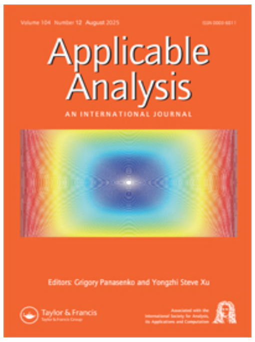

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Well-posedness and dynamical properties for a class of plate equation

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ABSTRACT

In this work, we consider an initial-boundary value problem of a plate equation in a bounded domain of $\mathbb{R}^n$, with memory and the time-weighted function $\alpha(t)$. We apply the Faedo-Galerkin method and the contraction mapping principle to establish the local existence of weak solutions. Subsequently, we explore the dynamics of these weak solutions, focusing on global existence and finite-time blow-up, using the Nehari manifold and modified concavity arguments. Furthermore, we derive the upper and lower bounds for the blow-up time of solutions with high-energy levels.

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1. Introduction

In this work, we are concerned with the following initial and boundary value problem

$$\begin{cases} u_{tt} + \Delta^2 u + u + \alpha(t)(g * \Delta u) + \Delta^2 u_{tt} = |u|^{p-2}u \ln|u|, & (x,t) \in \Omega \times \mathbb{R}^+ \\ u(x,t) = \dfrac{\partial u}{\partial \nu}(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is an open-bounded domain with a smooth boundary $\partial\Omega$, ν is the unit outer normal to $\partial\Omega$, $u_0(x), u_1(x)$ are given initial data and $g * \Delta u = \int_0^t g(t-s)\Delta u(s)\,ds$. The exponent p satisfies

$$2 < p < +\infty \quad \text{if } n \leq 4; \quad 2 < p < \frac{2(n-2)}{n-4} \quad \text{if } n \geq 5. \tag{2}$$

The study of plate equations has been extensively explored over the years, owing to their importance in several areas of physics, particularly in the theories of elasticity within solid mechanics and continuum mechanics [1].

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For plate models, the pioneering work of Berger [2] can be cited. Numerous significant solutions have been established for plate equations. For instance, Sugitani and Kawashima [3] examined the following initial value problem

$$\begin{aligned} &u_{tt} - \Delta u_{tt} + \Delta^2 u + u_t = f(u_t), && x \in \mathbb{R}^n,\ t > 0 \\ &u(x,0) = u_0(x), && u_t(x,0) = u_1(x). \end{aligned} \tag{3}$$

The authors established the global existence and optimal decay of solutions by imposing regularity conditions on the initial data and introducing a set of time-weighted Sobolev spaces. Liu and Kawashima [4] extended the results in [3] to a semi-linear plate equation with memory in multiple dimensions ($n \geq 1$):

$$u_{tt} + \Delta^2 u + u + g * \Delta u = f(u),$$

and proved the global in time existence and optimal decay estimates of solutions. For further studies related to plate equations, we refer to [5–10].

The memory term $g * \Delta u$, as discussed in [4, 7, 10], is weaker than the linear frictional damping term u_t in the model (3). This weaker dissipative mechanism is evident in the decay structure of the solutions.

Liu et al. [11] considered the following initial-boundary value problem with plate equation

$$\begin{aligned} &u_{tt} + \Delta^2 u + u + g * \Delta u = |u|^{p-2}u, && x \in \Omega, \quad t > 0 \\ &u = \frac{\partial u}{\partial \nu} = 0, && x \in \partial\Omega, \quad t > 0 \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), && x \in \Omega. \end{aligned} \tag{4}$$

The authors proved the local well-posedness of the solutions using the Faedo-Galerkin method. They established the existence of global solutions and provided an upper bound estimation for the blow-up time.

Li et al. [12] considered the stabilization of a weak viscoelastic wave equation with variable coefficients with an interior delay term:

$$\begin{aligned} &u_{tt} - Au - \alpha(t)(g * Au) = -\mu u_t(x, t-\tau) \quad \text{in } \Omega \times (0,\infty) \\ &u = 0 \quad \text{on } \Gamma_0 \times (0,\infty) \\ &\partial_{\nu_A} u - \alpha(t)(g * \partial_{\nu_A} u) = -\ell u_t \quad \text{on } \Gamma_1 \times (0,\infty) \\ &u_t(x, t-\tau) = f_0(x, t-\tau) \quad x \in \Omega, \end{aligned}$$

where $Au = -\sum_{i,j} = \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j})$, and they proved exponential decay rates for the energy, which depends on factors, such as the geometry, viscoelastic effects, the strength of the delay and the intensity of the mechanical boundary damping.

The logarithmic nonlinearity is of significant interest in physics, as it naturally emerges in various domains, suchas inflationary cosmology, supersymmetric field theories, quantum mechanics and nuclear physics [13, 14].

Al-Gharabli and Messaoudi [15] considered the plate equation

$$u_{tt} + \Delta^2 u + u + h(u_t) = u \ln |u|^k.$$

The authors established the global existence and decay rate of the solution's energy through the application of the multiplier method. Al-Gharabli et al. [16] extended the results in [15] to the following plate equation

$$u_{tt} + \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 u(s)\,\mathrm{d}s = u \ln |u|^k.$$

For more works related to plate equations with logarithmic source terms, we refer to [17–22].

Tahamtani et al. [23] considered the equation

$$u_{tt} - \Delta u + \alpha(t)\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s + u_t + u = u \ln |u|^k,$$

with acoustic boundary and initial conditions. The term $\alpha(t)\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s$ is called 'weak viscoelastic' when it comes with the time-weighted function $\alpha(t)$, which is considered as a dissipative term and stabilizes the system. The authors proved a global existence and general decay of solutions for appropriately chosen initial data. In this regard, specifically concerning weak viscoelastic equations, we refer to [24, 25].

Motivated by the above mentioned papers, our purpose in this research is to investigate the local well-posedness of solutions to the problem (1) with the kernel g and the time-weighted function α (see hypothesis (**H**)) using the Faedo-Galerkin approximation method and a the contraction mapping principle. We further study the dynamics of solutions, including the global existence and finite time blow-up of solutions for initial data in the stable set created by the Nehari manifold. Finally, by utilizing the modified concavity argument, we establish the blow-up of solutions with high initial energy and obtain upper estimate for the blow-up time (Theorem 2.3). Alongside this, we also provide a lower bound for the blow-up time.

We generalize the results in [11] in the following directions:

- The major nonlinearity in the governing problem (1) with the nonlinear logarithmic source term and weak memory term.
- Compared to the existing literature results and [11], this paper address a notable gap in the investigating the finite time blow-up phenomenon for high initial energy and give the upper and lower bounds for the blow-up time.

The organization of this paper is as follows. In Section 2, we introduce some notations, definitions and lemmas that will be used in the sequel. Next, in Section 3, using the Faedo-Galerkin approximation method and a contraction mapping principle, we prove the local existence of the solution. Moreover, the global existence result has been established in Section 4. In Section 5, the finite time blow-up of solutions for high initial energy will be considered. Finally, the upper and lower bounds of the blow-up times are derived by combining the concavity method.

2. Preliminaries

In this section, we present some materials that will be used throughout of this work. We use the standard Lebesgue space $L^2(\Omega)$-inner product $(\cdot,\cdot)$ and the Sobolev space $H_0^2(\Omega)$ with their usual scalar products and norms, $\|\phi\|_{H_0^2} := \sqrt{\|\phi\|_2^2 + \|\Delta\phi\|_2^2}$. By $\langle\cdot,\cdot\rangle$, we will represent the duality pairing between $H_0^2(\Omega)$ and $H^{-2}(\Omega)$.

To deal with the problem (1), we will make the following hypothesis:

(H) $g, \alpha : R^+ \to R^+$ are non-increasing differentiable functions satisfying

$$g(0) > 0, \quad 1 - C_1^2\alpha(t)\int_0^t g(s)\,\mathrm{d}s > \ell > 0, \quad \int_0^\infty g(s)\,\mathrm{d}s < +\infty,$$

$$\alpha(t) > 0 \quad \forall\, t > 0, \quad \lim_{t\to+\infty} \frac{-\alpha'(t)}{\alpha(t)} = 0.$$

Here, we assume that C_1 is the optimal embedding constant of $H_0^2(\Omega) \hookrightarrow H_0^1(\Omega)$, i.e.

$$C_1 = \sup_{\|\phi\|_{H_0^2}\neq 0} \frac{\|\nabla\phi\|_2}{\sqrt{\|\phi\|_2^2 + \|\Delta\phi\|_2^2}} \tag{5}$$

for any ϕ in a Hilbert space $H_0^2(\Omega)$.

