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A note on complete convergence for m-NOD random variables

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Abstract. In this paper, we extend Jajte's technique, to study the rate of complete convergence for weighted sequence of m-NOD random variables. In addition, we make a simulation study to illustrate the asymptotic behavior in the sense of the rate of complete convergence.

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1 Introduction

Hsu and Robbins [7] introduced the concept of complete convergence in the following way. A sequence $\{X_n, n \geq 1\}$ of random variables is completely convergent (c.c.) to a constant θ if for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|X_n - \theta| > \varepsilon\right) < \infty.$$

In view of the Borel-Cantelli lemma, the complete convergence to a constant θ implies that $X_n \to \theta$ almost surely (a.s.), and therefore complete convergence is a stronger concept than a.s. convergence. Hence the complete convergence is a very important tool in establishing almost sure convergence of sums and weighted sums of random variables. Hsu and Robbins [7] proved that the sequence of arithmetic means of i.i.d random variables converges completely to the expected value if the variance of the summands is finite. Erdős [4] proved the converse. There are many papers devoted to the study of complete convergence for sums and weighted sums of independent and dependent random variables and fields (see for example [10] and [15] where further references may be found).

Let us recall the definition of negative orthant dependence (NOD), which was introduced by Joag-Dev and Proschan [9] as follows.

Definition 1. *A finite collection of random variables $X_1, \dots, X_n$ is said to be negatively orthant dependent (NOD) if*

$$\mathbb{P}(X_1 > x_1, \dots, X_n > x_n) \le \prod_{i=1}^{n} \mathbb{P}(X_i > x_i)$$

and

$$\mathbb{P}(X_1 \le x_1, \dots, X_n \le x_n) \le \prod_{i=1}^{n} \mathbb{P}(X_i \le x_i), \text{ for any } x_1, \dots, x_n \in \mathbb{R}.$$

Inspired by the definition of NOD random variables, we recall the concept of m-NOD random variables which was introduced by Wang et al. [18] as follows.

Definition 2. *Let $m \ge 1$ be a fixed integer. A sequence $\{X_n, n \ge 1\}$ of random variables is said to be m-negatively orthant dependent (m-NOD) if, for any $n \ge 2$ and any $i_1, \dots, i_n$ such that $|i_k - i_j| \ge m$ for all $1 \le k \ne j \le n$, we have that $X_{i_1}, \dots, X_{i_n}$ are NOD.*

An array $\{X_{ni}, i \ge 1, n \ge 1\}$ of random variables is said to be rowwise m-NOD if, for every $n \ge 1$, $\{X_{ni}, i \ge 1\}$ is a sequence of m-NOD random variables. For $m = 1$, the concept m-NOD random variables reduces to the so-called NOD random variables. Hence, the concept of m-NOD random variables is a natural extension of NOD random variables which includes independent random variables and negatively associated (NA) random variables.

The m-NOD property is preserved under monotonic functions, this fact is stated as the following lemma, which will be used throughout the paper.

Lemma 1. *(cf. [18]) Let $\{X_n, n \ge 1\}$ be a sequence of m-NOD random variables. If $\{g_n(x), n \ge 1\}$ are all nondecreasing (or nonincreasing) functions, then a sequence of random random variables $\{g_n(X_n), n \ge 1\}$ is also m-NOD random sequence.*

Jajte [8] studied a large class of summability methods defined as follows: it is said that a sequence $\{X_n, n \ge 1\}$ of random variables is almost surely summable to a random variable X by the method (h, g) if

$$\frac{1}{g(n)} \sum_{k=1}^{n} \frac{1}{h(k)} X_k \to X \text{ a.s., } n \to \infty.$$

For a sequence $\{X_n, n \ge 1\}$ of i.i.d. random variables Jajte proved that $\{X_n - \mathbb{E}X_n \mathbb{I}[|X_n| \le \psi(n)], n \ge 1\}$ is almost surely summable to 0 by the method (h, g) iff $\mathbb{E}\psi^{-1}(|X_1|) < \infty$ ($\psi^{-1}(\cdot)$ is inverse of $\psi(\cdot)$), where g, h and $\psi(y) = g(y)h(y)$ are functions satisfying some additional conditions. The most up-to-date survey on this matter may be found in Fazakas et al. [5], Wang [19], Matuła and Seweryn [11], Shen [14], Tang [17], Son et al. [16] and Naderi et al. [12] and Naderi et al. [13].

Now we recall the concept of stochastic domination, which will be used in the sequel.

Definition 3. *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that*

$$\mathbb{P}(|X_n| > x) \leq C\mathbb{P}(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$.

The next section will be devoted to the study of the rate of complete convergence for weighted m-NOD random variables in format of method (h, g).

