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Mean Convergence for Weighted Sums of Negative Superadditive Dependent Random Variables

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Abstract—The mean convergence is obtained for weighted sums of negative superadditive dependent (NSD) random variables under the condition of uniform integrability concerning an array of constants. Furthermore, a simulation study is conducted to confirm the obtained results.

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1. INTRODUCTION

Throughout this paper, all random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Mean convergence for sums (with or without) dependent random variables are surveyed by many authors. The reader may refer to Antonini et al. [4], Ordòñez Cabrera et al. [13], Ordòñez Cabrera and Volodin [14] and Rosalsky and Thành [15]. The important notion of *uniform integrability* (UI) allows us to establish mean convergence and weak laws of large numbers. The classical notion of UI of a sequence $\{X_n, n \geq 1\}$ of integrable random variables is defined by the condition

$$\lim_{t\to\infty} \sup_{n\geq 1} \mathbb{E}|X_n|\mathbb{I}(|X_n| > t) = 0.$$

The concept of UI has been generalized in several directions. Motivated by the classical notion of UI, Chandra [6] used the condition

$$\lim_{t\to\infty} \sup_{n\geq 1} n^{-1} \sum_{i=1}^{n} \mathbb{E}|X_i|\mathbb{I}(|X_i| > t) = 0$$

to propose the notion of *Cesàro uniform integrability* (CUI). The notion of CUI was extended by Ordòñez Cabrera [12] to uniform integrability concerning an array of constants. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants. Then, a sequence $\{X_n, n \geq 1\}$ of random variables is said to be $\{a_{ni}\}$-*uniform integrable* ($\{a_{ni}\}$-UI) if

$$\sup_{n\geq 1} \sum_{i=1}^{n} |a_{ni}| < \infty \quad \text{and} \quad \lim_{t\to\infty} \sup_{n\geq 1} \sum_{i=1}^{n} |a_{ni}|\mathbb{E}|X_i|\mathbb{I}(|X_i| > t) = 0.$$

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It is obvious that when $a_{ni} = n^{-1}$, for $1 \leq i \leq n$ and $n \geq 1$, $\{a_{ni}\}$-UI reduces to CUI. Thus, $\{a_{ni}\}$-UI is weaker than CUI. For more details on the ways the various concepts of uniform integrability are related to each other, we refer the reader to Thành [16].

In what follows, we present an example of $\{a_{ni}^p\}$-*uniform integrability*.

Example 1. Let the probability space be the Lebesgue interval, that is, the unit interval with the Lebesgue measure. We define the sequence of random variables $\{X_n, n \geq 1\}$ and an array of constants $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ by

$$X_n(\omega) = \frac{1}{\lg n}\mathbb{I}_{A_n}(\omega), \quad n \geq 1 \quad \text{and} \quad a_{ni} = \begin{cases} 1/n & 1 \leq i \leq n \\ 0 & i > n, \end{cases}$$

where $A_n = [0, n^{-1}]$ and $\lg x = \ln \max(x, e)$. By using Markov's inequality and the fact that the series $\sum_{i=1}^{\infty} \frac{1}{i \lg^2 i}$ is convergent, for $p \geq 2$ we can write

$$\sup_{n\geq 1}\sum_{i=1}^{n}|a_{ni}|^p\mathbb{E}|X_i|^p\mathbb{I}(|X_i| > t)$$

$$\leq \sup_{n\geq 1}\sum_{i=1}^{n}|a_{ni}|^p t^p\mathbb{P}(|X_i| > t) + \sup_{n\geq 1}\sum_{i=1}^{n}|a_{ni}|^p\int_t^{\infty} px^{p-1}\mathbb{P}(|X_i| > x)dx$$

$$\leq \frac{1}{t}\sup_{n\geq 1}\sum_{i=1}^{n}\frac{1}{n^p}\frac{1}{i\lg^{p+1} i} + \frac{1}{(p-1)t^{p(p-1)}}\sup_{n\geq 1}\sum_{i=1}^{n}\frac{1}{n^p}\frac{1}{i\lg^{p^2} i}$$

$$\leq \frac{1}{t}\sup_{n\geq 1}\sum_{i=1}^{n}\frac{1}{i\lg^{p+1} i} + \frac{1}{(p-1)t^{p(p-1)}}\sup_{n\geq 1}\sum_{i=1}^{n}\frac{1}{i\lg^{p} i}$$