We introduce the modified energy, potential energy and Nehari functionals sequentially

$$\begin{aligned} E(t) = {} & \frac{1}{2}\left(\|u_t\|_{H_0^2}^2 + \|u\|_{H_0^2}^2 + \alpha(t)(g\circ\nabla u)(t) - \alpha(t)\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2\right) \\ & - \frac{1}{p}\int_\Omega |u|^p \ln|u|\,\mathrm{d}x + \frac{1}{p^2}\|u\|_p^p, \end{aligned} \tag{6}$$

$$\begin{aligned} J(u(t)) = {} & \frac{1}{2}\left(\|u\|_{H_0^2}^2 + \alpha(t)(g\circ\nabla u)(t) - \alpha(t)\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2\right) \\ & - \frac{1}{p}\int_\Omega |u|^p \ln|u|\,\mathrm{d}x + \frac{1}{p^2}\|u\|_p^p, \end{aligned} \tag{7}$$

$$\begin{aligned} I(u(t)) = {} & \|u\|_{H_0^2}^2 + \alpha(t)(g\circ\nabla u)(t) - \alpha(t)\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 \\ & - \int_\Omega |u|^p \ln|u|\,\mathrm{d}x, \end{aligned} \tag{8}$$

where $(g\circ\nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2\,\mathrm{d}s$. By a direct computation, we obtain

$$E(t) = \frac{1}{2}\|u_t\|_{H_0^2}^2 + J(u(t)), \tag{9}$$

and

$$\begin{aligned} J(u(t)) = {} & \frac{p-2}{2p}\left(\|u\|_{H_0^2}^2 + \alpha(t)(g\circ\nabla u)(t) - \alpha(t)\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2\right) \\ & + \frac{1}{p^2}\|u\|_p^p + \frac{1}{p}I(u(t)), \end{aligned} \tag{10}$$

According to Nehari functional, we define the potential well

$$\mathcal{N}^{+} = \{u \in H_0^2(\Omega); I(u) > 0\},$$

the complement of potential well

$$\mathcal{N}^{-} = \{u \in H_0^2(\Omega); I(u) < 0\},$$

and the Nehari manifold

$$\mathcal{N} = \{u \in H_0^2(\Omega) \setminus \{0\}; I(u) = 0\}.$$

Lemma 2.1: *Let $u = u(x,t)$ be the solution of (1), then the energy functional $E(t)$ satisfies the following inequality*

$$E'(t) \le -\frac{1}{2}\alpha(t)g(t)\|\nabla u\|_2^2 - \frac{1}{2}\alpha'(t)\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2. \tag{11}$$

Proof: By multiplying the first equation of (1) in u_t and integrating it over $\Omega \times (0,t)$, we obtain

$$E(t) + \frac{1}{2}\int_0^t \alpha(\tau)g(\tau)\|\nabla u\|_2^2\,\mathrm{d}\tau + \frac{1}{2}\int_0^t \alpha'(\tau)\left(\int_0^\tau g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2\,\mathrm{d}\tau$$
$$\le E(0),$$

for all $0 \le t \le T$, that holds (11) where

$$E(0) = \frac{1}{2}\left(\|u_1\|_{H_0^2}^2 + \|u_0\|_{H_0^2}^2\right) - \frac{1}{p}\int_\Omega |u_0|^p \ln|u_0|\,\mathrm{d}x + \frac{1}{p^2}\|u_0\|_p^p,$$

and thus the proof is completed. ∎

We note that $-\alpha'(t)(\int_0^t g(s)\,\mathrm{d}s)\|\nabla u\|_2^2 > 0$ (see (**H**)) maybe caused that $E(t)$ not to be a non-increasing function. Similar to the methods used in [26] (Remark 1) and [23] (Lemma 2.1), we can drop out this term from (11) and state the following Lemma. For details of the proof, see Lemma 2.1 in [23].

Lemma 2.2: *Suppose that* (**H**) *holds. Then, for $(u_0,u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$, $E(t)$ is a non-increasing function.*

The following theorem will be used for proving the blow-up result in Section 5.

Theorem 2.3: *[27] Assume that $\psi(t) \in C^2([0,T])$ is a positive function, satisfying the following inequality*

$$\psi\psi'' - \tilde{\alpha}(\psi')^2 + \tilde{\gamma}\psi'\psi + \tilde{\beta}\psi \ge 0, \quad \tilde{\alpha} > 1, \tilde{\beta} \ge 0, \tilde{\gamma} \ge 0,$$

and $\psi(0) > 0$. If

$$\psi'(0) > \frac{\tilde{\gamma}}{\tilde{\alpha}-1}\psi(0),$$

$$\left(\psi'(0) - \frac{\tilde{\gamma}}{\tilde{\alpha}-1}\psi(0)\right)^2 > \frac{2\tilde{\beta}}{2\tilde{\alpha}-1}\psi(0).$$

Then

$$\psi(t) \to +\infty \quad \text{as} \quad t \to T^* \le \psi^{1-\tilde{\alpha}}(0)A^{-1},$$
$$A^2 \equiv (\tilde{\alpha}-1)^2\psi^{-2\tilde{\alpha}}(0)\left[\left(\psi'(0) - \frac{\tilde{\gamma}}{\tilde{\alpha}-1}\psi(0)\right)^2 - \frac{2\tilde{\beta}}{2\tilde{\alpha}-1}\psi(0)\right].$$

Definition 2.4: For $T > 0$, a function

$$u \in C([0,T]; H_0^2(\Omega)) \cap C^1([0,T]; H_0^2(\Omega)) \cap C^2([0,T]; H^{-2}(\Omega))$$

is called a weak solution of the problem (1) if $u(x,0) = u_0(x)$ in $H_0^2(\Omega)$, $u_t(x,0) = u_1(x)$ in $H_0^2(\Omega)$, and for $t \in [0,T]$, the equality

$$\begin{aligned} &\langle u_{tt}, \phi\rangle + (\Delta u, \Delta\phi) + (u,\phi) + \alpha(t)(g * \Delta u, \phi) + (\Delta u_{tt}, \Delta\phi) \\ &\quad = (|u|^{p-2}u \ln|u|, \phi), \end{aligned} \tag{12}$$

holds for any $\phi \in H_0^2(\Omega)$.

Furthermore, the solution u can be extended to $[0, T_{\max})$ where $T_{\max}$ is the maximal existence time. We say that the solution u is either global when $T_{\max} = +\infty$ or it blows up in a finite time when $T_{\max} < +\infty$.

3. Local existence

In this section, we use the Faedo-Galerkin approximation method and a Contraction Mapping Principle to show that the local existence and uniqueness of solutions for problem (1).

First, we define the space

$$\mathcal{H} = C([0,T]; H_0^2(\Omega)) \cap C^1([0,T]; H_0^2(\Omega))$$

equipped with the norm

$$\|v\|_{\mathcal{H}}^2 = \max_{0\le t\le T}\left(\|v_t\|_{H_0^2}^2 + \ell\|v\|_{H_0^2}^2\right)$$

and establish the following lemma:

Lemma 3.1: *Suppose that* (**H**) *holds. For* $(u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$ *and* $v \in \mathcal{H}$, *there is a unique solution* $u \in \mathcal{H} \cap C^2([0,T]; H^{-2}(\Omega))$ *and* $u_t \in L^2([0,T]; H_0^2(\Omega))$ *that solves*

$$\begin{aligned} &u_{tt} + \Delta^2 u + u + \alpha(t)(g * \Delta u) + \Delta^2 u_{tt} = |v|^{p-2}v\ln|v|, \quad (x,t) \in \Omega \times \mathbb{R}^+ \\ &u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+ \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega \end{aligned} \tag{13}$$

Existence. To establish the existence of weak solutions for the problem (13), we use the Faedo-Galerkin approximations. $\{w_j\}_{j\in\mathbb{N}}$ is an orthogonal basis of the separable space $H_0^2(\Omega)$.