Throughout the paper, let us denote by C a positive constant not depending on n, which may be different in various places, $\lfloor . \rfloor$ is an integer part of a number and let $\mathbb{I}(A)$ be the indicator function of the set A.

2 Complete convergence

We begin with the assumptions which will be imposed on our weights. Let $g : [0, \infty) \to \mathbb{R}$ and $h : [0, \infty) \to \mathbb{R}$ be nonnegative functions let $\psi(y) = g(y)h(y)$ and we consider the class of all functions, $g(\cdot), h(\cdot)$ and $\psi(\cdot)$ which satisfies the following conditions:

(A1) h is nondecreasing and ψ is strictly increasing with $\psi([0, \infty)) = [0, \infty)$,

(A2) there exist constants $a, b > 0, r \geq 1, \alpha \in (1, 2]$ and a strictly increasing function $H(\cdot)$, such that

$$\psi^\alpha(s) \int_s^\infty \frac{x^{r-1}}{\psi^\alpha(x)} dx \leq aH(s) + b, \text{ for all } s > 0,$$

(A3) there exists a constant $C > 0$ such that for some $\alpha \in (1, 2]$

$$\sum_{i=1}^n \frac{1}{h^\alpha(i)} \leq C \frac{n}{h^\alpha(n)}.$$

At first we provide some lemmas which will be used in the proofs of our main results which were discussed in [1] and [2] . The first one is a basic, well known, property of stochastic domination. In the second lemma we state the Rosenthal-type maximal inequality for m-NOD random variables which may be found in [18].

Lemma 2. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\beta > 0$ and $b > 0$, the following two statements hold:*

$$\begin{aligned} \mathbb{E}|X_n|^\beta \mathbb{I}[|X_n| \leq b] &\leq C_1 \left[\mathbb{E}|X|^\beta \mathbb{I}[|X| \leq b] + b^\beta \mathbb{P}(|X| > b) \right], \\ \mathbb{E}|X_n|^\beta \mathbb{I}[|X_n| > b] &\leq C_2 \mathbb{E}|X|^\beta \mathbb{I}[|X| > b], \end{aligned}$$

where C_1 and C_2 are positive constants. It is also obvious that $\mathbb{E}(|X_n|^\beta) \leq C\mathbb{E}(|X|^\beta)$.

Lemma 3. *(Rosenthal-type inequality) Let $\{X_n, n \geq 1\}$ be a sequence of m-NOD random variables with $EX_n = 0$ and $E|X_n|^p < \infty$, for some $p \geq 1$ and every $n \geq 1$. Then there exist positive constants $C_{m,p}$ and $D_{m,p}$ depending only on m and p such that, for every $n \geq m$,*

$$\mathbb{E} \left| \sum_{i=1}^n X_i \right|^p \leq \begin{cases} C_{m,p} \sum_{i=1}^n \mathbb{E} |X_i|^p & \textit{for} \quad 1 \leq p \leq 2 \\ D_{m,p} \left\{ \sum_{i=1}^n \mathbb{E} |X_i|^p + \left(\sum_{i=1}^n \mathbb{E} X_i^2 \right)^{p/2} \right\} & \textit{for} \quad p > 2 \end{cases} \tag{2.1}$$

In the following, we mention one of the basic inequality for the m-NOD random variables in the form of a lemma.

Lemma 4. *Let $\{X_n, n \geq 1\}$ be a sequence of m-NOD random variables. Then there exists a positive constant C such that, for any $x \geq 0$ and all $n \geq 1$,*

$$\Big(1 - \mathbb{P}\big(\max_{1\leq i\leq n} |X_k| > x\big)\Big)^2 \sum_{i=1}^{n} \mathbb{P}\big(|X_i| > x\big) \leq C\,\mathbb{P}\big(\max_{1\leq i\leq n} |X_i| > x\big).$$

Proof The proof will be based on Lemma 1.10 in [20]. Let $B_i = (|X_i| > x)$ and $\beta_n = 1 - \mathbb{P}\left(\cup_{i=1}^{n} B_i\right)$. Without loss of generality, assume that $\beta_n > 0$. Note that $\{\mathbb{I}\{X_i > x\} - \mathbb{E}\mathbb{I}\{X_i > x\}, i \geq 1\}$ and $\{\mathbb{I}\{X_i < -x\} - \mathbb{E}\mathbb{I}\{X_i < -x\}, i \geq 1\}$ are still m-NOD by Lemma 1. From Lemma 3 for $p = 2$ and C_r inequality, we get

