it follows from (60) that

$$\begin{aligned} L'(t) \le {} & -\frac{4\varepsilon}{k+2}E(t) - \left(1 - \varepsilon - \frac{2\varepsilon}{k+2}\right)\|u_t\|_2^2 + \frac{\varepsilon}{k+2}\|\nabla u_t\|_2^2 \\ & - \frac{\varepsilon k}{k+2}\|u\|_{H^{**}}^2 + \frac{\varepsilon k}{k+2}\frac{2a^2}{\pi(k+2)}\|\Delta u\|_2^2 \\ & + \frac{\varepsilon k^2}{k+2}\left(\ln\|\nabla u\|_2^2 - \frac{n(1+\ln a)}{2}\right)\|\nabla u\|_2^2. \end{aligned} \tag{61}$$

Substituting the inequalities $\frac{\varepsilon}{k+2}\|\nabla u_t\|_2^2 \le \frac{2\varepsilon}{k+2}E(t)$ and $\|\Delta u\|_2^2 \le \gamma^{-1}\|u\|_{H^{**}}^2$ in (61). By using (10), (11), and the fact that $u \in \mathcal{N}^+$ (see Lemma 7 (i)), we obtain the following:

$$\begin{aligned} L'(t) \le {} & -\frac{2\varepsilon}{k+2}E(t) - \left(1 - \varepsilon - \frac{2\varepsilon}{k+2}\right)\|u_t\|_2^2 \\ & - \frac{\varepsilon k}{k+2}\left(1 - \frac{2a^2}{\pi\gamma(k+2)}\right)\|u\|_{H^{**}}^2 \\ & + \frac{\varepsilon k^2}{k+2}\left(\ln\|\nabla u\|_2^2 - \frac{n(1+\ln a)}{2}\right)\|\nabla u\|_2^2. \end{aligned} \tag{62}$$

To this end, from (10) and assumption (57), we conclude the following inequalities:

$$\begin{aligned} \|\nabla u\|_2^2 & \le \frac{4}{k+2}E(t) \le \frac{4}{k+2}E(0) \le \frac{4}{k+2}\alpha\ell \\ & = \frac{4}{k+2}\alpha\left(\frac{2\pi\gamma}{k}\right)^{\frac{n}{4}} e^{\frac{n}{2}}, \end{aligned}$$

or

$$\ln\|\nabla u\|_2^2 \le \ln\frac{4}{k+2}\alpha\left(\frac{2\pi\gamma}{k}\right)^{\frac{n}{4}} e^{\frac{n}{2}},$$

and by taking a satisfying $\left(\frac{4}{k+2}\right)^{\frac{2}{n}}\alpha^{\frac{2}{n}}\sqrt{\frac{2\pi\gamma}{k}} < a < \sqrt{\frac{\pi\gamma(k+2)}{2}}$ where α an be assured by assumption (57), we guarantee

$$L'(t) \le -\frac{2\varepsilon}{k+2}E(t) - \left(1 - \varepsilon - \frac{2\varepsilon}{k+2}\right)\|u_t\|_2^2. \tag{63}$$

Now, choosing $\varepsilon > 0$ sufficiently small such that $1 - \varepsilon - \frac{2\varepsilon}{k+2} > 0$, inequality (63) becomes

$$L'(t) \le -\frac{2\varepsilon}{k+2}E(t), \forall t > 0. \tag{64}$$

Further, by virtue of (56), let $c_2 = \frac{2\varepsilon}{k+2}$ then (64) becomes $\frac{dL(t)}{dt} \le -c_2 L(t)$ for all $t \in [0, T_{max})$. Through Gronwall inequality, we get the result (58), where $c_1 = \frac{\alpha_2}{\alpha_1}E(0)$. □

5.2 | Growth of Solution

In this subsection, we investigate that the solutions associated with problems (1–4) are unstable. In fact, we show the solution growth exponentially as t approaches to infinity for positive initial energy $E(0)$ or $E(0) < 0$.

Theorem 4. *Let $u = u(t)$ be a weak solution of problems (1–4).*

i. *For $(u_0, u_1) \in \mathcal{N}^- \times H^*$, $0 < E(0) < \frac{k+2}{4}\ell < d$, and $\|\nabla u\|_2^2 > \tilde{\alpha}\ell$ $(\tilde{\alpha} > 1)$, then $u = u(t)$ growth-up in H^{**}-norm.*

ii. *The solution growth-up in H^{**}-norm provided $E(0) < 0$.*

Proof. (i) For any $\varepsilon > 0$, let us define

$$\tilde{L}(t) = G(t) + \varepsilon \int_{\Omega} u u_t dx + \varepsilon\beta \int_{\Omega} \nabla u \nabla u_t dx + \frac{\varepsilon}{2}\|u\|_2^2. \tag{65}$$

Set

$$G(t) = E_1 - E(t), \tag{66}$$

where $E(0) < E_1 < \frac{k+2}{4}\ell$. From (21) and (66), we have $G'(t) = -E'(t) = \|u_t\|_2^2 > 0$ for all $t \geq 0$. According to Hölder's and Poincaré's inequalities, we can deduce the following from (65):

$$\begin{aligned} \tilde{L}(t) &\leq G(t) + \frac{\varepsilon}{2}\|u_t\|_{H^*}^2 + \varepsilon\|u\|_2^2 + \frac{\varepsilon}{2}\beta\lambda\|\Delta u\|_2^2 \\ &\leq \mu_1(\varepsilon)\left(G(t) + \frac{1}{2}\|u_t\|_{H^*}^2 + \frac{1}{2}\|u\|_{H^{**}}^2\right), \end{aligned} \tag{67}$$

where λ is the Poincaré constant and $\mu_1(\varepsilon) = \max\{1, 2\varepsilon, \varepsilon\beta\lambda\gamma^{-1}\}$.