$\mathcal{W}_m = \text{span}\{w_1, w_2, \ldots, w_m\}$, where $\{w_j\}$ is the complete orthogonal system of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$ and $\|w_j\| = 1$ for all j, i.e. $\Delta w_j + \lambda_j w_j = 0$, where λ_j is the related eigenvalues in $H_0^1(\Omega)$. The projection of the initial data on the finite dimensional subspace $\mathcal{W}_m$ is given by

$$u_0^m = \sum_{j=1}^{m} a_j w_j, \quad u_1^m = \sum_{j=1}^{m} b_j w_j,$$

where

$$u_0^m \to u_0 \quad \text{in } H_0^2(\Omega), \quad u_1^m \to u_1 \quad \text{in } H_0^2(\Omega), \quad \text{as } m \to \infty.$$

Let us construct

$$u^m(x,t) = \sum_{j=1}^{m} h_{jm}(t) w_j(x) \tag{14}$$

for $j = 1, 2, \ldots, m$ solves the problem

$$\begin{aligned} &\langle u_{tt}^m, w_j\rangle + (\Delta u^m, \Delta w_j) + (u^m, w_j) + \alpha(t)(g * \Delta u^m, w_j) + (\Delta u_{tt}^m, \Delta w_j) \\ &\quad = (|v|^{p-2} v \ln|v|, w_j), \end{aligned} \tag{15}$$

$$u^m(x,0) = \sum_{j=1}^{m} a_j w_j \to u_0 \quad \text{strongly as } m \to \infty, \tag{16}$$

$$u_t^m(x,0) = \sum_{j=1}^{m} b_j w_j \to u_1 \quad \text{strongly as } m \to \infty. \tag{17}$$

Inserting (14) into (15)–(17), we get the following Cauchy problem to the ordinary differential equation in terms of h_{jm} for $j = 1, 2, \ldots, m$:

$$\begin{aligned} &(1+\lambda_j^2) h_{jm}''(t) + (1+\lambda_j^2) h_{jm}(t) - \lambda_j \alpha(t)(g * h_{jm}) = (|v|^{p-2} v \ln|v|, w_j), \\ &h_{jm}(0) = a_j, \quad h_{jm}'(0) = b_j. \end{aligned} \tag{18}$$

Based on the theory of ordinary differential equations, Peano's theorem, for each m, there is $t_m > 0$ such that the problem (18) admits a solution $h_{jm} \in C^2[0, t_m]$ and, therefore, $u_m \in C^2([0, t_m]; H_0^2(\Omega))$.

We show $t_m = T$, and the local solution is uniformly bounded independently of m and t. Multiplying (15) by $h_{jm}'(t)$ and sum from $j = 1$ to m, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t^m\|_{H_0^2}^2 + \|u^m\|_{H_0^2}^2 + \alpha(t)(g \circ \nabla u^m)(t) - \alpha(t)\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u^m\|_2^2\right) \\ &\quad = \frac{1}{2}\left(\alpha(t)(g' \circ \nabla u^m)(t) + \alpha'(t)(g \circ \nabla u^m)(t) - \alpha(t) g(t) \|\nabla u^m\|_2^2\right. \end{aligned}$$

$$-\alpha'(t)\left(\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u^m\|_2^2\right)+\int_\Omega |v|^{p-2}v\ln|v|u_t^m\,\mathrm{d}x. \tag{19}$$

Integrating (19) on $(0,t)$, using Sobolev embedding (5) and (**H**), yields

$$\begin{aligned}&\|u_t^m\|_{H_0^2}^2+\ell\|u^m\|_{H_0^2}^2+\alpha(t)(g\circ\nabla u^m)(t)\\&\quad\le-\int_0^t\left(\alpha(\tau)g(\tau)+\alpha'(\tau)\int_0^\tau g(s)\,\mathrm{d}s\right)\mathrm{d}\tau\,\|\nabla u^m\|_2^2\\&\qquad+\|u_1^m\|_{H_0^2}^2+\|u_0^m\|_{H_0^2}^2+2\int_0^t\int_\Omega |v|^{p-2}v\ln|v|u_\tau^m\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

From the fact that $-\int_0^t(\alpha(\tau)g(\tau)+\alpha'(\tau)\int_0^\tau g(s)\,\mathrm{d}s)\,\mathrm{d}\tau\le 0$, it follows from above inequality that

$$\begin{aligned}&\|u_t^m\|_{H_0^2}^2+\ell\|u^m\|_{H_0^2}^2+\alpha(t)(g\circ\nabla u^m)(t)\\&\quad\le\|u_1^m\|_{H_0^2}^2+\|u_0^m\|_{H_0^2}^2+2\int_0^t\int_\Omega |v|^{p-2}v\ln|v|u_\tau^m\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{20}$$

To estimate the last term in the right-hand side of (20), we use the inequalities

$$|\phi^{p-1}\ln\phi||_{0<\phi<1}\le\frac{1}{e(p-1)};\quad \phi^{-\mu}\ln\phi|_{\phi\ge1}\le\frac{1}{e\mu}$$

as follows

$$\begin{aligned}\||v|^{p-2}v\ln|v|\|_2^2&=\int_{\{x\in\Omega,|v|<1\}}||v|^{p-2}v\ln|v||^2\,\mathrm{d}x+\int_{\{x\in\Omega,|v|\ge1\}}||v|^{p-2}v\ln|v||^2\,\mathrm{d}x\\&\le\frac{|\Omega|}{(e(p-1))^2}+\frac{1}{(e\mu)^2}\|v\|_{2(p-1+\mu)}^{2(p-1+\mu)}\\&\le\frac{|\Omega|}{(e(p-1))^2}+\frac{C_{2(p-1+\mu)}^{2(p-1+\mu)}}{(e\mu)^2}\|v\|_{H_0^2}^{2(p-1+\mu)},\end{aligned} \tag{21}$$

where $C_{2(p-1+\mu)}^{2(p-1+\mu)}$ is the optimal embedding constant of $H_0^2(\Omega)\hookrightarrow L^{2(p-1+\mu)}(\Omega)$. Here, we choose $0<\mu<\frac{n}{n-4}-(p-1)$.

By (21), Hölder inequality and Young's inequality into (20), we deduce

$$\begin{aligned}&\|u_t^m\|_{H_0^2}^2+\ell\|u^m\|_{H_0^2}^2+\alpha(t)(g\circ\nabla u^m)(t)\\&\le\|u_1^m\|_{H_0^2}^2+\|u_0^m\|_{H_0^2}^2+2\int_0^t\left[\||v|^{p-2}v\ln|v|\|_2^{\frac{2}{p}}\cdot\|u_\tau^m\|_2^{\frac{2}{p}}\right]^{\frac{p}{2}}\mathrm{d}\tau\\&\le\|u_1^m\|_{H_0^2}^2+\|u_0^m\|_{H_0^2}^2+2\int_0^t\left[\frac{p-1}{p}\||v|^{p-2}v\ln|v|\|_2^{\frac{2}{p-1}}+\frac{1}{p}\|u_\tau^m\|_2^2\right]^{\frac{p}{2}}\mathrm{d}\tau\\&\le\|u_1^m\|_{H_0^2}^2+\|u_0^m\|_{H_0^2}^2\end{aligned}$$

$$+2\int_0^t \left[\frac{p-1}{p}\left(\frac{|\Omega|}{(e(p-1))^2} + \frac{C_{2(p-1+\mu)}^{2(p-1+\mu)}}{(e\mu)^2}\|v\|_{H_0^2}^{2(p-1+\mu)} \right)^{\frac{1}{p-1}} + \frac{1}{p}\|u_\tau^m\|_2^2 \right]^{\frac{p}{2}} \mathrm{d}\tau$$
$$\leq \|u_1^m\|_{H_0^2}^2 + \|u_0^m\|_{H_0^2}^2 + \beta t$$
$$+ \frac{2^{\frac{p}{2}}}{p}\int_0^t \left[\|u_\tau^m\|_{H_0^2}^2 + \ell\|u^m\|_{H_0^2}^2 + \alpha(\tau)(g\circ\nabla u^m)(\tau) \right]^{\frac{p}{2}} \mathrm{d}\tau, \tag{22}$$

where

$$\beta = 2^{\frac{p}{2}}\left[\frac{p-1}{p}\left(\frac{|\Omega|}{(e(p-1))^2} + \frac{C_{2(p-1+\mu)}^{2(p-1+\mu)}}{(e\mu)^2}\|v\|_{H_0^2}^{2(p-1+\mu)} \right) \right]^{\frac{p}{2(p-1)}}.$$

Here we used the fact that $v \in \mathcal{H}$.

$$\mathcal{K}_1(t) = \|u_t^m\|_{H_0^2}^2 + \ell\|u^m\|_{H_0^2}^2 + \alpha(t)(g\circ\nabla u^m)(t),$$
$$\mathcal{K}_1(0) = \|u_1^m\|_{H_0^2}^2 + \|u_0^m\|_{H_0^2}^2.$$