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{n}(\mathbb{I}_{B_i} - \mathbb{E}\mathbb{I}_{B_i})\right)^2 &= \mathbb{E}\left(\sum_{i=1}^{n}(\mathbb{I}_{\{X_i>x\}} - \mathbb{E}\mathbb{I}_{\{X_i>x\}}) + (\mathbb{I}_{\{X_i<-x\}} - \mathbb{E}\mathbb{I}_{\{X_i<-x\}})\right)^2 \\
&\leq 2\mathbb{E}\left(\sum_{i=1}^{n}(\mathbb{I}_{\{X_i>x\}} - \mathbb{E}\mathbb{I}_{\{X_i>x\}})\right)^2 + 2\mathbb{E}\left(\sum_{i=1}^{n}(\mathbb{I}_{\{X_i<-x\}} - \mathbb{E}\mathbb{I}_{\{X_i<-x\}})\right)^2 \\
&\leq C\sum_{i=1}^{n}\mathbb{P}(B_i). \qquad (2.2)
\end{aligned}$$

By (2.2) and the Cauchy-Schwarz inequality, we can write

$$\begin{aligned}
\sum_{i=1}^{n}\mathbb{P}(B_i) &= \sum_{i=1}^{n}\mathbb{P}\left(B_i \cap (\cup_{j=1}^{n}B_j)\right) = \sum_{i=1}^{n}\mathbb{E}\left(\mathbb{I}_{B_i}\mathbb{I}_{(\cup_{j=1}^{n}B_j)}\right) \\
&= \sum_{i=1}^{n}\mathbb{E}(\mathbb{I}_{B_i} - \mathbb{E}\mathbb{I}_{B_i})\mathbb{I}_{(\cup_{j=1}^{n}B_j)} + \sum_{i=1}^{n}\mathbb{E}\mathbb{I}_{B_i}\mathbb{I}_{(\cup_{j=1}^{n}B_j)} \\
&= \mathbb{E}\left(\sum_{i=1}^{n}(\mathbb{I}_{B_i} - \mathbb{E}\mathbb{I}_{B_i})\mathbb{I}_{(\cup_{j=1}^{n}B_j)}\right) + \sum_{i=1}^{n}\mathbb{P}(B_i)\mathbb{P}(\cup_{j=1}^{n}B_j) \\
&\leq \left(\mathbb{E}\left(\sum_{i=1}^{n}(\mathbb{I}_{B_i} - \mathbb{E}\mathbb{I}_{B_i})\right)^2\mathbb{E}\mathbb{I}_{\cup_{j=1}^{n}B_j}\right)^{1/2} + (1-\beta_n)\sum_{i=1}^{n}\mathbb{P}(B_i) \\
&\leq \left(\frac{C(1-\beta_n)}{\beta_n}\beta_n\sum_{i=1}^{n}\mathbb{P}(B_i)\right)^{1/2} + (1-\beta_n)\sum_{i=1}^{n}\mathbb{P}(B_i) \\
&\leq \frac{1}{2}\left(\frac{C(1-\beta_n)}{\beta_n} + \beta_n\sum_{i=1}^{n}\mathbb{P}(B_i)\right) + (1-\beta_n)\sum_{i=1}^{n}\mathbb{P}(B_i).
\end{aligned}$$

Then we get $\beta_n^2\sum_{i=1}^{n}\mathbb{P}(B_i) \leq C(1-\beta_n)$, this completes the proof.

In the following $\{X_n, n \geq 1\}$ is a sequence of m-NOD random variables which is dominated by the random variable Y and we will also use the notations $\hat{X}_i = -\psi(n)\mathbb{I}\{X_i < -\psi(n)\} + X_i\mathbb{I}\{|X_i| \leq \psi(n)\} + \psi(n)\mathbb{I}\{X_i > \psi(n)\}$ and $m(n,i) = \mathbb{E}X_i\mathbb{I}\{|X_i| \leq \psi(n)\}$, for each $i, n \geq 1$.

Lemma 5. *Let $\{X_n, n \geq 1\}$ be a sequence of m-NOD random variables stochastically dominated by a random variable Y. Moreover assume that the functions $g(\cdot), h(\cdot)$ and $\psi(\cdot)$ satisfy the conditions (A1) and (A3) and $\lim_{n\to\infty} n\mathbb{P}(|Y| > \psi(n)) = 0$, then*

$$\lim_{n\to\infty} \frac{1}{g(n)} \left| \sum_{i=1}^{n} \frac{\mathbb{E}(\hat{X}_i) - m(n,i))}{h(i)} \right| = 0.$$