Taking the time derivative of the functional $\tilde{L}(t)$ together with the Equation (1), we get the following:

$$\begin{aligned} \tilde{L}'(t) &= G'(t) + \varepsilon\|u_t\|_2^2 + \varepsilon\int_{\Omega} u u_{tt} dx + \varepsilon\beta\|\nabla u_t\|_2^2 \\ &\quad + \varepsilon\beta\int_{\Omega} \nabla u \nabla u_{tt} dx + \varepsilon\int_{\Omega} u u_t dx \\ &= (1+\varepsilon)\|u_t\|_2^2 + \varepsilon\beta\|\nabla u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 - \varepsilon\|u\|_{H^{**}}^2 \\ &\quad + \varepsilon k\int_{\Omega} |\nabla u|^2 \ln|\nabla u| dx. \end{aligned} \tag{68}$$

By (7), (66), and for any positive constant $\tilde{N}$, it follows from (68) that

$$\begin{aligned} \tilde{L}'(t) &= \varepsilon\tilde{N}G(t) - \varepsilon\tilde{N}E_1 + \left(1 + \varepsilon + \varepsilon\frac{\tilde{N}}{2}\right)\|u_t\|_2^2 \\ &\quad + \varepsilon\beta\left(\frac{\tilde{N}}{2} + 1\right)\|\nabla u_t\|_2^2 \\ &\quad - \varepsilon\|\nabla u\|_2^2 + \varepsilon\left(\frac{\tilde{N}}{2} - 1\right)\|u\|_{H^{**}}^2 + \varepsilon\frac{k+2}{2}\frac{\tilde{N}}{2}\|\nabla u\|_2^2 \\ &\quad - \varepsilon k\left(\frac{\tilde{N}}{2} - 1\right)\int_{\Omega} |\nabla u|^2 \ln|\nabla u| dx. \end{aligned} \tag{69}$$

Since by Lemma 7 (ii) (i.e., $u \in \mathcal{N}^-$ and $I(u(t)) < 0$). Noticing $E_1 < \frac{k+2}{4}\ell < \frac{k+2}{4\tilde{\alpha}}\|\nabla u\|_2^2 < \frac{k+2}{4}\|\nabla u\|_2^2$, combining it with (69), we have the following:

$$\begin{aligned} \tilde{L}'(t) &\geq \varepsilon\tilde{N}G(t) + 2\left(1 + \varepsilon + \varepsilon\frac{\tilde{N}}{2}\right)\frac{1}{2}\|u_t\|_2^2 \\ &\quad + 2\varepsilon\beta\left(\frac{\tilde{N}}{2} + 1\right)\frac{1}{2}\|\nabla u_t\|_2^2 \\ &\quad - \varepsilon\|\nabla u\|_2^2 + \varepsilon\left(\frac{\tilde{N}}{2} - 1\right)\|u\|_{H^{**}}^2 \\ &\quad - \varepsilon k\left(\frac{\tilde{N}}{2} - 1\right)\int_{\Omega} |\nabla u|^2 \ln|\nabla u| dx. \end{aligned} \tag{70}$$

By virtue of logarithmic Sobbolev inequality (16), it follows from (70) that

$$\begin{aligned} \tilde{L}'(t) &\geq \varepsilon\tilde{N}G(t) + 2\left(1 + \varepsilon + \varepsilon\frac{\tilde{N}}{2}\right)\frac{1}{2}\|u_t\|_2^2 \\ &\quad + 2\varepsilon\beta\left(\frac{\tilde{N}}{2} + 1\right)\frac{1}{2}\|\nabla u_t\|_2^2 \\ &\quad + \varepsilon\left[k\left(\frac{\tilde{N}}{2} - 1\right)\frac{n(1+\ln a)}{2} - k\left(\frac{\tilde{N}}{2} - 1\right)\ln\|\nabla u\|_2^2 - 1\right] \\ &\quad \|\nabla u\|_2^2 \\ &\quad + \varepsilon\left(\frac{\tilde{N}}{2} - 1\right)\left(1 - \frac{ka^2}{2\pi\gamma}\right)\|u\|_{H^{**}}^2. \end{aligned} \tag{71}$$

Since $\|\nabla u\|_2^2 > \tilde{\alpha}\ell = \tilde{\alpha}\left(\frac{2\pi\gamma}{k}\right)^{\frac{n}{4}} e^{\frac{n}{2}}$, take $a = \sqrt{\frac{\pi\gamma}{k}}$ with $\tilde{\alpha}$ satisfying $1 \leq \tilde{\alpha}^k < e\left(\frac{1}{2}\right)^{\frac{nk}{4}}$, choosing $\tilde{N} > 4$, and then $k\left(\frac{n\left(1+\ln\sqrt{\frac{\pi\gamma}{k}}\right)}{2} - \ln\|\nabla u\|_2^2\right) - 1 > 0$. Thus, (71) becomes

$$\begin{aligned} \tilde{L}'(t) &\geq 4\varepsilon G(t) + 2(1+3\varepsilon)\frac{1}{2}\|u_t\|_2^2 + 6\varepsilon\beta\frac{1}{2}\|\nabla u_t\|_2^2 + \frac{\varepsilon}{2}\|u\|_{H^{**}}^2 \\ &\geq 4\varepsilon G(t) + 6\varepsilon\frac{1}{2}\|u_t\|_{H^*}^2 + \frac{\varepsilon}{2}\|u\|_{H^{**}}^2 \\ &\geq \varepsilon\left(G(t) + \frac{1}{2}\|u_t\|_{H^*}^2 + \frac{1}{2}\|u\|_{H^{**}}^2\right). \end{aligned} \tag{72}$$