Then, from (22), we have

$$\mathcal{K}_1(t) \leq \mathcal{K}_1(0) + \beta t + \frac{2^{\frac{p}{2}}}{p}\int_0^t \mathcal{K}_1^{\frac{p}{2}}(\tau)\,\mathrm{d}\tau.$$

We define

$$\mathcal{K}_2(t) = \mathcal{K}_1(0) + \beta t + \frac{2^{\frac{p}{2}}}{p}\int_0^t \mathcal{K}_1^{\frac{p}{2}}(\tau)\,\mathrm{d}\tau,$$

thus, $\mathcal{K}_1(t) = \frac{p^{\frac{2}{p}}}{2}(\mathcal{K}_2'(t)-\beta)^{\frac{2}{p}}$ and $\mathcal{K}_1(t) \leq \mathcal{K}_2(t)$ implies

$$\mathcal{K}_2'(t) \leq \frac{p^{\frac{2}{p}}}{2}\mathcal{K}_2^{\frac{p}{2}}(t) + \beta \leq \frac{p^{\frac{2}{p}}}{2}\left(\mathcal{K}_2(t) + \frac{1}{p}(p\beta)^{\frac{2}{p}} \right)^{\frac{p}{2}},$$
$$\mathcal{K}_2(0) = \mathcal{K}_1(0).$$

We readily obtain

$$\mathcal{K}_1(t) \leq \mathcal{K}_2(t) \leq \left(\tilde{\mathcal{K}}_1^{\frac{2}{2-p}}(0) - \frac{p-2}{2p}2^{\frac{p}{2}}t \right)^{\frac{2-p}{2}}, \tag{23}$$

where $\tilde{\mathcal{K}}_1(0) = \mathcal{K}_2(0) + \frac{1}{p}(p\beta)^{\frac{2}{p}}$. The right-hand side of (23) blow-up as $t \nearrow \frac{2p}{p-2}2^{\frac{p}{2}}$ $\tilde{\mathcal{K}}_1^{\frac{2}{2-p}}(0)$. But for $T = \frac{2p}{2-p}2^{-(\frac{p}{2}+1)}\tilde{\mathcal{K}}_1^{\frac{2}{2-p}}(0)$, we get

$$\|u_t^m\|_{H_0^2}^2 + \ell\|u^m\|_{H_0^2}^2 + \alpha(t)(g\circ\nabla u^m)(t) \leq 2^{\frac{2}{p-2}}\tilde{\mathcal{K}}_1(0).$$

Consequently, for large m, this implies

$$\sup_{t\in[0,t_m]}\left(\|u_t^m\|_{H_0^2}^2+\ell\|u^m\|_{H_0^2}^2+\alpha(t)(g\circ\nabla u^m)(t)\right)<2^{\frac{2}{p-2}}\tilde{\mathcal{K}}_1(0). \tag{24}$$

Hence, the approximation solution (see (14)) is uniformly bounded in m on the interval $[0,T]$ and h_{jm} can extend in $[0,T]$. Combining this fact and the problem (18) admits a unique solution $h_{jm}\in C^2[0,T]$ and, therefore, $u^m\in C^2([0,T];H_0^2(\Omega))$. It follows from (24) that

$$\begin{aligned}&(u^m)_{m\in\mathbb{N}}\quad \text{is uniformly bounded in } L^\infty(0,T;H_0^2(\Omega)),\\&(u_t^m)_{m\in\mathbb{N}}\quad \text{is uniformly bounded in } L^\infty(0,T;H_0^2(\Omega)),\end{aligned}\tag{25}$$

therefore, there is a subsequence $\{u^m\}$ (denote by $\{u^m\}$ again) and pass to the limit as $m\to\infty$,

$$\begin{aligned}u^m&\to u\quad \text{weakly* in } L^\infty(0,T;H_0^2(\Omega))\\u_t^m&\to u\quad \text{weakly* in } L^\infty(0,T;H_0^2(\Omega)),\end{aligned}\tag{26}$$

$$\begin{aligned}u^m&\to u\quad \text{weakly in } L^2(0,T;H_0^2(\Omega))\\u_t^m&\to u\quad \text{weakly in } L^2(0,T;H_0^2(\Omega)).\end{aligned}\tag{27}$$

Using (24), (25), So by Lions-Aubin's Lemma (see [28], Corollary 4), we get u^m and u_t^m approach strongly into u and u_t in $C([0,T];L^2(\Omega))$, therefore, $u^m(0)$ and $u_t^m(0)$ make sense i.e. $u^m(0)\to u(0)$, $u_t^m(0)\to u_1(0)$ in $L^2(\Omega)$. Next, multiply (15) by $\Phi\in C_0^\infty(0,T)$, integrate on $(0,T)$ and pass to the limit as $m\to\infty$, we get for any $w_j\in\mathcal{W}_m$

$$\begin{aligned}&-\int_0^T\left(u_t+\Delta u_t,w_j\Phi'\right)\,\mathrm{d}t\\&=\int_0^T\Big[(|v|^{p-2}v\ln|v|-u,w_j\Phi)-(\Delta u,\Delta w_j\Phi)\\&\quad-\alpha(t)\int_0^t g(t-s)(\nabla u(s),\nabla w_j\Phi)\,\mathrm{d}s\Big]\,\mathrm{d}t.\end{aligned}$$

This yields $u_{tt}\in L^2(0,T;H^{-2}(\Omega))$ and since $u_t\in L^2(0,T;H_0^2(\Omega))$, so we have $u_t\in C(0,T;H^{-2}(\Omega))$.

(**Uniqueness**). u_1 and u_2 are strong solutions. Then, the equation

$$(u_1-u_2)_{tt}+\Delta^2(u_1-u_2)+(u_1-u_2)+\alpha(t)(g*\Delta(u_1-u_2))+\Delta^2(u_1-u_2)_{tt}=0$$

in $\Omega\times\mathbb{R}^+$ share homogenous initial and boundary conditions. By (20), we have

$$\|(u_1-u_2)_t\|_{H_0^2}^2+\ell\|u_1-u_2\|_{H_0^2}^2+\alpha(t)(g\circ\nabla(u_1-u_2))(t)\le 0$$

i.e. $u_1-u_2=0$ and the proof is complete.

Now, based on the Lemma 3.1 and the Contraction Mapping Principle, we are going to show that the problem (1) has a unique local solution.

Theorem 3.2: *Suppose that (**H**) holds. For $(u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$, problem* (1) *admits a unique solution u in* $[0, T]$ *for some* $T > 0$.

Proof: For any $T > 0$, let us define the convex closed subset of $\mathcal{H}$ by

$$\mathcal{M}_T = \{u \in \mathcal{H}; \|u\|_{\mathcal{H}} \le M\}.$$

We cover $\mathcal{M}_T$ by a ball with radius $M > 0$ and denote

$$B_M(\mathcal{M}_T) = \{v \in \mathcal{M}_T; \|v\|_{\mathcal{H}} \le M\},$$

provided that M is large and T is small. We denote $u = \mathcal{S}(v)$ which is the unique solution of the problem (13) that corresponds to v, so $\mathcal{S} : \mathcal{M}_T \to \mathcal{M}_T$. We shall see that the fixed point theorem (see [[29], Corolary (3.6.2)]) can be applied to $\mathcal{S} : B_M(\mathcal{M}_T) \to B_M(\mathcal{M}_T)$. We show that the map $\mathcal{S}$ is contractive, which implies $\mathcal{S}(B_M(\mathcal{M}_T)) \subset B_M(\mathcal{M}_T)$.