*Proof B*y (A3), Definition 3 and Hölder's inequality we get

$$\begin{aligned}
& \frac{1}{g(n)} \left| \sum_{i=1}^{n} \frac{\mathbb{E}(\hat{X}_i) - \mathbb{E}(X_i \mathbb{I}[|X_i| \leq \psi(n)])}{h(i)} \right| \\
\leq \quad & \frac{1}{g(n)} \sum_{i=1}^{n} \frac{\mathbb{E}(\psi(n)\mathbb{I}[|X_i| > \psi(n)])}{h(i)} \\
= \quad & \frac{1}{g(n)} \sum_{i=1}^{n} \frac{\mathbb{E}(\psi(n)\mathbb{I}[|X_i| > \psi(n))}{h(i)} = h(n) \sum_{i=1}^{n} \frac{\mathbb{P}(|X_i| > \psi(n)])}{h(i)} \\
\leq \quad & Ch(n)\mathbb{P}(|Y| > \psi(n)) \sum_{i=1}^{n} \frac{1}{h(i)} \\
\leq \quad & Ch(n)\mathbb{P}(|Y| > \psi(n)) n^{\frac{\alpha-1}{\alpha}} \left(\sum_{i=1}^{n} \frac{1}{h^{\alpha}(i)} \right)^{\frac{1}{\alpha}} \leq Cn\mathbb{P}(|Y| > \psi(n)) \to 0.
\end{aligned}$$

Let us state our main result.

Theorem 1. *Let $\{X_n, n \geq 1\}$ be a sequence of m-NOD random variables stochastically dominated by a random variable Y. Moreover assume that the functions g, h and ψ satisfy the conditions (A1), (A2) and (A3) and $\mathbb{E}\left(H(\psi^{-1}(|Y|))\right) < \infty$. If for some $r \geq 1$*

$$\mathbb{E}\big(\psi^{-1}(|Y|)\big)^r < \infty. \tag{2.3}$$

Then

$$\sum_{n=1}^{\infty} n^{r-2} \mathbb{P}\left(\left| \sum_{i=1}^{n} \frac{X_i - m(n,i)}{h(i)} \right| > \varepsilon g(n) \right) < \infty \qquad \forall \varepsilon > 0 \quad . \tag{2.4}$$

*Proof F*or any $n \geq 1$, define $S_n = \sum_{i=1}^n \frac{X_i - m(n,i)}{h(i)}$ and $\hat{S}_n = \sum\limits_{i=1}^{n} \frac{\hat{X}_i - m(n,i)}{h(i)}$.

It is easy to see that

$$\begin{aligned}
\{|S_n| > \varepsilon g(n)\} &= \left\{|S_n| > \varepsilon g(n), S_n \neq \hat{S}_n\right\} \cup \left\{|S_n| > \varepsilon g(n), S_n = \hat{S}_n\right\} \\
&\subset \left\{ \bigcup_{i=1}^{n} [|X_i| > \psi(n)] \right\} \cup \left\{ \left|\hat{S}_n\right| > \varepsilon g(n) \right\}.
\end{aligned}$$

Then for every $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2}\mathbb{P}\left(|S_n| > \varepsilon g(n)\right) \\
&\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} n^{r-2}\mathbb{P}\left(|X_i| > \psi(n)\right) + \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}\left(\left|\hat{S}_n\right| > \varepsilon g(n)\right) =: I + II.
\end{aligned}$$

Now, we prove that the series I and II are finite. Since X_n is stochastically dominated by the random variable Y, we obtain for $r \geq 1$

$$\begin{aligned}
I &= \sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n}\mathbb{P}(|X_i| > \psi(n)) \ \leq C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n}\mathbb{P}(|Y| > \psi(n)) \\
&= C\sum_{n=1}^{\infty} n^{r-1}\mathbb{P}(|Y| > \psi(n)) = \ C\sum_{n=1}^{\infty} n^{r-1}\mathbb{P}(\psi^{-1}(|Y|) > n) \\
&\leq C\,\mathbb{E}\big(\psi^{-1}(|Y|)\big)^r < \infty.
\end{aligned}$$

Since $\mathbb{E}(\psi^{-1}(|Y|)) < \infty$, by dominated convergence theorem we can show that $\lim_{n\to\infty} n\mathbb{P}(|Y| > \psi(n)) = 0$, therefore from Lemma 5 and Markov's inequality, we have

$$\begin{aligned}
II &= \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}\left(\left|\sum_{i=1}^{n}\frac{\hat{X}_i - m(n,i)}{h(i)}\right| > \varepsilon g(n)\right) \\
&= \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}\left(\left|\sum_{i=1}^{n}\frac{\hat{X}_i - \mathbb{E}(\hat{X}_i)}{h(i)}\right| + \left|\sum_{i=1}^{n}\frac{\mathbb{E}(\hat{X}_i) - m(n,i)}{h(i)}\right| > \varepsilon g(n)\right) \\
&\leq \sum_{n=1}^{\infty} n^{r-2}\mathbb{P}\left(\left|\sum_{i=1}^{n}\frac{\hat{X}_i - \mathbb{E}(\hat{X}_i)}{h(i)}\right| > \frac{\varepsilon}{2}g(n)\right) \\
&\leq \sum_{n=1}^{\infty}\frac{n^{r-2}}{(\frac{\varepsilon}{2})^{\alpha}g^{\alpha}(n)}\mathbb{E}\left[\left|\sum_{i=1}^{n}\frac{\hat{X}_i - \mathbb{E}(\hat{X}_i)}{h(i)}\right|^{\alpha}\right].
\end{aligned}$$