$$\leq \frac{1}{t}\sup_{n\geq 1}\sum_{i=1}^{n}\frac{1}{i\lg^{2} i} + \frac{1}{(p-1)t^{p(p-1)}}\sup_{n\geq 1}\sum_{i=1}^{n}\frac{1}{i\lg^{2} i}$$

$$\leq \frac{1}{t}\sum_{i=1}^{\infty}\frac{1}{i\lg^{2} i} + \frac{1}{(p-1)t^{p(p-1)}}\sum_{i=1}^{\infty}\frac{1}{i\lg^{2} i} \leq C\left(\frac{1}{t} + \frac{1}{(p-1)t^{p(p-1)}}\right) \to 0 \quad \text{as} \quad t \to \infty.$$

Thus, the sequence $\{|X_n|^p, n \geq 1\}$ of random variables is $\{a_{ni}^p\}$-UI for $p \geq 2$.

The following counterexample shows that $\{|X_n|^p, n \geq 1\}$ is $\{|a_{ni}^p|\}$-UI but $\{|X_n|^q, n \geq 1\}$ is not valid $\{|a_{ni}^q|\}$-UI for $0 < q < p$.

Example 2. Let $\{X_n, n \geq 1\}$ be a sequence of identically and bounded ($\mathbb{P}(|X| \leq C) = 1$) random variables. Since $\mathbb{E}|X|^\alpha\mathbb{I}[|X| > t] \to 0$ *as* $t \to \infty$ for $\alpha > 0$ and $\sum_{i=1}^{\infty} i^{-2} < \infty$, $\sum_{i=1}^{\infty} i^{-1} = \infty$, then by taking $a_{ni} = 1/i,\ 1 \leq i \leq n$ we can say $\{|X_n|^2, n \geq 1\}$-UI is $\{|a_{ni}^2|\}$-UI but $\{|X_n|, n \geq 1\}$ it is not $\{|a_{ni}|\}$-UI.

Negatively associated (NA) random variables were introduced by Alam and Saxena [1] and carefully studied by Joag and Proschan [11] and Block et al. [3]. As pointed out and proved by Joag and Proschan, a number of well-known multivariate distributions possess the NA property. Negative association has found important and wide applications in multivariate statistical analysis and reliability theory. The applications of NA to probability, stochastic processes and statistics have been discussed by many researchers.

Definition 1. *A sequence $X_1, X_2, \dots, X_n$ of random variables is called NA if for every pair of disjoint subsets A_1 and A_2* of $\{1, 2, \dots, n\}$,

$$\mathbb{C}ov(f_1(X_i; i \in A_1), f_2(X_j; j \in A_2)) \leq 0,$$

where the functions $f_1(X_i; i \in A_1)$ and $f_2(X_j; j \in A_2))$ are increasing in any variable (or decreasing in any variable) and the covariance exists. An infinite family of random variables is NA if each of its finite subfamilies is NA.

The next dependence notion is the concept of negative superadditive dependence, which is weaker than NA. *Negatively superadditive-dependent* (NSD) random variables were introduced by Hu [9] as follows.

Definition 2. *A random vector* $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$ *is said to be NSD if*

$$\mathbb{E}\phi(X_1, X_2, \dots, X_n) \leq \mathbb{E}\phi(X_1^*, X_2^*, \dots, X_n^*),$$

where $X_1^*, X_2^*, \dots, X_n^*$ *are independent,* X_i^* *and* X_i *have the same distribution for each* i*, and* ϕ *is a superadditive function such that the expectations given in the above equation exist. A sequence* $\{X_n, n \geq 1\}$ *of random variables is said to be NSD if for each* $n \geq 1$*, the vector* $(X_1, X_2, \dots, X_n)$ *is NSD.*

As shown by Hu [9], NSD does not necessarily imply NA. He also proposed an open problem asking about the truth of the reverse implication: Is it true that NA implies NSD? Furthermore, he provided some basic properties of NSD random vectors and established three structural theorems for such vectors. Later on, Christofides and Vaggelatou [7] solved the open problem, showing that NA implies NSD, we refer more details Amini et al. [2]. Therefore, the NSD structure is an extension of the NA structure and is sometimes more useful.

We provide definition according to the Farlie–Gumbel–Morgenstern (FGM) random sequences that we can apply to NSD.