Multiplying the first equation in problem (13) by u_t, integrating on $\Omega \times (0, t)$ and using (**H**), we obtain

$$\begin{aligned} &\|u_t\|_{H_0^2}^2 + \ell\|u\|_{H_0^2}^2 + \alpha(t)(g \circ \nabla u)(t) \\ &\quad \le \|u_1\|_{H_0^2}^2 + \|u_0\|_{H_0^2}^2 + 2\int_0^t \int_\Omega |v|^{p-2} v \ln|v| u_\tau \, \mathrm{d}x \, \mathrm{d}\tau, \end{aligned} \tag{28}$$

where $u = \mathcal{S}(v)$ is the corresponding solution to problem (1) for fix $v \in \mathcal{M}_T$. By similar arguments in (21) and (22), it follows from (28)

$$\begin{aligned} &\|u_t\|_{H_0^2}^2 + \ell\|u\|_{H_0^2}^2 + \alpha(t)(g \circ \nabla u)(t) \\ &\quad \le \|u_1\|_{H_0^2}^2 + \|u_0\|_{H_0^2}^2 + \frac{2^{\frac{p}{2}}}{p}\int_0^t \left(\|u_\tau\|_2^2\right)^{\frac{p}{2}} \mathrm{d}\tau \\ &\qquad + 2^{\frac{p}{2}}\int_0^t \left[\frac{p-1}{p}\left(\frac{|\Omega|}{(e(p-1))^2} + \frac{C_{2(p-1+\mu)}^{2(p-1+\mu)}}{(e\mu)^2}\|v\|_{H_0^2}^{2(p-1+\mu)}\right)\right]^{\frac{p}{2(p-1)}} \mathrm{d}\tau \\ &\quad \le \|u_1\|_{H_0^2}^2 + \|u_0\|_{H_0^2}^2 + \beta_M T + \frac{2^{\frac{p}{2}}}{p}\int_0^t \left[\|u_\tau\|_{H_0^2}^2 + \ell\|u\|_{H_0^2}^2 + \alpha(\tau)(g \circ \nabla u)(\tau)\right]^{\frac{p}{2}} \mathrm{d}\tau, \end{aligned} \tag{29}$$

where

$$\beta_M = 2^{\frac{p}{2}}\left[\frac{p-1}{p}\left(\frac{|\Omega|}{(e(p-1))^2} + \frac{C_{2(p-1+\mu)}^{2(p-1+\mu)}}{(e\mu)^2}M^{2(p-1+\mu)}\right)\right]^{\frac{p}{2(p-1)}}.$$

According to (23) and from the fact that

$$\mathcal{K}_1(0) = \mathcal{K}_2(0) = \|u_1^m\|_{H_0^2}^2 + \|u_0^m\|_{H_0^2}^2 \le \|u_1\|_{H_0^2}^2 + \|u_0\|_{H_0^2}^2,$$

(see (24)), by simple computation, it follows from (29)

$$
\begin{aligned}
&\|u_t\|^2_{H_0^2} + \ell\|u\|^2_{H_0^2} + \alpha(t)(g\circ\nabla u)(t)\\
&\quad \le \|u_1\|^2_{H_0^2} + \|u_0\|^2_{H_0^2} + \left(\beta_M + \frac{2^{\frac{p^2}{2(p-2)}}}{p}\tilde{\mathcal{K}}_1^{\frac{p}{2}}(0)\right)T.
\end{aligned}
$$

Choose $T > 0$ sufficiently small and take $M > 0$ large enough so that

$$
\begin{aligned}
\left(\beta_M + \frac{2^{\frac{p^2}{2(p-2)}}}{p}\tilde{\mathcal{K}}_1^{\frac{p}{2}}(0)\right)T \le \frac{M^2}{2},\\
\|u_1\|^2_{H_0^2} + \|u_0\|^2_{H_0^2} \le \frac{M^2}{2},
\end{aligned}
$$

we get

$$
\|u_t\|^2_{H_0^2} + \ell\|u\|^2_{H_0^2} + \alpha(t)(g\circ\nabla u)(t) \le M^2
$$

implies $\|u\|_{\mathcal{H}} \le M$ so $\mathcal{S}(\mathcal{M}_T) \subseteq \mathcal{M}_T$.

Finally, we show that for $v_1, v_2 \in \mathcal{M}_T$ and $u_1 = \mathcal{S}(v_1)$, $u_2 = \mathcal{S}(v_2)$, we have $\|u_1 - u_2\|_{\mathcal{H}} \le k\|v_1 - v_2\|_{\mathcal{H}}$ for $0 < k < 1$. To do this, taking $z = u_1 - u_2$ which solves

$$
\begin{aligned}
&z_{tt} + \Delta^2 z + z + \alpha(t)(g*\Delta z) + \Delta^2 z_{tt} = |v_1|^{p-2}v_1\ln|v_1|\\
&\quad - |v_2|^{p-2}v_2\ln|v_2|, \quad (x,t)\in\Omega\times\mathbb{R}^+\\
&z(x,t) = \frac{\partial z}{\partial \nu}(x,t) = 0, \quad (x,t)\in\partial\Omega\times\mathbb{R}^+\\
&z(x,0) = z_t(x,0) = 0, \quad x\in\Omega.
\end{aligned}
\tag{30}
$$

Multiply (30) by z_t, integrate on $\Omega\times(0,t)$ and use (**H**), we obtain

$$
\begin{aligned}
&\|z_t\|^2_{H_0^2} + \ell\|z\|^2_{H_0^2} + \alpha(t)(g\circ\nabla z)(t)\\
&\quad \le 2\int_0^t\int_\Omega \left(|v_1|^{p-2}v_1\ln|v_1| - |v_2|^{p-2}v_2\ln|v_2|\right) z_\tau \,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
\tag{31}
$$

The mean-value theorem yields

$$
\begin{aligned}
&\|z_t\|^2_{H_0^2} + \ell\|z\|^2_{H_0^2} + \alpha(t)(g\circ\nabla z)(t)\\
&\le 2\int_0^t\int_\Omega |\xi|^{p-2}(v_1 - v_2)z_\tau\,\mathrm{d}x\,\mathrm{d}\tau + 2(p-1)\int_0^t\int_\Omega |\xi|^{p-2}\ln|\xi|(v_1 - v_2)z_\tau\,\mathrm{d}x\,\mathrm{d}\tau\\
&:= \mathcal{E}_1 + \mathcal{E}_2,
\end{aligned}
\tag{32}
$$

where $|\xi| = |\theta v_1 + (1-\theta)v_2| \le |v_1| + |v_2|, \quad 0 < \theta < 1.$

Since $v_1, v_2 \in \mathcal{M}_T$, according to Cauchy's inequality, Hölder inequality and Sobolev embedding, we obtain

$$\begin{aligned}
\mathcal{E}_1 &\le 2\int_0^t \|\xi\|^{p-2}_{\frac{n}{2}(p-2)} \cdot \|v_1 - v_2\|_{\frac{2n}{n-4}} \cdot \|z_\tau\|_2 \,\mathrm{d}\tau \\
&\le 2\int_0^t \||v_1| + |v_2|\|^{p-2}_{\frac{n}{2}(p-2)} \cdot \|v_1 - v_2\|_{\frac{2n}{n-4}} \cdot \|z_\tau\|_2 \,\mathrm{d}\tau \cdot \\
&\le C_{\frac{2n}{n-4}} M^{2(p-2)} T \|v_1 - v_2\|^2_{\mathcal{H}} + \int_0^t \|z_\tau\|_2^2 \,\mathrm{d}\tau \\
&\le C_{\frac{2n}{n-4}} M^{2(p-2)} T \|v_1 - v_2\|^2_{\mathcal{H}} \\
&\quad + \int_0^t \left(\|z_\tau\|^2_{H_0^2} + \ell \|z\|^2_{H_0^2} + \alpha(\tau)(g \circ \nabla z)(\tau) \right) \mathrm{d}\tau, \qquad (33)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_2 &\le 2(p-1)\int_0^t \||\xi|^{p-2} \ln|\xi|\|_{\frac{n}{2}} \cdot \|v_1 - v_2\|_{\frac{2n}{n-4}} \cdot \|z_\tau\|_2 \,\mathrm{d}\tau \\
&\le (p-1)\left(\int_0^t \||\xi|^{p-2} \ln|\xi|\|^2_{\frac{n}{2}} \cdot \|v_1 - v_2\|^2_{\frac{2n}{n-4}} \,\mathrm{d}\tau + \int_0^t \|z_\tau\|_2^2 \,\mathrm{d}\tau \right). \qquad (34)
\end{aligned}$$