Combining (67) and (72) gives $\frac{d\tilde{L}}{dt} \geq \hat{\mu}\tilde{L}$, ($\hat{\mu} = \varepsilon^{-1}\mu_1(\varepsilon)$), then integrating on $(0, t)$ yields

$$\tilde{L}(t) \geq \tilde{L}(0)e^{\hat{\mu}t}, \quad \forall t \in [0, T_{max}) \tag{73}$$

where $\tilde{L}(0) = E_1 - E(0) + \varepsilon(u_0, u_1) + \varepsilon\beta(\nabla u_0, \nabla u_1) + \frac{\varepsilon}{2}\|u_0\|_2^2 > 0$.

On the other hand, by (7), (66), (67), $\|\nabla u\|_2^2 > \tilde{\alpha}\ell \geq \ell$ (see Lemma 3 (ii)) and the fact that $G(t) > G(0) > 0$, we have the following:

$$\begin{aligned} 0 &< \tilde{L}(0)e^{\mu t} \leq \tilde{L}(t) \\ &\leq \mu_1(\varepsilon)\left(G(t) + \frac{1}{2}\|u_t\|_{H^*}^2 + \frac{1}{2}\|u\|_{H^{**}}^2\right) \\ &= \mu_1(\varepsilon)\left(E_1 - E(t) + \frac{1}{2}\|u_t\|_{H^*}^2 + \frac{1}{2}\|u\|_{H^{**}}^2\right) \\ &\leq \mu_1(\varepsilon)\left(\frac{k+2}{4}\ell - \frac{k+2}{4}\|\nabla u\|_2^2 + \frac{k}{2}\int_{\Omega} |\nabla u|^2 \ln|\nabla u| dx\right) \\ &\leq \frac{k}{2}\mu_1(\varepsilon)\int_{\Omega} |\nabla u|^2 \ln|\nabla u| dx. \end{aligned} \tag{74}$$

By virtue of Sobolev embedding inequality and from the fact that $s^{-\delta}\ln s < (e\delta)^{-1}$ for any $s \geq 1$, $\delta > 0$, it follows from (74) that

$$\begin{aligned} \tilde{L}(0)e^{\hat{\mu}t} \leq \tilde{L}(t) &\leq \frac{k}{2e\delta}\int_{\Omega} |\nabla u|^{2+\delta} dx \leq B(\gamma\|\Delta u\|_2^2)^{\frac{2+\delta}{2}} \\ &\leq B(\|u\|_{H^{**}}^2)^{\frac{2+\delta}{2}}, \end{aligned} \tag{75}$$

where $B = \frac{k}{2e\delta}C_{2+\delta}^{2+\delta}\mu_1(\varepsilon)(\gamma\lambda)^{\frac{2+\delta}{2}}$ (λ is Poincaré constant, $C_{2+\delta}$ is the optimal Sobolev embedding constant), and arbitrary constant

δ satisfies $2 < 2+\delta < \begin{cases} +\infty, n=1,2 \\ \frac{2n}{n-2}, \quad n\geq 3 \end{cases}$ gives $\lim_{t\to+\infty} \|u\|^2_{H^{**}} = +\infty$ that implies the desired conclusion.

(ii) In (i), we showed that the solution of problems (1–4) grows as an exponential function in the H^{**}-norm (i.e., the weak solution u can extend over time, the whole half line). The technique of proof (ii) is similar provided the initial energy $E(0) < 0$ with some necessary modifications. Let us define (65) by setting $G(t) = -E(t)$, then

$$\begin{aligned}\tilde{L}(t) &= \varepsilon\tilde{N}G(t) + (1+\varepsilon+\varepsilon\frac{\tilde{N}}{2})\|u_t\|_2^2 + \varepsilon\beta\left(\frac{\tilde{N}}{2}+1\right)\|\nabla u_t\|_2^2 \\ &+ \varepsilon\left(\frac{\tilde{N}}{2}-1\right)\|u\|^2_{H^{**}} + \varepsilon\frac{k+2}{2}\frac{\tilde{N}}{2}\|\nabla u\|_2^2 \\ &- \varepsilon k\left(\frac{\tilde{N}}{2}-1\right)\int_\Omega |\nabla u|^2 \ln|\nabla u|dx.\end{aligned}$$

Since $E(t) \leq E(0) < 0$, so

$$\int_\Omega |\nabla u|^2 \ln|\nabla u|dx > \frac{1}{k}\|u_t\|^2_{H^*} + \frac{1}{k}\|u\|^2_{H^{**}} + \frac{k+2}{2k}\|\nabla u\|_2^2 > 0,$$

and $0 < G(t) = -E(t) < \int_\Omega |\nabla u|^2 \ln|\nabla u|dx$. Then, for a constant $\check{C}$,

$$\tilde{L}(t) \geq \check{C}\left(G(t) + \|u_t\|^2_{H^*} + \|u\|^2_{H^{**}}\right).$$

Thus, the proof of the opposite estimate is the same as in (i); we omit the details. □

6 | Blow-Up of Solution

This section aims to prove the blow-up of solutions for problems (1–4) with positive initial energy. First, by providing the sufficient assumptions, we prove the blow-up of solutions with appropriate positive initial energy and obtain the upper bound for the blow-up time. Next, by introducing an appropriate functional, we obtain a lower bound for the blow-up time.