Definition 3. *A sequence* $\{X_n, n \geq 1\}$ *of random variables is called a FGM random sequence if for* $n \geq 1$ *and* $(x_1, \dots, x_n) \in \mathbb{R}^n$*, we have*

$$F(x_1, \dots, x_n) = \prod_{i=1}^{n} F_i(x_i) \left(1 + \sum_{1 \leq i < j \leq n} \alpha_{ij}[1 - F_k(x_k)][1 - F_j(x_j)]\right),$$

where the parameters α_{ij} *satisfy in conditions* $|\alpha_{ij}| \leq 1$ *and* $1 + \sum_{1 \leq i < j \leq n} \alpha_{ij} \geq 0$*. Also,* F_i *is cumulative distribution function of* X_i*.*

Remark 1. For the parameters $\alpha_{ij} \leq 0$, FGM random sequence is NSD. On more details we refer the reader to Mari and Kotz [8].

We also need basic definitions from the theory of regularly varying function. For a complete exposition on the subject, the reader may consult Bingham et al. [5].

Definition 4. *A measurable function* $U : [a, \infty) \to (0, \infty)$ *,* $a \in \mathbb{R}$*, is called regularly varying at infinity (zero) with exponent* ρ*, denoted as* $U \in \mathcal{RV}_\infty(\rho)$ $(U \in \mathcal{RV}_0(\rho))$*, if for all* $t > 0$*,*

$$\frac{U(tx)}{U(x)} \to t^\rho \quad as \quad x \to \infty \; (x \to 0).$$

If $\rho = 0$*, then we say that it is slowly varying at infinity (zero) and write* $U \in \mathcal{SV}_\infty$ $\left(U \in \mathcal{SV}_0\right)$*.*

Throughout this paper, let C denote a positive constant not depending on n, let $\mathbb{I}(A)$ be the indicator function of a set A and $\lg x = \ln \max(e, x)$.

2. THE MAIN RESULT

We begin with some preliminary facts which are needed for the proof of our main result. The first lemma is due to Hu [9].

Lemma 1. *If* $X_1, X_2, \dots, X_n$ *are NSD random variables and* $g_1(\cdot), g_2(\cdot), \dots, g_n(\cdot)$ *are non-decreasing functions, then* $g_1(X_1), g_2(X_2), \dots, g_n(X_n)$ *are also NSD random variables.*

In the second lemma, we state a Rosenthal-type maximal inequality for NSD random variables. It can be found in Wang et al. [17].

Lemma 2 (A Rosenthal-type maximal inequality). *Let* $\{X_n, n \geq 1\}$ *be a sequence of NSD random variables with* $E|X_n|^p < \infty$*, for some* $p > 1$ *and every* $n \geq 1$*. Then, there exist positive constants* C_p *and* D_p *depending only on* p *such that for every* $n \geq 1$*,*

$$\mathbb{E}\left(\max_{1 \leq k \leq n} \left|\sum_{i=1}^{k} X_i\right|^p\right) \leq C_p \sum_{i=1}^{n} \mathbb{E}\left|X_i\right|^p \quad \text{for} \quad 1 < p \leq 2$$

and

$$\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}X_i\right|^p\right)\le D_p\left\{\sum_{i=1}^{n}\mathbb{E}\,|X_i|^p+\left(\sum_{i=1}^{n}\mathbb{E}X_i^2\right)^{p/2}\right\}\quad \text{for}\quad p>2.$$

Now, we present the main result that states the mean convergence for weighted sums of NSD random variables.

Theorem 1. *Let $\{X_n, n\ge 1\}$ be a sequence of NSD random variables with $\mathbb{E}X_n=0$ for each $n\ge 1$.*

(i) *Let $1\le p\le 2$ and $\{a_{ni}, 1\le i\le n, n\ge 1\}$ be an array of real numbers such that $\max_{1\le i\le n}|a_{ni}|\to 0$ as $n\to\infty$. If $\{|X_n|^p, n\ge 1\}$ is $\{|a_{ni}|^p\}$-UI, then*

$$\max_{1\le i\le n}\left|\sum_{i=1}^{k}a_{ni}X_i\right|\xrightarrow{Lp}0,\quad \text{as}\quad n\to\infty. \tag{1}$$

(ii) *Let $p>2$ and $\{a_{ni}, 1\le i\le n, n\ge 1\}$ be an array of real numbers such that $a_{ni}\le 1$, $\sup_{n\ge1}\sum_{i=1}^{n}|a_{ni}|<\infty$ and $\max_{1\le i\le n}|a_{ni}|\to 0$ as $n\to\infty$. If $\{|X_n|^p, n\ge 1\}$ is $\{a_{ni}^2\}$-UI, then*

$$\max_{1\le i\le n}\left|\sum_{i=1}^{k}a_{ni}X_i\right|\xrightarrow{Lp}0,\quad \text{as}\quad n\to\infty. \tag{2}$$