Since $p - 2 < \frac{n}{n-4} - 1$, there is $\mu > 0$ such that, $\frac{n}{2}(p - 2 + \mu) < \frac{2n}{n-4}$. By similar arguments in (21), we deduce

$$\begin{aligned}
\||\xi|^{p-2} \ln|\xi|\|^2_{\frac{n}{2}} &\le \left(\frac{|\Omega|}{(e(p-1))^{\frac{n}{2}}} + \frac{C^{\frac{n}{2}(p-1+\mu)}_{\frac{n}{2}(p-1+\mu)}}{(e\mu)^{\frac{n}{2}}} \||v_1| + |v_2|\|^{\frac{n}{2}(p-1+\mu)}_{\frac{n}{2}(p-1+\mu)} \right)^{\frac{4}{n}} \\
&\le \left(\frac{|\Omega|}{\left(e(p-1)\right)^{\frac{n}{2}}} + \frac{C^{\frac{n}{2}(p-1+\mu)}_{\frac{n}{2}(p-1+\mu)}}{(e\mu)^{\frac{n}{2}}} M^{\frac{n}{2}(p-1+\mu)} \right)^{\frac{4}{n}} \\
&:= \tilde{\beta}_M.
\end{aligned}$$

Applying this to (34), we get

$$\begin{aligned}
\mathcal{E}_2 &\le (p-1) C_{\frac{2n}{n-4}} \tilde{\beta}_M T \|v_1 - v_2\|^2_{\mathcal{H}} \\
&\quad + (p-1)\int_0^t \left(\|z_\tau\|^2_{H_0^2} + \ell \|z\|^2_{H_0^2} + \alpha(\tau)(g \circ \nabla z)(\tau) \right) \mathrm{d}\tau. \qquad (35)
\end{aligned}$$

Combining (32), (33) and (35), we get

$$\begin{aligned}
&\|z_t\|^2_{H_0^2} + \ell \|z\|^2_{H_0^2} + \alpha(t)(g \circ \nabla z)(t) \\
&\quad \le C_{\frac{2n}{n-4}} \left((p-1)\tilde{\beta}_M + M^{2(p-2)} \right) T \|v_1 - v_2\|^2_{\mathcal{H}}
\end{aligned}$$

$$+ (p-1)\int_0^t \left(\|z_\tau\|_{H_0^2}^2 + \ell\|z\|_{H_0^2}^2 + \alpha(\tau)(g\circ\nabla z)(\tau)\right)\,\mathrm{d}\tau,$$

Gronwall's inequality gives

$$\|z_t\|_{H_0^2}^2 + \ell\|z\|_{H_0^2}^2 + \alpha(t)(g\circ\nabla z)(t)$$
$$\le \mathrm{e}^{(p-1)T}\left(C_{\frac{2n}{n-4}}\left((p-1)\tilde{\beta}_M + M^{2(p-2)}\right)T\right)\|v_1 - v_2\|_{\mathcal{H}}^2.$$

Choose T sufficiently small and take

$$k := \mathrm{e}^{(p-1)T}\left(C_{\frac{2n}{n-4}}\left((p-1)\tilde{\beta}_M + M^{2(p-2)}\right)T\right) < 1,$$

implies that $\mathcal{S}$ is a contract map in $\mathcal{M}_T$. Thus, using the Contraction Mapping Principle, we conclude that problem (1) admits a unique solution. ■

4. Global existence

In this section, we are concerned with the existence of a global weak solution to problem (1).

$$u \in C([0,T_{\max});H_0^2(\Omega)) \cap C^1([0,T_{\max});H_0^2(\Omega)) \cap C^2([0,T_{\max});H^{-2}(\Omega))$$

holds for any $T \in (0, T_{\max})$ and $T_{\max} = +\infty$ gives the solution exists globally.

Lemma 4.1: *Suppose that* (**H**) *holds. For* $(u_0,u_1) \in \mathcal{N}^+ \times H_0^2(\Omega)$ *satisfy*

$$C_{p+\mu}^{p+\mu}\left(\frac{2p}{(p-2)\ell}E(0)\right)^{\frac{p+\mu-2}{2}} < (e\mu)^p\ell, \quad \mu > 0. \tag{36}$$

Then, $u = u(t) \in \mathcal{N}^+$ *for all* $t \in [0, T_{\max})$.

Proof: Since $u(t)$ is a weak solution of problem (1), so $u \in C([0,T_{\max});H_0^2(\Omega))$ implies $I(u) \in C[0,T_{\max})$. On the contrary, we suppose that there is $t_0 \in (0,T_{\max})$ such that $u(t_0) \in \mathcal{N}$. For $\varepsilon > 0$ sufficiently small there exists t_ε such that $u(t_\varepsilon) \in \mathcal{N}^-$, $0 < t_0 < t_\varepsilon$ and

$$I(u(t_\varepsilon)) = -p\varepsilon < 0. \tag{37}$$

Applying the functional $J(u(t))$ (see (10)), using Sobolev embedding (5) and (**H**) yields

$$J(u(t_\varepsilon)) = \frac{1}{p}I(u(t_\varepsilon)) + \frac{p-2}{2p}\left(\|u(t_\varepsilon)\|_{H_0^2}^2 + \alpha(t_\varepsilon)(g\circ\nabla u)(t_\varepsilon)\right.$$
$$\left. -\alpha(t_\varepsilon)\left(\int_0^{t_\varepsilon} g(s)\,\mathrm{d}s\right)\|\nabla u(t_\varepsilon)\|_2^2\right) + \frac{1}{p^2}\|u(t_\varepsilon)\|_p^p$$
$$\ge -\varepsilon + \frac{(p-2)\ell}{2p}\|u(t_\varepsilon)\|_{H_0^2}^2. \tag{38}$$

We recall Lemma 2.2 and (9) and (37) implies

$$\begin{aligned}\|u(t_\varepsilon)\|^2_{H^2_0} &\le \frac{2p}{(p-2)\ell}\left(J(u(t_\varepsilon))+\varepsilon\right)\\ &\le \frac{2p}{(p-2)\ell}\left(E(t_\varepsilon)+\varepsilon\right)\\ &\le \frac{2p}{(p-2)\ell}\left(E(0)+\varepsilon\right). \end{aligned} \tag{39}$$

By $I(u(t))$ (see (8)), we get

$$\begin{aligned} I(u(t_\varepsilon)) = {} & \|u(t_\varepsilon)\|^2_{H^2_0} + \alpha(t_\varepsilon)(g\circ\nabla u)(t_\varepsilon) - \alpha(t_\varepsilon)\left(\int_0^{t_\varepsilon} g(s)\,\mathrm{d}s\right)\|\nabla u(t_\varepsilon)\|_2^2\\ & - \int_\Omega |u(t_\varepsilon)|^p \ln|u(t_\varepsilon)|\,\mathrm{d}x\\ \ge {} & \ell\|u(t_\varepsilon)\|^2_{H^2_0} - \int_\Omega |u(t_\varepsilon)|^p \ln|u(t_\varepsilon)|\,\mathrm{d}x. \end{aligned} \tag{40}$$

From $\phi^{-\mu}\ln\phi \le \frac{1}{e\mu}$ for any $\phi > 1$, choose $0 < \mu < \frac{2n}{n-4} - p$ to guarantees $p+\mu < \frac{2n}{n-4}$ and use (39), we obtain

$$\begin{aligned} \int_\Omega |u(t_\varepsilon)|^p \ln|u(t_\varepsilon)|\mathrm{d}x &\le \frac{1}{(e\mu)^p}\|u(t_\varepsilon)\|^{p+\mu}_{p+\mu}\\ &\le \frac{C^{p+\mu}_{p+\mu}}{(e\mu)^p}\left(\|u(t_\varepsilon)\|^2_{H^2_0}\right)^{\frac{p-2+\mu}{2}}\|u(t_\varepsilon)\|^2_{H^2_0}\\ &\le \frac{C^{p+\mu}_{p+\mu}}{(e\mu)^p}\left(\frac{2p}{(p-2)\ell}\left(E(0)+\varepsilon\right)\right)^{\frac{p-2+\mu}{2}}\|u(t_\varepsilon)\|^2_{H^2_0}, \end{aligned} \tag{41}$$

where $C^{p+\mu}_{p+\mu}$ is the optimal embedding constant of $H^2_0(\Omega) \hookrightarrow L^{p+\mu}(\Omega)$. Thanks to the hypothesis (36) and $\varepsilon > 0$ sufficiently small

$$\frac{C^{p+\mu}_{p+\mu}}{(e\mu)^p}\left(\frac{2p}{(p-2)\ell}\left(E(0)+\varepsilon\right)\right)^{\frac{p-2+\mu}{2}} \le \ell. \tag{42}$$

Hence, (40)–(42) shows that $I(u(t_\varepsilon)) > 0$ contradicts with (37). Therefore, $u(t) \in \mathcal{N}^+$ for all $t \in [0, T_{\max})$. ∎