From (2.1) and Lemma 2, we get

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{n^{r-2}}{(\frac{\varepsilon}{2})^{\alpha}g^{\alpha}(n)}\mathbb{E}\left[\left|\sum_{i=1}^{n}\frac{\hat{X}_i-\mathbb{E}(\hat{X}_i)}{h(i)}\right|^{\alpha}\right]\\
\leq\ & C\sum_{n=1}^{\infty}\frac{n^{r-2}}{g^{\alpha}(n)}\sum_{i=1}^{n}\frac{\mathbb{E}\left|\hat{X}_i-\mathbb{E}(\hat{X}_i)\right|^{\alpha}}{h^{\alpha}(i)}\\
\leq\ & C\sum_{n=1}^{\infty}\frac{n^{r-2}}{g^{\alpha}(n)}\sum_{i=1}^{n}\frac{\mathbb{E}\left|\hat{X}_i\right|^{\alpha}}{h^{\alpha}(i)}\\
\leq\ & C\sum_{n=1}^{\infty}\frac{n^{r-2}}{g^{\alpha}(n)}\sum_{i=1}^{n}\frac{\psi^{\alpha}(n)\mathbb{P}(|Y|>\psi(n))}{h^{\alpha}(i)}+C\sum_{n=1}^{\infty}\frac{n^{r-2}}{g^{\alpha}(n)}\sum_{i=1}^{n}\frac{\mathbb{E}|Y|^{\alpha}\mathbb{I}[|Y|\leq\psi(n)]}{h^{\alpha}(i)}\\
\leq\ & C\sum_{n=1}^{\infty}n^{r-1}\mathbb{P}(|Y|>\psi(n))+C\sum_{n=1}^{\infty}\frac{n^{r-1}\mathbb{E}|Y|^{\alpha}\mathbb{I}[|Y|\leq\psi(n)}{\psi^{\alpha}(n)}\\
\leq\ & C\mathbb{E}\big(\psi^{-1}(|Y|)\big)^{r}+C\sum_{n=1}^{\infty}\frac{n^{r-1}\mathbb{E}|Y|^{\alpha}\mathbb{I}[|Y|\leq\psi(n)}{\psi^{\alpha}(n)}\\
=\ & C\mathbb{E}\big(\psi^{-1}(|Y|)\big)^{r}+C\mathbb{E}\left(\sum_{n=1}^{\infty}\frac{n^{r-1}|Y|^{\alpha}\mathbb{I}\left\{|Y|\leq\psi(n)\right\}}{\psi^{\alpha}(n)}\right)
\end{aligned}
$$

By our assumption, the first part of the last equality is finite. Now, the assumption (A2) allows us to write

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{n^{r-1}|Y|^{\alpha}\mathbb{I}\left\{|Y|\leq\psi(n)\right\}}{\psi^{\alpha}(n)} &= \sum_{n=1}^{\lfloor\psi^{-1}(|Y|)\rfloor+1}\frac{n^{r-1}|Y|^{\alpha}\mathbb{I}\left\{|Y|\leq\psi(n)\right\}}{\psi^{\alpha}(n)}\\
&\quad+\sum_{n=\lfloor\psi^{-1}(|Y|)\rfloor+2}^{\infty}\frac{n^{r-1}|Y|^{\alpha}\mathbb{I}\left\{|Y|\leq\psi(n)\right\}}{\psi^{\alpha}(n)}\\
\leq \sum_{n=1}^{\lfloor\psi^{-1}(|Y|)\rfloor+1}&\big(\left\lfloor\psi^{-1}(|Y|)\right\rfloor+1\big)^{r-1}+2^{r-1}|Y|^{\alpha}\sum_{n=\lfloor\psi^{-1}(|Y|)\rfloor+2}^{\infty}\frac{(n-1)^{r-1}}{\psi^{\alpha}(n)}\\
&\leq\big(\left\lfloor\psi^{-1}(|Y|)\right\rfloor+1\big)^{r}+C|Y|^{\alpha}\int_{\psi^{-1}(|Y|)}^{\infty}\frac{x^{r-1}}{\psi^{\alpha}(n)}dx\\
&\leq 2^{r}\big(\psi^{-1}(|Y|)\big)^{r}+2^{r}+C\left(aH(\psi^{-1}(|Y|))+b\right)
\end{aligned}
$$

which, in the light of (2.3), implies $II<\infty$. The proof is completed.