Proof. Without loss of generality, we suppose that $a_{ni}\ge 0$, because we can use the identity $a_{ni}=a_{ni}^{+}-a_{ni}^{-}$ in the general case. Since the sequence $\{|X_n|^p, n\ge 1\}$ is $\{|a_{ni}|^p\}$-UI and $a_{ni}\ge 0$,

$$\lim_{t\to\infty}\sup_{n\ge1}\sum_{i=1}^{n}a_{ni}^p\mathbb{E}|X_i|^p\mathbb{I}(|X_i|>t)=0.$$

For $i\ge 1$ and fixed $t>0$, set

$$Y_i=-t\mathbb{I}[X_i<-t]+X_i\mathbb{I}[|X_i|\le t]+t\mathbb{I}[X_i>t],$$

$$Z_i=X_i-Y_i=(X_i+t)\mathbb{I}[X_i<-t]+(X_i-t)\mathbb{I}[X_i>t].$$

Since $a_{ni}\ge 0$, by Lemma 1, we observe that $\{a_{ni}(Y_i-\mathbb{E}Y_i), i\ge 1\}$ and $\{a_{ni}(Z_i-\mathbb{E}Z_i), i\ge 1\}$ are sequences of NSD random variables. Moreover, for $k\ge 1$ let

$$S_k=\sum_{i=1}^{k}a_{ni}X_i,\quad S'_k=\sum_{i=1}^{k}a_{ni}(Y_i-\mathbb{E}Y_i),\quad S''_k=\sum_{i=1}^{k}a_{ni}(Z_i-\mathbb{E}Z_i).$$

Since $\mathbb{E}X_i=0$, we can write $S_k=S'_k+S''_k$.

First, we prove (i). By C_r and Lyapunov inequalities,

$$\mathbb{E}\left(\max_{1\le i\le n}|S_k|^p\right)\le C\left(\mathbb{E}\left(\max_{1\le i\le n}|S'_k|^2\right)\right)^{p/2}+C\mathbb{E}\left(\max_{1\le i\le n}|S''_k|^p\right)\doteq I_1+I_2.$$

To prove (1), we show that $I_1\to 0$ and $I_2\to 0$ as $n\to\infty$. For I_1, by Lemma 2 and the fact that $|Y_i|\le t$ we obtain

$$I_1\le C\left[\sum_{i=1}^{n}a_{ni}^2\mathbb{E}|Y_i-\mathbb{E}Y_i|^2\right]^{p/2}\le C\left[\sum_{i=1}^{n}a_{ni}^2\mathbb{E}|Y_i|^2\right]^{p/2}$$

$$\le Ct^p\left[\sum_{i=1}^{n}a_{ni}^2\right]^{p/2}=Ct^p\left[\sum_{i=1}^{n}a_{ni}^pa_{ni}^{2-p}\right]^{p/2}\le Ct^p\left[\sup_{n\ge1}\sum_{i=1}^{n}a_{ni}^p\right]^{p/2}\left(\max_{1\le i\le n}a_{ni}\right)^{p(1-p/2)}. \tag{3}$$

For I_2, note that $|Z_i| \le 2|X_i|\mathbb{I}[|X_i| > t]$ and Lemma 2 allow us to write

$$I_2 \le C\sum_{i=1}^{n} a_{ni}^p \mathbb{E}|Z_i - \mathbb{E}Z_i|^p \le C\sum_{i=1}^{n} a_{ni}^p \mathbb{E}|X_i|^p \mathbb{I}[|X_i| > t]$$
$$\le C\sup_{n\ge 1}\sum_{i=1}^{n} a_{ni}^p \mathbb{E}|X_i|^p \mathbb{I}[|X_i| > t]. \quad (4)$$

Now, (3) and (4) show that

$$\mathbb{E}\left(\max_{1\le i\le n}|S_k|^p\right) \le Ct^p\left[\sup_{n\ge 1}\sum_{i=1}^{n} a_{ni}^p\right]^{p/2}\left(\max_{1\le i\le n} a_{ni}\right)^{p(1-p/2)} + C\sup_{n\ge 1}\sum_{i=1}^{n} a_{ni}^p \mathbb{E}|X_i|^p \mathbb{I}[|X_i| > t]. \quad (5)$$

Since $\sup_{n\ge 1}\sum_{i=1}^{n} a_{ni}^p < \infty$, $\max_{1\le i\le n} a_{ni} \to 0$ as $n \to \infty$ and $\{|X_n|^p, n \ge 1\}$ is $\{a_{ni}^p\}$-UI, we immediately deduce (i) by letting $n \to \infty$ and $t \to \infty$, respectively, in (5).