Theorem 4.2: *Suppose that exponent p satisfies* (2) *and* **(H)** *holds. Then, for* $(u_0, u_1) \in \mathcal{N}^+ \times H^2_0(\Omega)$ *the local weak solution u got in Theorem* 3.2 *exists globally.*

Proof: Let $u(t)$, $t \in [0, T_{\max})$ be the local weak solution of the problem (1). To show $T_{\max} = +\infty$, it is enough to find a constant $C > 0$ such that

$$\sup_{t\in[0,T_{\max})} \left(\|u_t\|_{H_0^2}^2 + \ell \|u\|_{H_0^2}^2 + \alpha(t)(g \circ \nabla u)(t) \right) \leq C. \tag{43}$$

Because of Lemma 2.2, Sobolev embedding (5), (9), (10) and **(H)**, we obtain

$$\begin{aligned} E(0) &\geq \frac{1}{2}\|u_t\|_{H_0^2}^2 + \frac{1}{p} I(u(t)) + \frac{1}{p^2}\|u\|_p^p \\ &\quad + \frac{p-2}{2p}\left(\|u\|_{H_0^2}^2 + \alpha(t)(g \circ \nabla u)(t) - \alpha(t)\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 \right) \\ &\geq \frac{p-2}{2p}\min\{1, \frac{p}{p-2}\}\left(\|u_t\|_{H_0^2}^2 + \ell\|u\|_{H_0^2}^2 + \alpha(t)(g \circ \nabla u)(t) \right). \end{aligned}$$

Take $C^{-1} = \frac{p-2}{2pE(0)}\min\{1, \frac{p}{p-2}\}$. Then, (43) holds and consequently, the solution is global. ■

5. Finite time blow-up for high initial energy

In this section, we are concerned with the finite time blow-up criterion for the problem (1) at a high initial energy level. We give a pivotal Lemma, which can be used to prove our results. The crucial Lemma is as follows:

Lemma 5.1: *Suppose that* **(H)** *holds and* $\ell > \frac{1}{(p-1)^2}$. *Let* $(u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$ *satisfy*

$$0 < E(0) < \frac{\gamma}{p}((u_0, u_1) + (\Delta u_0, \Delta u_1)), \quad \gamma > 0. \tag{44}$$

Then, the map $\{t \to (u, u_t) + (\Delta u, \Delta u_t)\}$ *is strictly increasing on* $[0, T_{\max})$, *when*

$$u \in C([0, T_{\max}); H_0^2(\Omega)) \cap C^1([0, T_{\max}); H_0^2(\Omega)) \cap C^2([0, T_{\max}); H^{-2}(\Omega)).$$

Proof: We used the idea of [18, 30] to prove this Lemma. Using directly the first equation of problem (1), one has

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}((u, u_t) + (\Delta u, \Delta u_t)) \\ &\quad = \|u_t\|_2^2 + (u, u_{tt}) + \|\Delta u_t\|_2^2 + (\Delta u, \Delta u_{tt}) \\ &\quad = \|u_t\|_{H_0^2}^2 - \|u\|_{H_0^2}^2 + \alpha(t)\int_\Omega\int_0^t g(t-s)\nabla u(t)(\nabla u(s) - \nabla u(t))\,\mathrm{d}s\,\mathrm{d}x \\ &\qquad + \alpha(t)\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 + \int_\Omega |u|^p \ln|u|\,\mathrm{d}x. \end{aligned} \tag{45}$$

Using the full advantage of Young's inequality, we get

$$\int_\Omega\int_0^t g(t-s)\nabla u(t)(\nabla u(s) - \nabla u(t))\,\mathrm{d}s\,\mathrm{d}x$$

$$\geq -\frac{1}{4p}\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 - \frac{p}{4}(g\circ\nabla u)(t). \tag{46}$$

Combining (45) and (46) with the definition of modified energy functional $E(t)$ (see (6)), (**H**) and Sobolev embedding (5), it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left((u,u_t)+(\Delta u,\Delta u_t)\right)\\
&\geq \left(\frac{p}{2}+1\right)\|u_t\|_{H_0^2}^2 + \left(\frac{p}{2}-1\right)\|u\|_{H_0^2}^2 + \frac{p}{4}\alpha(t)(g\circ\nabla u)(t)\\
&\quad -\left(\frac{p}{2}+\frac{1}{4p}-1\right)\alpha(t)\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 - pE(t)\\
&\geq \left(\frac{p}{2}+1\right)\|u_t\|_{H_0^2}^2 + \left[\left(\frac{p}{2}-1\right)+\left(\frac{p}{2}+\frac{1}{4p}-1\right)(\ell-1)\right]\|u\|_{H_0^2}^2\\
&\quad +\frac{p}{4}\alpha(t)(g\circ\nabla u)(t) - pE(t)\\
&\geq \frac{\gamma}{2}\left(\|u_t\|_{H_0^2}^2 + \|u\|_{H_0^2}^2 + \alpha(t)(g\circ\nabla u)(t) - \frac{2p}{\gamma}E(t)\right)\\
&\geq \frac{\gamma}{2}\left(\|u_t\|_{H_0^2}^2 + \|u\|_{H_0^2}^2 - \frac{2p}{\gamma}E(t)\right),
\end{aligned} \tag{47}$$

where $\gamma = \min\{p+2, (p-2)+(p+\frac{1}{2p}-2)(\ell-1), \frac{p}{2}\}$. One can easily verify that since $\ell > \frac{1}{(p-1)^2}$, then $(p-2)+(p+\frac{1}{2p}-2)(\ell-1) > 0$.

Let

$$F(t) = (u,u_t) + (\Delta u,\Delta u_t) - \frac{p}{\gamma}E(t). \tag{48}$$

Thanks to the Schwarz's inequality, yields

$$\begin{aligned}
F'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left((u,u_t)+(\Delta u,\Delta u_t)\right) - \frac{p}{\gamma}E'(t)\\
&\geq \frac{\gamma}{2}\left(\|u_t\|_{H_0^2}^2 + \|u\|_{H_0^2}^2 - \frac{2p}{\gamma}E(t)\right)\\
&\geq \gamma\left((u,u_t)+(\Delta u,\Delta u_t) - \frac{p}{\gamma}E(t)\right)\\
&= \gamma F(t).
\end{aligned}$$

Applying Gronwall's inequality in the above inequality, we get

$$F(t) \geq \mathrm{e}^{\gamma t}F(0), \quad t\in[0,T_{\max}) \tag{49}$$

where $F(0) = (u_0,u_1) + (\Delta u_0,\Delta u_1) - \frac{p}{\gamma}E(0)$, and this completes the proof. ■

Lemma 5.2: *Suppose that* (**H**) *and* (44) *hold. Let* $(u_0,u_1)\in\mathcal{N}^-\times H_0^2(\Omega)$ *and* $\ell > \frac{1}{(p-1)^2}$. *Then, the weak solution* $u=u(t)$ *of the problem* (1) *belongs to* $\mathcal{N}^-$ *for all* $t\in[0,T_{\max})$.

Proof: We assume that $u_0 \in \mathcal{N}^-$. Because of Lemma 5.1, we have $u = u(t) \notin \mathcal{N}^-$ for all $t \in [0, T_{\max})$. In other words, u is a global solution of the problem (1) and $E(t) \geq 0$ for all $t \in [0, T_{\max})$. We use the contradiction method to reach this conclusion.

With the continuity of $I(u)$, there would be a first time $t_1 \in [0, T_{\max})$ such that $u = u(t_1) \in \mathcal{N}$ and $I(u(t)) < 0$ for $t \in [0, t_1)$. From the definition of $F(t)$ (see (48)) which is strictly increasing on $[0, t_1)$ and according to its continuity and monotonicity, we have

$$F(t) \geq e^{\gamma t}\left[(u_0, u_1) + (\Delta u_0, \Delta u_1) - \frac{p}{\gamma}E(0)\right] > 0, \quad t \in [0, t_1), \tag{50}$$

where (44) has been used.