Theorem 5. *Let the conditions of Theorem 4 (part (i)) hold. If we choose $\varepsilon > 0$ small enough such that*

$$(u_0, u_1) + \beta(\nabla u_0, \nabla u_1) - 8\varepsilon\|u_0\|^2_{H^*} > 0,$$

then there exists a finite time $T^* < T_{max}$ such that the solutions blow-up at T^*, that is,

$$\lim_{t\to T^{*-}} \|u\|^2_{H^*} = +\infty,$$

with the following upper bound

$$T^* < \frac{2(\|u_0\|_2^2 + \beta\|\nabla u_0\|_2^2)^5}{\sqrt[4]{\|u_0\|^2_{H^*}((u_0,u_1)+\beta(\nabla u_0,\nabla u_1) - 8\varepsilon\|u_0\|^2_{H^*})}}.$$

Proof. Let us define the following auxiliary functional with its derivatives as follows:

$$\psi(t) = \|u\|^2_{H^*}, \tag{76}$$

$$\psi'(t) = 2(u,u_t) + 2\beta(\nabla u, \nabla u_t), \tag{77}$$

$$\begin{aligned}\psi''(t) &= 2\|u_t\|^2_{H^*} - 2\|u\|^2_{H^{**}} - 2\|\nabla u\|_2^2 - 2(u,u_t) \\ &+ 2k\int_\Omega |\nabla u|^2 \ln|\nabla u|dx,\end{aligned} \tag{78}$$

where in $\psi''(t)$, (1) has been used.

By virtue of (65) and (77), for any $\delta_0 > 0$, (78) can be rewritten as

$$\begin{aligned}\psi''(t) &= \delta_0\left(\tilde{L}(t) - E_1 - \varepsilon\int_\Omega uu_t dx - \varepsilon\beta\int_\Omega \nabla u\nabla u_t dx - \frac{\varepsilon}{2}\|u\|_2^2\right) \\ &+ \frac{\delta_0+4}{2}\|u_t\|^2_{H^*} + \frac{\delta_0-4}{2}\|u\|^2_{H^*} \\ &+ \left(\frac{\delta_0(k+2)}{4} - 2\right)\|\nabla u\|_2^2 \\ &- k\frac{\delta_0-4}{2}\int_\Omega |\nabla u|^2\ln|\nabla u|dx - 2(u,u_t) \\ &\geq \frac{\delta_0\varepsilon}{2}\left(-2\int_\Omega uu_tdx - 2\beta\int_\Omega \nabla u\nabla u_t dx\right) \\ &+ \frac{\delta_0-4}{2}\|u\|^2_{H^*} - 2(u,u_t) \\ &+ \left(\frac{\delta_0(k+2)}{4} - 2\right)\|\nabla u\|_2^2 \\ &- \frac{k(\delta_0-4)}{2}\int_\Omega |\nabla u|^2\ln|\nabla u|dx \\ &= -\frac{\delta_0\varepsilon}{2}\psi'(t) + \frac{\delta_0+4}{2}\|u_t\|^2_{H^*} + \frac{\delta_0-4}{2}\|u\|^2_{H^*} - 2(u,u_t) \\ &+ \left(\frac{\delta_0(k+2)}{4} - 2\right)\|\nabla u\|_2^2 \\ &- \frac{k(\delta_0-4)}{2}\int_\Omega |\nabla u|^2\ln|\nabla u|dx,\end{aligned} \tag{79}$$

where we used the fact that $\tilde{L}(t) \geq \tilde{L}(0)e^{\hat{\mu}t} > 0, \quad \forall t \in [0, T_{max})$.

Thanks to the logarithmic Sobolev inequality (16), it follows from (79) that

$$\begin{aligned}\psi''(t) &\geq -\frac{\delta_0\varepsilon}{2}\psi'(t) + \frac{\delta_0+4}{2}\|u_t\|^2_{H^*} + \frac{\delta_0-4}{2}\|u\|_2^2 + \frac{\delta_0-4}{2}(1-\frac{ka^2}{2\pi\gamma})\|u\|^2_{H^{**}} \\ &+ \left(\frac{\delta_0(k+2)-8}{4} + \frac{nk(\delta_0-4)}{4}(1+\ln a) - \frac{k(\delta_0-4)}{2}\ln\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 \\ &- 2(u,u_t).\end{aligned}$$

By using Young's inequality, we have

$$\begin{aligned}(u,u_t) &\leq \frac{\delta_0-4}{8}\|u\|_2^2 + \frac{2}{\delta_0-4}\|u_t\|_2^2 \\ &\leq \frac{\delta_0-4}{8}\|u\|_2^2 + \frac{2}{\delta_0-4}\|u_t\|^2_{H^*}.\end{aligned}$$

Thus, from the above inequalities, we obtain the following:

$$\begin{aligned}\psi''(t) \geq & -\frac{\delta_0\varepsilon}{2}\psi'(t) + \left(\frac{\delta_0+4}{2} - \frac{4}{\delta_0-4}\right)\|u_t\|_{H^*}^2 + \frac{\delta_0-4}{4}\|u\|_2^2 \\ & + \frac{\delta_0-4}{2}\left(1-\frac{ka^2}{2\pi\gamma}\right)\|u\|_{H^{**}}^2 \\ & + \left(\frac{\delta_0(k+2)-8}{4} + \frac{nk(\delta_0-4)}{4}(1+\ln a) \right. \\ & \left. - \frac{k(\delta_0-4)}{2}\ln\|\nabla u\|_2^2\right)\|\nabla u\|_2^2.\end{aligned} \tag{80}$$