In what follows we shall use the concept of regularly varying functions (see [3]).

Definition 4. *A measurable function* $U:[a,\infty)\to(0,\infty)$, $a\in\mathbb{R}$, *is called regularly varying at infinity with exponent* ρ, *denoted as* $U(\cdot)\in\mathcal{RV}(\rho)$, *if for all* $t>0$,

$$\lim_{x\to\infty}\frac{U(tx)}{U(x)}=t^{\rho}.$$

If $\rho=0$ *then we say that* U *is slowly varying at infinity and write* $U\in\mathcal{SV}$.

Theorem 2. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed m-NOD random variables such that $\mathbb{P}\left(|X_k| > x\right) \in \mathcal{RV}(\rho)$. Moreover assume that the functions g, h and ψ satisfy the condition (A1). If*

$$\sum_{n=1}^{\infty} n^{r-2} \mathbb{P}\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} \frac{X_i - m(n,i)}{h(i)} \right| > \varepsilon g(n) \right) < \infty, \qquad \forall \varepsilon > 0 \tag{2.5}$$

then

$$\mathbb{E}\big[\psi^{-1}(|Y|)\big]^r < \infty \; for \; r \geq 2. \tag{2.6}$$

*Proof B*y a similar proof as the proof of Theorem 1 in [12], the desired results can be obtained. Let us recall that $m(n,1) = \mathbb{E}X_1\mathbb{I}\left[|X_1| \leq \psi(n)\right]$. Since $h(\cdot)$ is nondecreasing we have

$$\max_{1\leq k\leq n} \frac{|X_k - m(n,1)|}{\psi(n)} \leq \max_{1\leq k\leq n} \frac{1}{\mathrm{g}(n)} \frac{|X_k - m(n,1)|}{h(k)} \leq 2 \max_{1\leq k\leq n} \frac{1}{\mathrm{g}(n)} |S_k|.$$

Therefore

$$\mathbb{P}\bigg(\max_{1\leq k\leq n} |X_k - m(n,1)| > \psi(n)\varepsilon \bigg) \leq \mathbb{P}\bigg(\max_{1\leq k\leq n} |S_k| > \frac{\varepsilon}{2}\mathrm{g}(n) \bigg). \tag{2.7}$$

Because of that $|m(n,1)| \leq \psi(n)$, making use of the inequality $|x - y| \geq |x| - |y|$ for $x, y \in \mathbb{R}$ we get

$$\begin{aligned} \left\{ \max_{1\leq k\leq n} |X_k| > (\varepsilon+1)\psi(n) \right\} &\subset \left\{ \max_{1\leq k\leq n} |X_k| - |m(n,1)| > \varepsilon\psi(n) \right\} \\ &\subset \left\{ \max_{1\leq k\leq n} |X_k - m(n,1)| > \varepsilon\psi(n) \right\}, \end{aligned} \tag{2.8}$$

from (2.7) and (2.8), we have

$$\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| > \frac{\varepsilon}{2}\mathrm{g}(n) \right) \geq \mathbb{P}\left(\max_{1\leq k\leq n} |X_k| > (\varepsilon+1)\psi(n) \right). \tag{2.9}$$

It follows immediately that (2.5) implies

$$\mathbb{P}\left(\max_{1\leq k\leq n} |X_k| > (\varepsilon+1)\psi(n) \right) \to 0, \; as \; n \to \infty.$$

Thus, for sufficiently large n

$$\mathbb{P}\left(\max_{1\leq k\leq n} |X_k| > (\varepsilon+1)\psi(n) \right) < \frac{1}{2}.$$

According to Lemma 4 and (2.9) we obtain

$$\sum_{k=1}^{n}\mathbb{P}\left(|X_k|>(\varepsilon+1)\psi(n)\right)\le\frac{C\,\mathbb{P}\left(\max\limits_{1\le k\le n}|X_k|>(\varepsilon+1)\psi(n)\right)}{\left(1-\mathbb{P}\left(\max\limits_{1\le k\le n}|X_k|>(\varepsilon+1)\psi(n)\right)\right)^2}$$
$$\le 4C\,\mathbb{P}\left(\max_{1\le k\le n}|X_k|>(\varepsilon+1)\psi(n)\right)\qquad(2.10)$$
$$\le 4C\mathbb{P}\left(\max_{1\le k\le n}|S_k|>\frac{(\varepsilon+1)}{2}g(n)\right).$$