Next, we prove (ii). Similar to the proof of (i), by C_r and Lyapunov inequalities we obtain

$$\mathbb{E}\left(\max_{1\le i\le n}|S_k|^p\right) \le C\mathbb{E}\left(\max_{1\le i\le n}|S'_k|^p\right) + C\mathbb{E}\left(\max_{1\le i\le n}|S''_k|^p\right) \doteq J_1 + J_2. \quad (6)$$

For J_1, by the C_r inequality, Lemma 2 and the fact that $|Y_i| \le t$ we can write

$$J_1 \le C\sum_{i=1}^{n} a_{ni}^p \mathbb{E}|Y_i - \mathbb{E}Y_i|^p + C\left(\sum_{i=1}^{n} a_{ni}^2 \mathbb{E}|Y_i - \mathbb{E}Y_i|^2\right)^{p/2}$$
$$\le C\sum_{i=1}^{n} a_{ni}^p \mathbb{E}|Y_i|^p + C\left(\sum_{i=1}^{n} a_{ni}^2 \mathbb{E}|Y_i|^2\right)^{p/2}$$
$$\le Ct^p\left(\sum_{i=1}^{n} a_{ni}^p\right) + Ct^p\left(\sum_{i=1}^{n} a_{ni}^2\right)^{p/2} \le Ct^p\left(\sum_{i=1}^{n} a_{ni}^2\right) + Ct^p\left(\sum_{i=1}^{n} a_{ni}^2\right)^{p/2} \quad (by 0 < a_{ni} \le 1)$$
$$\le Ct^p\left\{\left((\sup_{n\ge 1}\sum_{i=1}^{n} a_{ni})(\max_{1\le i\le n} a_{ni})\right) + \left((\sup_{n\ge 1}\sum_{i=1}^{n} a_{ni})(\max_{1\le i\le n} a_{ni})\right)^{p/2}\right\}. \quad (7)$$

For J_2, similar to (4) we obtain

$$J_2 \le C\left\{\sum_{i=1}^{n} a_{ni}^p \mathbb{E}|Z_i - \mathbb{E}Z_i|^p + \left(\sum_{i=1}^{n} a_{ni}^2 \mathbb{E}|Z_i - \mathbb{E}Z_i|^2\right)^{p/2}\right\}$$
$$\le C\left\{\sum_{i=1}^{n} a_{ni}^p \mathbb{E}|Z_i|^p + C\left(\sum_{i=1}^{n} a_{ni}^2 \mathbb{E}|Z_i|^2\right)^{p/2}\right\}$$
$$\le C\sum_{i=1}^{n} a_{ni}^p \mathbb{E}|X_i|^p \mathbb{I}(|X_i| > t) + C\left(\sum_{i=1}^{n} a_{ni}^2 \mathbb{E}|X_i|^2 \mathbb{I}(|X_i| > t)\right)^{p/2}$$
$$\le C\sum_{i=1}^{n} a_{ni}^2 \mathbb{E}|X_i|^p \mathbb{I}(|X_i| > t) + C\left(t^{2-p}\sum_{i=1}^{n} a_{ni}^2 \mathbb{E}|X_i|^p \mathbb{I}(|X_i| > t)\right)^{p/2} \quad (\text{by } 0 < a_{ni} \le 1)$$
$$\le C\sup_{n\ge 1}\sum_{i=1}^{n} a_{ni}^2 \mathbb{E}|X_i|^p \mathbb{I}(|X_i| > t) + Ct^{p(1-p/2)}\left(\sup_{n\ge 1}\sum_{i=1}^{n} a_{ni}^2 \mathbb{E}|X_i|^p \mathbb{I}(|X_i| > t)\right)^{p/2}. \quad (8)$$