At this point, if we choose $a=\sqrt{\frac{2\pi\gamma}{k}}$, $\delta_0=8$, and since $1 \leq \tilde{\alpha}^k < e\left(\frac{1}{2}\right)^{\frac{nk}{4}}$ (c.f. Theorem 4), then (80) is as follows:

$$\psi''(t) \geq -4\varepsilon\psi'(t) + 5\|u_t\|_{H^*}^2,$$

and therefore,

$$\psi(t)\psi''(t) \geq -4\varepsilon\psi(t)\psi'(t) + 5\|u_t\|_{H^*}^2\psi(t). \tag{81}$$

On the other hand, by using the Cauchy–Schwarz and Young inequalities, we have

$$\begin{aligned}\left(\psi'(t)\right)^2 &= 4\left((u,u_t)+\beta(\nabla u,\nabla u_t)\right)^2 \\ &= 4\left((u,u_t)^2+\beta^2(\nabla u,\nabla u_t)^2+2\beta(u,u_t)(\nabla u,\nabla u_t)\right) \\ &\leq 4\left(\|u\|_2^2\|u_t\|_2^2+\beta^2\|\nabla u\|_2^2\|\nabla u_t\|_2^2\right. \\ &\quad \left.+2\beta\|u\|_2\|u_t\|_2\|\nabla u\|_2\|\nabla u_t\|_2\right) \\ &\leq 4\left(\|u\|_2^2\|u_t\|_2^2+\beta^2\|\nabla u\|_2^2\|\nabla u_t\|_2^2\right. \\ &\quad \left.+\beta\|u\|_2^2\|\nabla u_t\|_2^2+\beta\|u_t\|_2^2\|\nabla u\|_2^2\right) \\ &= 4\left(\|u\|_2^2(\|u_t\|_2^2+\beta\|\nabla u_t\|_2^2)\right. \\ &\quad \left.+\beta\|\nabla u\|_2^2(\|u_t\|_2^2+\beta\|\nabla u_t\|_2^2)\right) \\ &= 4(\|u_t\|_2^2+\beta\|\nabla u_t\|_2^2)(\|u\|_2^2+\beta\|\nabla u\|_2^2) \\ &= 4\|u_t\|_{H^*}^2\|u\|_{H^*}^2 \\ &= 4\|u_t\|_{H^*}^2\psi(t).\end{aligned} \tag{82}$$

Combining (81) and (82) yields

$$\psi(t)\psi''(t) \geq -4\varepsilon\psi(t)\psi'(t) + \frac{5}{4}\left(\psi'(t)\right)^2. \tag{83}$$

Now, if we choose ε sufficiently small such that $(u_0,u_1)+\beta(\nabla u_0,\nabla u_1) > 8\varepsilon\|u_0\|_{H^*}^2$, then the hypotheses of Lemma 4 with $\tilde{\alpha}=\frac{5}{4}$, $\tilde{\gamma}=4\varepsilon$ and $\tilde{\beta}=0$ are fulfilled, that is, $\psi'(0) > 16\varepsilon\psi(0)$, and thus, there exists a finite time $T^* < T_{max}$ such that the solutions blow-up at T^*, that is,

$$\lim_{t\to T^{*-}}\|u\|_{H^*}^2 = +\infty,$$

with

$$T^* < \frac{2(\|u_0\|_2^2+\beta\|\nabla u_0\|_2^2)^5}{\sqrt[4]{\|u_0\|_{H^*}^2((u_0,u_1)+\beta(\nabla u_0,\nabla u_1)-8\varepsilon\|u_0\|_{H^*}^2)}}.$$

□

Now, at the end of this section, we shall derive a lower bound for the blow-up time by modified differential inequalities when blow-up occurs.

Theorem 6. *Assume that the conditions of Theorem 5 hold. Then the blow-up time T^* has the following lower bound*

$$\int_{\phi(0)}^{\infty}\frac{dy}{\xi_1 y^{\frac{\mu+1}{2}}+y+\xi_2} \leq T^*,$$

where ξ_1, μ, and ξ_2 are positive constants that will be determined in (87) and

$$\phi(0) = \frac{1}{2}\left(\|u_1\|_{H^*}^2+\|u_0\|_{H^{**}}^2+\|\nabla u_0\|_2^2+E_1-E(0)\right).$$

Proof. Define the following auxiliary functional

$$\phi(t) = \frac{1}{2}\left(\|u_t\|_{H^*}^2+\|u\|_{H^{**}}^2+\|\nabla u\|_2^2+G(t)\right), \tag{84}$$

where $G(t)$ satisfies (66). First, we show that $\phi(t)$ is also blow-up at T^*. To achieve this result, we prove that $\phi(t) \geq \delta_1\tilde{L}(t)$. Suppose that $\delta_1 > 0$ and from (65), we obtain