Also, since $\mathbb{P}\left(|X_k|>x\right)\in\mathcal{RV}(\rho)$ we have

$$\mathbb{P}\big(\,|X_1|>(\varepsilon+1)\psi(n)\big)\sim(\varepsilon+1)^{\rho}\,\mathbb{P}\big(\,|X_1|>\psi(n)\big).$$

Now, by using (2.5) and (2.10), we get

$$\sum_{n=1}^{\infty}n^{r-2}\sum_{k=1}^{n}\mathbb{P}\left(|X_k|>\psi(n)\right)\ \sim\ \sum_{n=1}^{\infty}n^{r-2}\sum_{k=1}^{n}\mathbb{P}\left(|X_k|>(\varepsilon+1)\psi(n)\right)$$
$$=\ \sum_{n=1}^{\infty}n^{r-1}\mathbb{P}\left(|X_1|>\psi(n)\right)<\infty,$$

which implies (2.6) and completes the proof.

Let us present some functions satisfying the assumptions (A1)-(A3).

Remark 1. Let us take the power functions $h(x)=x^p$, $g(x)=x^q$ and $H(x)=x^r$, then all the requirements (A1)-(A3) are valid with $0\le\alpha p<1$ and $\alpha(p+q)\ge r$. Using Proposition 1.5.10 in [3] we can extend this example of weights to $h(x)=x^p$ and $g(x)=x^qL^{p+q}(x)$, where p,q satisfy the above constraints and $L(x)$ is a slowly varying function.

From this remark we get the following corollary.

Corollary 1. *Let $\{X_n,n\ge1\}$ be a sequence of m-NOD random variables stochastically dominated by a random variable Y and $0<\beta\le2$. If $\mathbb{E}|Y|^{\beta}<\infty$, then*

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\left|\sum_{i=1}^{n}X_i-\mathbb{E}X_i\mathbb{I}[|X_i|\le n^{2/\beta}]\right|>\varepsilon n^{2/\beta}\right)<\infty.$$

Conversely, if $\sum_{n=1}^{\infty}\mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}X_i-\mathbb{E}X_i\mathbb{I}[|X_i|\le n^{2/\beta}]\right|>\varepsilon n^{2/\beta}\right)<\infty$ and $\mathbb{P}\left(|X_k|>x\right)\in\mathcal{RV}(\rho)$ for $k\ge1$, then $\mathbb{E}|Y|^{\beta}<\infty$.

*Proof I*t is enough, we use the Theorem 1 and Theorem 2 for $r=2$, and functions $h(x)=1$, $g(x)=x^{2/\beta}$, $\psi(x)=x^{2/\beta}$ and $H(x)=x^2$.

Now, following Theorem 11.2 of [6] we can restate type the Hsu-Robbins theorem for m-NOD sequences.

Corollary 2. *Let $\{X_n,n\ge1\}$ be a sequence of identically distributed m-NOD random variables. If $\mathbb{E}X=0$ and $\mathbb{E}X^2<\infty$, then*

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\left|\sum_{k=1}^{n}X_k\right|\ge\varepsilon n\right)<\infty,\ \textit{for all }\varepsilon>0.$$

*Proof T*o prove, we apply Corollary 1 with $\beta = 2$ and we get

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - \mathbb{E}X\mathbb{I}[|X| \le n]) \to 0 \; c.c., \; n \to \infty,$$

since $\mathbb{E}X = 0$ and $\mathbb{E}X^2 < \infty$, by the dominated convergence theorem $\mathbb{E}X\mathbb{I}[|X| \le n] \to \mathbb{E}X = 0$ as $n \to \infty$, hence , $n^{-1}\sum_{k=1}^{n}\mathbb{E}X\mathbb{I}[|X| \le n] \to 0$ as $n \to \infty$, and we get the conclusion.

3 Simulation study

In this section, we illustrate the efficiency and rate of complete convergence in Theorem 1 through two numerical examples. According to the Remark 1, we set $h(n) = n^p$, $g(n) = n^q$ and $H(x) = x^r$ (where $p = 0.5$, $q = 1$, $r \ge 1$, $\alpha = 2$) in Theorem 1, and for each $r = 1, 2, 3$, we take the sample size $n = 3(1)200$. For each n, we simulate m-NOD random variables $X_1 = x_1, ..., X_n = x_n$ for $m = 1$ in Example 1 and $m = 2$ in Example 2. We then compute $s_n = \frac{1}{n^q}\left|\sum_{i=1}^{n}\frac{x_i}{i^p}\right|$. By repeating this procedure $B = 20000$ times, we observe the vector $\left\{S_n^1, ..., S_n^{B=20000}\right\}$ and finally compute $P_n = \frac{1}{B}\sum_{i=1}^{B} I\{S_n^i > \varepsilon\}$ as an estimation of $\mathbb{P}(\frac{1}{n^q}\left|\sum_{i=1}^{n}\frac{x_i}{i^p}\right| > \varepsilon)$. Now by taking the cumulative sum of $n^{r-2}P_n$'s and plotting the scatter plots of $(n, \sum_{j=1}^{n} n^{r-2}\mathbb{P}(\frac{1}{j^q}\left|\sum_{i=1}^{j}\frac{x_i}{i^p}\right| > \varepsilon))$, we can analyze the behavior of complete convergence.