By (7) and (8),

$$\mathbb{E}\left(\max_{1\le i\le n}|S_k|^p\right)\le Ct^p\left\{\left((\sup_{n\ge1}\sum_{i=1}^n a_{ni})(\max_{1\le i\le n}a_{ni})\right)+\left((\sup_{n\ge1}\sum_{i=1}^n a_{ni})(\max_{1\le i\le n}a_{ni})\right)^{p/2}\right\}$$

$$+C\sup_{n\ge1}\sum_{i=1}^n a_{ni}^2\mathbb{E}|X_i|^p\mathbb{I}(|X_i|>t)+Ct^{p(1-p/2)}\left(\sup_{n\ge1}\sum_{i=1}^n a_{ni}^2\mathbb{E}|X_i|^p\mathbb{I}(|X_i|>t)\right)^{p/2}. \tag{9}$$

Since $\sup_{n\ge1}\sum_{i=1}^n a_{ni}<\infty$ and $\max_{1\le i\le n}a_{ni}\to0$ as $n\to\infty$, $\{|X_n|^p,n\ge1\}$ is $\{a_{ni}^2\}$-UI, (ii) immediately follows by letting $n\to\infty$ and $t\to\infty$, respectively, in (9). □

Remark 2. It is obvious that $\{a_{ni}\}$-UI is weaker than CUI. So, under the conditions of Theorem 1, if $\{|X_n|^p,n\ge1\}$ is CUI, then the results (1) and (2) follow.

Remark 3. Since convergence in mean implies convergence in probability, all the obtained results are also true for convergence in probability.

Let us present an example of a FGM sequence of random variables for which Theorem 1 be applied.

Example 3. Let $\{X_n,n\ge1\}$ be a sequence of FGM random variables with the parameters $\alpha_{ij}\le0$. Suppose Stationary sequence $\{X_n=Y\cdot\varepsilon_n,n\ge1\}$ random variables, where $\{\varepsilon_n,n\ge1\}$ is a sequence of i.i.d., independent of Y such that for each $i\ge1$, $P(\varepsilon_i=1)=P(\varepsilon_i=-1)=\frac{1}{2}$, and Y be the Pareto distribution $F_Y(y)=1-y^{-\alpha}$, $y\in[1,\infty)$, $\alpha>0$. Since $\mathbb{P}(Y>t)\in\mathcal{RV}_\infty(-\alpha)$, then by Theorem 1.5.11 (ii) in Bingham et al. (1987) we can write for large t

$$p\int_t^\infty y^{p-1}\mathbb{P}(Y>y)dy=\frac{1}{\alpha-p}t^p\mathbb{P}(Y>t),$$

now we have for $p<\alpha$,

$$\mathbb{E}|X|^p\mathbb{I}[|X|>t]=\mathbb{E}Y^p\mathbb{I}[Y>t]=t^p\mathbb{P}(Y>t)+p\int_t^\infty y^{p-1}\mathbb{P}(Y>y)dy$$

$$=\frac{(\alpha-p)+1}{(\alpha-p)}t^p\mathbb{P}(Y>t)=\frac{(\alpha-p)+1}{(\alpha-p)}t^{p-\alpha}\to0\quad\text{as}\quad t\to\infty. \tag{10}$$

Let $a_{ni}=n^{-\theta},1\le i\le n,n>1$ and $\theta p\ge1$. Since $\max_{1\le i\le n}|a_{ni}|=n^{-\theta}\to0$ as $n\to\infty$, $\sup_{n\ge1}\sum_{i=1}^n|a_{ni}|=\sup_{n\ge1}\frac{1}{n^{\theta-1}}<\infty$ and by (10)

$$\lim_{t\to\infty}\sup_{n\ge1}\sum_{i=1}^n|a_{ni}|\mathbb{E}|X_i|\mathbb{I}(|X_i|>t)=\lim_{t\to\infty}\sup_{n\ge1}\frac{1}{n^{\theta-1}}\mathbb{E}|X|\mathbb{I}(|X|>t)\le\lim_{t\to\infty}\mathbb{E}|X|\mathbb{I}(|X|>t)=0,$$

hence, random sequence $\{X_n=Y\cdot\varepsilon_n,n\ge1\}$ and real number sequence $a_{ni}=n^{-\theta},1\le i\le n$ satisfy the assumptions of Theorem 1 (i). We can conclude for $p<\alpha$, $\alpha>0$, and $\theta p\ge1$

$$\frac{S_n}{n^\theta}\xrightarrow{Lp}0\quad\text{as}\quad n\to\infty.$$

Also, according to the above it is easy to show that these sequence satisfy in Theorem 1 (ii), and we can write for $p<\alpha$, $\alpha>0$, and $\theta\ge1/2$

$$\frac{S_n}{n^\theta}\xrightarrow{Lp}0\quad\text{as}\quad n\to\infty.$$

In what follows, the approach to the weighted law of large numbers the idea of Jajte [10] applied and we discuss the assumptions that will be imposed on our weights in Theorem 1.