$$\begin{aligned}\phi(t) &= \delta_1\tilde{L}(t) - \delta_1\left(G(t)+\varepsilon\int_\Omega uu_t dx\right. \\ &\quad \left.+\varepsilon\beta\int_\Omega \nabla u\nabla u_t dx + \frac{\varepsilon}{2}\|u\|_2^2\right) \\ &\quad + \frac{1}{2}\left(\|u_t\|_{H^*}^2+\|u\|_{H^{**}}^2+\|\nabla u\|_2^2+G(t)\right) \\ &= \delta_1\tilde{L}(t) + \left(\frac{1}{2}-\delta_1\right)G(t) - \frac{\varepsilon\delta_1}{2}\|u\|_2^2 \\ &\quad - \varepsilon\delta_1\left(\int_\Omega uu_t dx + \beta\int_\Omega \nabla u\nabla u_t dx\right) \\ &\quad + \frac{1}{2}\left(\|u_t\|_{H^*}^2+\|u\|_{H^{**}}^2+\|\nabla u\|_2^2\right) \\ &\geq \delta_1\tilde{L}(t) + \left(\frac{1}{2}-\delta_1\right)G(t) - \varepsilon\delta_1\|u\|_2^2 \\ &\quad - \varepsilon\delta_1\left(\|u\|_2\|u_t\|_2+\beta\|\nabla u\|_2\|\nabla u_t\|_2\right) \\ &\quad + \frac{1}{2}\left(\|u_t\|_{H^*}^2+\|u\|_{H^{**}}^2+\|\nabla u\|_2^2\right) \\ &\geq \delta_1\tilde{L}(t) + \left(\frac{1}{2}-\delta_1\right)G(t) - \varepsilon\delta_1\|u\|_2^2 \\ &\quad - \frac{\varepsilon\delta_1}{2}\|u_t\|_2^2 - \frac{\varepsilon\delta_1\beta}{2}\|\nabla u\|_2^2 \\ &\quad - \frac{\varepsilon\delta_1\beta}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\left(\|u_t\|_{H^*}^2+\|u\|_{H^{**}}^2+\|\nabla u\|_2^2\right) \\ &\geq \delta_1\tilde{L}(t) + \left(\frac{1}{2}-\delta_1\right)G(t) + \frac{1}{2}(1-\varepsilon\delta_1)\|u_t\|_{H^*}^2 \\ &\quad + \left(\frac{1}{2}-\varepsilon\delta_1\right)\|u\|_{H^{**}}^2 \\ &\quad + \frac{1}{2}(1-\varepsilon\delta_1\beta)\|\nabla u\|_2^2.\end{aligned} \tag{85}$$

Let $\delta_1 < \min\left\{\frac{1}{2}, \frac{1}{2\varepsilon}, \frac{1}{\varepsilon\beta}\right\}$, and since $G(t) > G(0) > 0$, then we get $\phi(t) \geq \delta_1 \tilde{L}(t)$, and by Theorem 4 part (i), we deduce that $\phi(t)$ growth exponentially. By similar calculation, one can easily find that there exists a constant $\delta_2 > 0$ such that $\phi(t) \geq \delta_2 \psi(t)$ and thus $\lim\limits_{t\to T^{*-}} \phi(t) = +\infty$.

On the other hand, by using (21), we have

$$\phi'(t) = \frac{1}{2}E'(t) + k\int_\Omega \nabla u \nabla u_t \ln|\nabla u| dx \leq k\int_\Omega \nabla u \nabla u_t \ln|\nabla u| dx. \tag{86}$$

In order to estimate the integral term in the right-hand side of (86), similar to (23), we let use the inequalities $|\phi\ln\phi|\Big|_{0<\phi<1} \leq \frac{1}{e}$; $\phi^{-\mu}\ln\phi\Big|_{\phi\geq 1} \leq \frac{1}{e\mu}$ and Young's inequality; we obtain the following:

$$\begin{aligned}
& k\int_\Omega \nabla u \nabla u_t \ln|\nabla u| dx \\
& \leq k\left(\int_{\{x\in\Omega;|\nabla u|\geq 1\}} \nabla u \nabla u_t \ln|\nabla u| dx + \int_{\{x\in\Omega;|\nabla u|<1\}} \nabla u \nabla u_t \ln|\nabla u| dx\right) \\
& \leq \frac{k}{e\mu}\int_{\{x\in\Omega;|\nabla u|\geq 1\}} |\nabla u|^{\mu+1}|\nabla u_t| dx + \frac{k}{2e}\int_{\{x\in\Omega;|\nabla u|<1\}} |\nabla u_t| dx \\
& \leq \frac{k}{e\mu}\||\nabla u|^{\mu+1}\|_2\|\nabla u_t\|_2 + \frac{k\sqrt{|\Omega|}}{2e}\|\nabla u_t\|_2 \\
& \leq \frac{k}{e\mu}\|\nabla u_t\|_2\left(\int_\Omega |\nabla u|^{2(\mu+1)} dx\right)^{\frac{1}{2}} + \frac{\beta}{2}\|\nabla u_t\|_2^2 + \frac{k^2|\Omega|}{4\beta e^2} \\
& \leq \beta\|\nabla u_t\|_2^2 + \frac{k^2}{\beta e^2\mu^2}\int_\Omega |\nabla u|^{2(\mu+1)} dx + \frac{k^2|\Omega|}{4\beta e^2} \\
& \leq \beta\|\nabla u_t\|_2^2 + \frac{k^2 C_{2(\mu+1)}^2}{\beta e^2\mu^2}\|\Delta u\|_2^{\mu+1} + \frac{k^2|\Omega|}{4\beta e^2} \\
& \leq \|u_t\|_{H^*}^2 + \frac{k^2 C_{2(\mu+1)}^2}{\beta e^2\mu^2\gamma^{\frac{\mu+1}{2}}}\left(\|u\|_{H^{**}}^2\right)^{\frac{\mu+1}{2}} + \frac{k^2|\Omega|}{4\beta e^2} \\
& \leq \phi(t) + \xi_1(\phi(t))^{\frac{\mu+1}{2}} + \xi_2,
\end{aligned} \tag{87}$$

where the embedding $H_0^2(\Omega) \hookrightarrow L^{2(\mu+1)}(\Omega)$ with constant $C_{2(\mu+1)}$ has been used and also $\xi_1 = \frac{k^2 C_{2(\mu+1)}^2}{\beta e^2\mu^2\gamma^{\frac{\mu+1}{2}}}$ and $\xi_2 = \frac{k^2|\Omega|}{4\beta e^2}$.