Example 1. In this example to create an $m - NOD$ sequence of random variables with $m = 1$ we use of multivariate normal distribution. For any fixed $n \ge 3$, we take a n-dimensional random vector $\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \sim$ $N_n(\underline{0}, \Sigma)$ where $\underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1}$ represents a zero vector and covariance matrix

$$\sum = \begin{pmatrix} 1+\theta^2 & -\theta & 0 & \cdots & 0 & 0 & 0 \\ -\theta & 1+\theta^2 & -\theta & \cdots & 0 & 0 & 0 \\ 0 & -\theta & 1+\theta^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\theta^2 & -\theta & \\ 0 & 0 & 0 & \cdots & -\theta & 1+\theta^2 & -\theta \\ 0 & 0 & 0 & \cdots & 0 & -\theta & 1+\theta^2 \end{pmatrix}_{n\times n},$$

where $0 < \theta < 1$ (we take $\theta = 0.5$). From [9] it is obvious that $\{X_n, n \ge 1\}$ is a NOD sequence (m-NOD with $m = 1$) and we can see that this sequence is stochastically dominated by the random variable Y where $Y \sim N(0, 1+\theta^2)$. It is clear that $\mathbb{E}(H\left[\psi^{-1}(|Y|)\right]) = \mathbb{E}(\psi^{-1}(|Y|))^r = \mathbb{E}(|Y|^{\frac{r}{p+q}}) < \infty$. Now all the conditions of Theorem 1 are satisfied and we can easily show that for each $n \ge 3$ and $1 \le i \le n$, $m(n,i) = 0$. The results of this example are shown in the first part of Figure 1.

Example 2. In this example, we proceed exactly as in Example 1, with the difference that the covariance matrix of the multivariate normal distribution will be as

$$\sum = \begin{pmatrix} 1-\theta^2 & 0 & -\theta & 0 & 0 & \dots & 0 \\ 0 & 1-\theta^2 & 0 & -\theta & 0 & \dots & 0 \\ -\theta & 0 & 1-\theta^2 & 0 & -\theta & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & -\theta & 0 & 1-\theta^2 & 0 & -\theta \\ 0 & \dots & 0 & -\theta & 0 & 1-\theta^2 & 0 \\ 0 & \dots & 0 & 0 & -\theta & 0 & 1-\theta^2 \end{pmatrix}_{n\times n},$$

to create an $m-NOD$ sequence of random variables with $m=2$. The results of this example are shown in the second part of Figure 1.

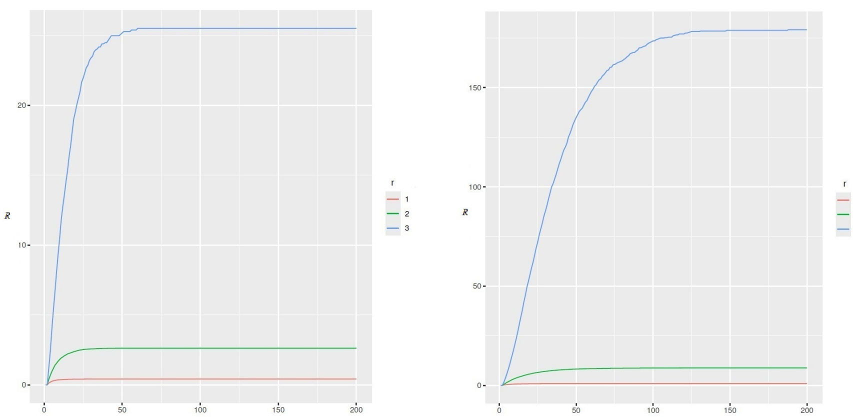

Figure 1.

Figure 1 exhibits the scatter plots of $(n, R = \sum_{j=1}^{n} n^{r-2}\mathbb{P}(\frac{1}{j^q}\left|\sum_{i=1}^{j}\frac{x_i}{i^p}\right| > \varepsilon))$ for $r=1,2,3$. It is observed that R is a increasing function of n but tends to a fixed value and is dominated to it for each $r=1,2,3$.

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