Let $g:[0,\infty)\to\mathbb{R}$ and $h:[0,\infty)\to\mathbb{R}$ be non-negative functions, and let $\phi(y)=g(y)h(y)$. Also, assume that the following conditions are satisfied.

(A1) $h(\cdot)$ is non-decreasing and for some $d\ge0$, $\phi(\cdot)$ is strictly increasing on $[d,\infty]$ with range $[d,\infty]$.

(A2) A constant $b > 0$ exists such that for $\alpha \geq 0$, $\sum_{i=1}^{n} \frac{1}{h^{\alpha}(i)} \leq b \frac{n}{h^{\alpha}(n)}$.

Remark 4. For the power functions $h(x) = x^q$ and $g(x) = x^p$ ($\phi(x) = x^{p+q}$), the requirements (A1) and (A2) are valid for $p + q > 0$ and $\alpha q < 1$. Also the requirements (A1) and (A2) are valid for functions $h(x) = x^q$ and $g(x) = \log^p x$ ($\phi(x) = x^q \log^p x$) for $p > 0$ and $\alpha q < 1$.

If we let $a_{ni} = \frac{1}{g(n)h(i)}$ for $n \geq 1$ and $1 \leq i \leq n$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let $\{X_n, n \geq 1\}$ be a sequence of NSD random variables with $\mathbb{E}X_n = 0$ for each $n \geq 1$, and assume that $g(\cdot)$, $h(\cdot)$, and $\phi(\cdot)$ satisfy the conditions (A1)−(A2) and $g(\cdot)$ is strictly increasing.*

(*i*) *Let $1 \leq p \leq 2$. If for $t > 0$,*

$$\lim_{t\to\infty} \sup_{n\geq 1} \frac{1}{g^p(n)} \sum_{i=1}^{n} \frac{1}{h^p(i)} \mathbb{E}|X_i|^p \mathbb{I}(|X_i| > t) = 0, \tag{11}$$

then

$$\max_{1\leq i\leq n} \left| \frac{1}{g^p(n)} \sum_{i=1}^{k} \frac{1}{h(i)} X_i \right| \xrightarrow{Lp} 0, \quad as \quad n \to \infty.$$

(*ii*) *Let $p > 2$, $a_{ni} = \frac{1}{g(n)h(i)} \leq 1$, $1 \leq i \leq n$, $\sup_{n\geq 1} \frac{n}{\phi(n)} < \infty$ and $\frac{n}{\phi(n)} \to 0$ as $n \to \infty$. If for $t > 0$,*

$$\lim_{t\to\infty} \sup_{n\geq 1} \frac{1}{g^p(n)} \sum_{i=1}^{n} \frac{1}{h^2(i)} \mathbb{E}|X_i|^p \mathbb{I}(|X_i| > t) = 0, \tag{12}$$

then

$$\max_{1\leq i\leq n} \left| \frac{1}{g^p(n)} \sum_{i=1}^{k} \frac{1}{h(i)} X_i \right| \xrightarrow{Lp} 0, \quad as \quad n \to \infty.$$

Corollary 2. *Let $\{X_n, n \geq 1\}$ be a sequence of NSD random variables with $\mathbb{E}X_n = 0$ for $n \geq 1$. If $\{|X_n|^p, n \geq 1\}$ is CUI for $1 \leq p < 2$, then*

$$\max_{1\leq k\leq n} \left| \frac{\sum_{i=1}^{n} X_i}{n^{1/p}} \right| \xrightarrow{Lp} 0, \quad as \quad n \to \infty. \tag{13}$$

Proof. In Corollary 1, let $g(n) = n^{1/p}, h(n) = 1$ ($\phi(n) = n^{1/p}$) for $1 \leq p < 2$. Since $\{|X_n|^p, n \geq 1\}$ is CUI, easily (11) follows. Now, by applying Corollary 1 (i) we conclude (13). □