Integrating (87) over $[0, t]$ yields

$$\int_0^t \frac{\phi'(s)}{\xi_1\phi(s)^{\frac{\mu+1}{2}} + \phi(s) + \xi_2} ds \leq t,$$

let $t \to T^*$ and since $\lim\limits_{t\to T^{*-}} \phi(t) = +\infty$, thus, for $y := \phi(s)$, we get the following:

$$\int_{\phi(0)}^{+\infty} \frac{1}{\xi_1 y^{\frac{\mu+1}{2}} + y + \xi_2} dy \leq T^*, \tag{88}$$

with $\phi(0) = \frac{1}{2}\left(\|u_1\|_{H^*}^2 + \|u_0\|_{H^{**}}^2 + \|\nabla u_0\|_2^2 + G(0)\right)$. Therefore, (88) provides a lower bound for the blow-up time, and proof of Theorem 6 is completed. □

Author Contributions

Faramarz Tahamtani: conceptualization, methodology, software, data curation, supervision, formal analysis, project administration, writing – review and editing, visualization, validation, investigation, funding acquisition, writing – original draft, resources. **Mohammad Shahrouzi:** conceptualization, methodology, software, data curation, supervision, formal analysis, validation, investigation, funding acquisition, writing – original draft, visualization, writing – review and editing, project administration, resources.

Acknowledgments

The authors received no specific funding for this work.

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

References

1. J. T. Kirby, "Nonlinear, Dispersive Long Waves in Water of Variable Depth," in *Gravity waves in water of finite depth, Advances in Fluid Mechanics*, vol. 10, ed. J. N. Hunt (Computational Mechanics Publications, 1997), 55–125.

2. J. T. Kirby, "Boussinesq Models and Applications to Nearshore Wave Propagation, Surfzone Progress and Wave-Induced Currents," in *Advances in coastal modeling* (Elsevier, 2003), 1–41.

3. P. A. Modsen and D. R. Fuhrman, *Higher-Order Boussinesq-Type Modeling of Nonlinear Wave Phenomena in Deep and Shallow Water, In Advances in Numerical Simulation of Nonlinear Water Waves* (World Scientific, 2010).

4. J. Boussinesq, "Théorie des Ondes et de Remous Qui se Propagent le Long d'un canal Rectangulaire horizontal, en Communiquant au Liquide Contene Dans Cecanal Des Vitesses Sensiblement Pareilles de la Surface au Found," *Journal de Mathématiques Pures et Appliquées* 217 (1872): 55–108.

5. V. Varlamov, "Eigenfunctions Expansion Method and the Long-Time Asymptotic for the Damped for the Damped Boussinesq Equation," *Discrete and Continuous Dynamical Systems* 7, no. 4 (2001): 675–702.

6. L. J. Bona and R. L. Sachs, "Global Existence of Smooth Solutions and Stability of Solitary Waves for a Generalized Boussinesq Equation," *Communications in Mathematical Physics* 118, no. 1 (1988): 15–29.

7. R. Xue, "Local and Global Existence of Solutions for the Cauchy Problem of a Generalized Boussinesq Equation," *Journal of Mathematical Analysis and Applications* 316, no. 1 (2006): 307–327.

8. S. Wang and G. Chen, "Cauchy Problem of the Generlized Double Dispersion Equation," *Nonlinear Analysis: Theory Methods & Applications* 64, no. 1 (2006): 159–173.

9. Y. Wang and C. Mu, "Global Existence and Blow Up of Solutions for the Multi-Dimensional Generalized Boussinesq Equation," *Mathematical Methods in the Applied Sciences* 30, no. 12 (2007): 1403–1417.

10. H. Babaoglu, H. A. Erbay, and A. Erkip, “Global Existence and Blow-Up of Solutions for a General Class of Doubly Dispersive Nonlocal Nonlinear Wave Equations,” *Nonlinear Analysis* 77 (2013): 82–93.

11. N. Kutev, N. Kolkovska, and M. Dimova, “Global Existence of Cauchy Problem for Boussinesq Equation With Combined Power-Type Nonlinearities,” *Journal of Mathematical Analysis and Applications* 410, no. 1 (2014): 427–444.

12. Y. Liu, “Decay and Scattering of Small Solutions of a Generalized Boussinesq Equation,” *Journal of Functional Analysis* 147, no. 1 (1997): 51–68.

13. Y. Liu and R. Xu, “Global Existence and Blow Up of Solutions for Cauchy Problem of Generalized Boussinesq Equation,” *Journal of Physica D: Nonlinear Phenomena* 237, no. 6 (2008): 721–731.

14. S. Wang and F. Da, “On the Asymptotic Behavior of Solutions for the Generalized Double Dispersion Equation,” *Applicable Analysis* 92, no. 6 (2013): 1179–1193.

15. H. Zhang and Q. Hu, “Global Existence and Nonexistence of Solution for Cauchy Problem of Two-Dimensional Generalized Boussinesq Equation,” *Journal of Mathematical Analysis and Applications* 422, no. 2 (2015): 1116–1130.

16. E. Pişkin and N. Polat, “Existence, Global, Nonexistence and Asymptotic Behavior of Solutions for a Multi-Dimensional Generalized Damped Boussinesq-Type Equation,” *Turkish Journal of Mathematics* 38, no. 4 (2014): 706–727.