Corollary 3. *Let $\{X_n, X, n \geq 1\}$ be a sequence of identically NSD random variables with $\mathbb{E}X = 0$ for $n \geq 1$. If $E|X|^p \mathbb{I}[|X| > t] \to 0$, $t \to \infty$ for $p > 1$, then*

$$\max_{1\leq k\leq n} \left| \frac{1}{\lg^p n} \sum_{i=1}^{n} \frac{X_i}{i^p} \right| \xrightarrow{Lp} 0, \quad as \quad n \to \infty. \tag{14}$$

Proof. Because $\sum_{i=1}^{n} \frac{1}{i^p} < \infty$ for $p > 1$, $\sup_{n\geq 1} \frac{n}{\phi(n)} = \sup_{n\geq 1} \frac{1}{\lg n} < \infty$ and $\frac{n}{\phi(n)} = \frac{1}{\lg n} \to 0$ as $n \to \infty$, then by according Corollary 1 (i) and (ii), we obtain (14). □

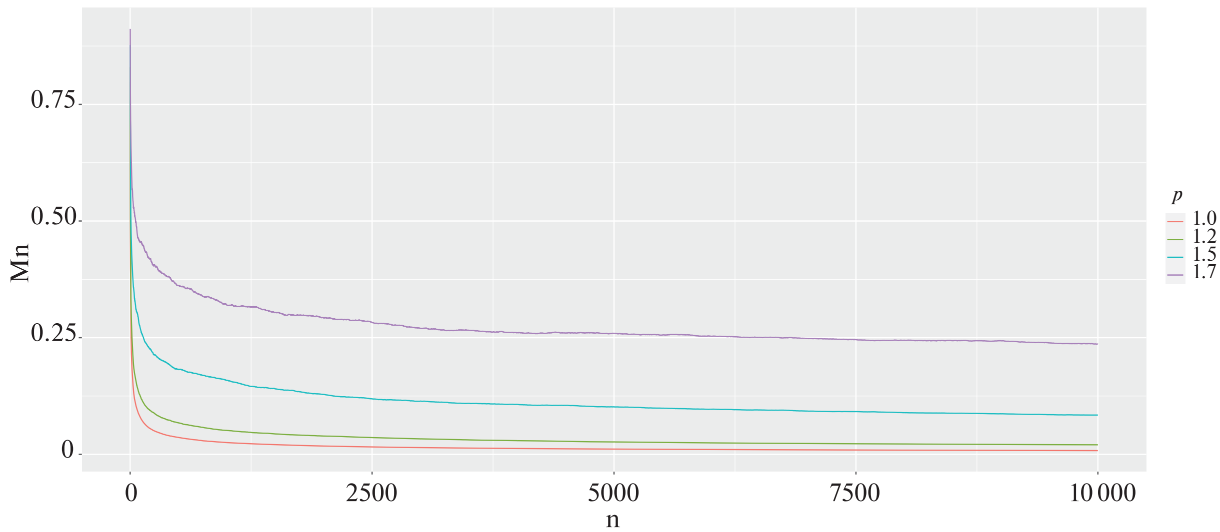

Fig. 1. The plot of M_n against n for Corollary 2.

3. SIMULATION STUDY

In this section, based on Corollary 2, we evaluate the numerical performance of

$$M_n = \max_{1 \le k \le n} \left| \frac{\sum_{i=1}^{n} X_i}{n^{1/p}} \right| \xrightarrow{Lp} 0, \quad \text{as} \quad n \to \infty. \tag{15}$$

We generate an NSD sequence by

$$X_n = a_m Y_n + b_m Z_n, \quad n \ge 1,$$

where a_m and b_m are positive sequences, Y_n and Z_n are negatively dependent random variables (corresponding to $\rho < 0$) with bivariate Normal distribution as

$$(Y, Z) \sim \mathcal{N}(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho),$$

where the sequence $\{X_n, n \ge 1\}$ is NSD, as shown by Yu et al. [18].

The simulation procedure of the sequence $\{X_n, n \ge 1\}$ was straightforward. We used the R software to compute M_n for n=10000 using 5000 replications of the sequence $\{X_n, n \ge 1\}$ for $p = 1, 1.2, 1.5, 1.7$ and $a_m = b_m = 1, \mu_1 = \mu_2 = 0, \sigma_1 = \sigma_2 = 1$, and $\rho = -0.5$, where the conditions of Corollary 2 are satisfied.

The convergence behavior of the sequence $\{M_n,\ n \ge 1\}$ is shown in Fig. 1. According to the figure, the terms M_n fluctuate around zero and the range of changes is significantly reduced.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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