17. M. Shahrouzi, F. Tahamtani, J. Ferreira, and M. M. Freitas, “Blow-Up Results for a Boussinesq-Type Plate Equation With a Logarithmic Damping Term and Variable-Exponent Nonlinearities,” *Applications of Mathematics* 50, no. 1 (2023): 81–96.

18. X. Wu, X. Yang, and L. Cheng, “Global Well-Posedness of Wave Equation With Weak and Strong Damping Terms p-Laplacian and Logarithmic Nonlinearity Source Term,” *Nonlinear Analysis: Real World Applications* 74 (2023): 103958.

19. H. Buljan, A. Šiber, M. Soljačić, T. Schwartz, M. Segev, and D. N. Christodoulides, “Incoherent White Light Solutions in Logarithmically Saturable Noninstantanous Nonlinear Media,” *Physical Review E* 68, no. 3 (2003): 036607.

20. J. D. Barrow and P. Parson, “Inflactionary Models With Logarithmic Potentials,” *Physical Review D* 52, no. 10 (1995): 5576–5587.

21. W. Królikwski, D. Edumundson, and O. Bang, “Unified Model for Partially Choernet Solutions in Logarithmically Nonlinear Media,” *Physical Review E* 61, no. 3 (2000): 3122–3126.

22. A. Linde, “Strings, Textures, Inflation and Spectrum Bending,” *Physics Letters B* 284, no. 3-4 (1992): 215–222.

23. G. Liu, “The Existence, General Decay and Blow-Up for a Plate Equation With Nonlinear Damping and a Logarithmic Source Term,” *Eras* 28, no. 1 (2020): 263–289.

24. X. K. Shao, N. J. Huang, and D. O’Regan, “Infinite Time Blow-up of Solutions for a Plate Equation With Weak Damping and Logarithmic Nonlinearity,” *Journal of Mathematical Analysis and Applications* 535, no. 2 (2024): 128144.

25. M. M. Al-Gharabli, A. Guesmia, and S. Messaoudi, “Existence and a General Decay Results for a Viscoelastic Plate Equation With a Logarithmic Nonlinearity,” *Communications on Pure and Applied Analysis* 18, no. 1 (2019): 159–180.

26. Y. Pang, X. Wang, and F. Wu, “Global Existence and Blowup in Infinite Time for a Fourth Order Wave Equation With Damping and Logarithmic Strain Terms,” *Discrete and Continuous Dynamical Systems - Series S* 14, no. 12 (2021): 4439–4463.

27. X. Li and Z. B. Fang, “Infinite Time Blow-Up With Arbitrary Initial Energy for a Damped Plate Equation,” *Mathematische Nachrichten* 297, no. 6 (2024): 2148–2174.

28. M. M. Al-Gharabli, A. Guesmia, and S. Messaoudi, “Well-Posedness and Asymptotic Stability Results for a Viscoelastic Plate Equation With a Logarithmic Nonlinearity,” *Applicable Analysis* 99, no. 1 (2020): 50–74.

29. M. M. Al-Gharabli, “New General Decay Results for a Viscoelastic Plate Equation With a Logarithmic Nonlinearity,” *Boundary Value Problems* 2019, no. 194 (2019): 1–21.

30. M. M. Al-Gharabli and S. Messaoudi, “Existence and a General Decay Result for a Plate Equation With Nonlinear Damping and a Logarithmic Source Term,” *Journal of Evolution Equations* 18 (2018): 105–152.

31. M. M. Al-Gharabli, A. Guesmia, and S. Messaoudi, “Some Existence and Exponential Stability Results for a Plate Equation With Strong Damping and a Logarithmic Source Term,” *Differential Equations and Dynamical Systems* 32 (2024): 1225–1239.

32. A. M. Wazwaz, “Gaussian Solitary Waves for the Logarithmic Boussinesq Equation and the Logarithmic Regularized Boussinesq Equation,” *Ocean Engineering* 94 (2015): 111–115.

33. Q. Hu, H. Zhang, and G. Liu, “Global Existence and Exponential Growth of Solution for the Logarithmic Boussinesq-Type Equation,” *Journal of Mathematical Analysis and Applications* 436, no. 2 (2016): 990–1001.

34. H. Ding and J. Zhou, “Well-Posedness of Solutions for the Dissipative Boussinesq Equation With Logarithmic Nonlinearity,” *Nonlinear Analysis: Real World Applications* 67 (2022): 103587.

35. A. Cotsiolis and N. K. Tavoularis, “On Logarithmic Sobolev Inequalities for Higher Order Fractional Derivatives,” *Comptes Rendus Mathematique* 340, no. 3 (2005): 205–208.

36. H. Chen, P. Luo, and G. Liu, “Global Solution and Blow-Up of a Semi-Linear Heat Equation With Logarithmic Nonlinearity,” *Journal of Mathematical Analysis and Applications* 422 (2015): 84–98.

37. L. E. Payne and D. H. Sattinger, “Saddle Points and Instability of Nonlinear Hyperbolic Equations,” *Israel Journal of Mathematics* 22, no. 3-4 (1975): 273–303.

38. C. Yang, V. D. Rădulescu, R. Xu, and M. Zhang, “Global Well-Posedness Analysis for the Nonlinear Existence Beam Equations in a Class of Modifed Woinowsky-Krieger Models,” *Advanced Nonlinear Studies* 22 (2022): 436–468.

39. M. Korpusov, “On Blowup of Solutions to a Kirchhoff Type Dissipative Wave Equation With a Source and Positive Energy,” *Siberian Mathematical Journal* 53, no. 4 (2012): 702–717.