From Lemma 2.2, $E(t_1) \leq E(0)$. Using (9), (10), **(H)** and Sobolev embedding (5), we get

$$\begin{aligned} E(0) &\geq \frac{1}{2}\|u_t(t_1)\|_{H_0^2}^2 + \frac{1}{p}I(u(t_1)) + \frac{1}{p^2}\|u\|_p^p \\ &\quad + \frac{p-2}{2p}\left(\|u(t_1)\|_{H_0^2}^2 + \alpha(t_1)(g \circ \nabla u)(t_1) - \alpha(t_1)\left(\int_0^{t_1} g(s)\,\mathrm{d}s\right)\|\nabla u(t_1)\|_2^2\right) \\ &\geq \tilde{\gamma}\left(\frac{1}{2}\|u_t(t_1)\|_{H_0^2}^2 + \frac{1}{2}\|u(t_1)\|_{H_0^2}^2\right) \\ &\geq \tilde{\gamma}\left((u, u_t) + (\Delta u, \Delta u_t)\right), \end{aligned} \tag{51}$$

where $\tilde{\gamma} = \min\{1, \frac{(p-2)\ell}{p}\}$. Using (48), (50) and (51) yields

$$\begin{aligned} E(0) &\geq \tilde{\gamma}\left(F(t_1) + \frac{p}{2\gamma}E(t_1)\right) \\ &\geq \tilde{\gamma}\, \mathrm{e}^{\gamma t_1}\left[(u_0, u_1) + (\Delta u_0, \Delta u_1) - \frac{p}{\gamma}E(0)\right] \\ &\geq \tilde{\gamma}\, e^{\gamma t_1}[(u_0, u_1) + (\Delta u_0, \Delta u_1)] \\ &:= \frac{\gamma}{p}[(u_0, u_1) + (\Delta u_0, \Delta u_1)], \end{aligned}$$

that contradict with (44) and the proof is completed. ■

Now we can state and prove the blow-up result and obtain an upper bound for the blow-up time.

Theorem 5.3: *Assume that* **(H)** *holds and let* $(u_0, u_1) \in \mathcal{N}^- \times H_0^2(\Omega)$. *If* $E(0) > \frac{2\gamma^2(p+2)\|u_0\|_{H_0^2}^2}{p^2(p-2)}$ *and* $\ell > \frac{1}{(p-1)^2}$, *then the weak solution* $u = u(t)$ *of problem* (1) *blows up at a finite time* $T^* = T_{\max}$ *such that*

$$T^* \leq \frac{4\|u_0\|_{H_0^2}^2}{(p-2)\left[(u_0, u_1) + (\Delta u_0, \Delta u_1) - \frac{2\gamma(p+2)\|u_0\|_{H_0^2}^2}{p(p-2)}\right]}.$$

Proof: Define the following auxiliary function

$$\psi(t) = \|u\|_{H_0^2}^2 > 0, \quad \text{for all } t \in [0, T_{\max}). \tag{52}$$

We have

$$\psi'(t) = 2(u, u_t) + 2(\Delta u, \Delta u_t). \tag{53}$$

From the first equation of the problem (1) and energy functional (6), we get

$$\begin{aligned}
\psi''(t) &= 2\|u_t\|_2^2 + 2(u, u_{tt}) + 2\|\Delta u_t\|_2^2 + 2(\Delta u, \Delta u_{tt}) \\
&= \frac{p+6}{2}\|u_t\|_{H_0^2}^2 + \frac{p-2}{2}\|u\|_{H_0^2}^2 \\
&\quad + \frac{p+2}{2}\alpha(t)(g \circ \nabla u)(t) - \frac{p-2}{2}\alpha(t)\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 \\
&\quad + 2\alpha(t)\int_\Omega \int_0^t g(t-s)\nabla u(t)(\nabla u(s) - \nabla u(t))\,\mathrm{d}s\,\mathrm{d}x \\
&\quad + \frac{p-2}{p}\int_\Omega |u|^p \ln|u|\,\mathrm{d}x + \frac{p+2}{p^2}\|u\|_p^p - (p+2)E(t).
\end{aligned} \tag{54}$$

Using Young's inequality (46), Sobolev embedding (5) and (**H**), from (54), we obtain

$$\begin{aligned}
\psi''(t) &\geq \frac{p+6}{2}\|u_t\|_{H_0^2}^2 + \left[\frac{(p-2)\ell}{2} - \frac{1-\ell}{2p}\right]\|u\|_{H_0^2}^2 + \alpha(t)(g \circ \nabla u)(t) \\
&\quad + \frac{p-2}{p}\int_\Omega |u|^p \ln|u|\,\mathrm{d}x + \frac{p+2}{p^2}\|u\|_p^p - (p+2)E(t).
\end{aligned} \tag{55}$$

By the definitions of $F(t)$ and Nehari functional $I(u(t))$ (c.f. (48) and (8)), from (55), we get

$$\begin{aligned}
\psi''(t) &\geq \frac{\gamma(p+2)}{p}[F(t) - ((u, u_t) + (\Delta u, \Delta u_t))] + \frac{p+6}{2}\|u_t\|_{H_0^2}^2 \\
&\quad + \left(\frac{(p-2)\ell}{2} + \frac{\ell-1}{2p} + \frac{p-2}{p}(\ell-1) + \frac{p-2}{p}\right)\|u\|_{H_0^2}^2 \\
&\quad + \left(\frac{p-2}{p} + 1\right)\alpha(t)(g \circ \nabla u)(t) - \frac{p-2}{p}I(u(t)) \\
&\geq \frac{\gamma(p+2)}{p}[F(t) - ((u, u_t) + (\Delta u, \Delta u_t))] \\
&\quad + \frac{p+6}{2}\|u_t\|_{H_0^2}^2 - \frac{p-2}{p}I(u(t)),
\end{aligned} \tag{56}$$

where $\ell > \frac{1}{(p-1)^2} > \frac{1}{p^2-3}$ has been used.

According to (49) in Lemma 5.1, Lemma 5.2 ($u \in \mathcal{N}^-$) and (53), it holds from (56) that

$$\psi''(t) \geq -\frac{\gamma(p+2)}{2p}\psi'(t) + \frac{p+6}{2}\|u_t\|_{H_0^2}^2. \tag{57}$$

Using Schwartz inequality and Young's inequality, we get from (53) that

$$\begin{aligned}(\psi'(t))^2 &= 4(((u,u_t) + (\Delta u, \Delta u_t))^2 \\ &= 4(u,u_t)^2 + 4(\Delta u, \Delta u_t)^2 + 8(u,u_t)(\Delta u, \Delta u_t) \\ &\leq 4(\|u\|_2^2\|u_t\|_2^2 + \|\Delta u\|_2^2\|\Delta u_t\|_2^2 + 2(\|u\|_2\|\Delta u_t\|_2)(\|u_t\|_2\|\Delta u\|_2)) \\ &\leq 4(\|u\|_2^2\|u_t\|_2^2 + \|\Delta u\|_2^2\|\Delta u_t\|_2^2 + \|u\|_2^2\|\Delta u_t\|_2^2 + \|u_t\|_2^2\|\Delta u\|_2^2) \\ &= 4\left(\|u_t\|_2^2 + \|\Delta u_t\|_2^2\right)\left(\|u\|_2^2 + \|\Delta u\|_2^2\right) \\ &= 4\|u_t\|_{H_0^2}^2\|u\|_{H_0^2}^2 \\ &= 4\|u_t\|_{H_0^2}^2\psi(t). \end{aligned} \tag{58}$$

Multiplying (57) by $\psi(t)$ and using (58), yields

$$\psi(t)\psi''(t) \geq -\frac{\gamma(p+2)}{2p}\psi(t)\psi'(t) + \frac{p+6}{8}(\psi'(t))^2. \tag{59}$$

Finally, the hypotheses of Theorem 2.3 are fulfilled with

$$\tilde{\alpha} = \frac{p+6}{8}, \quad \tilde{\gamma} = \frac{\gamma(p+2)}{2p}, \quad \tilde{\beta} = 0,$$

and if we choose $E(0) > \frac{2\gamma^2(p+2)\|u_0\|_{H_0^2}^2}{p^2(p-2)}$, then (44) yields

$$\psi'(0) = 2(u_0,u_1) + 2(\Delta u_0, \Delta u_1) > \frac{4\gamma(p+2)}{p(p-2)}\psi(0),$$

and, therefore, solution u of the problem (1) blows up at T^* with the following upper bound

$$T^* \leq \frac{4\|u_0\|_{H_0^2}^2}{(p-2)\left[(u_0,u_1) + (\Delta u_0, \Delta u_1) - \frac{2\gamma(p+2)\|u_0\|_{H_0^2}^2}{p(p-2)}\right]},$$

and the proof of Theorem 5.3 is completed. ■

We now proceed to establish a lower bound for the blow-up time, which is according to Theorem 5.3.

Theorem 5.4: *Assume that* (**H**) *holds and* $\ell > \frac{1}{(p-1)^2}$. *Let* $(u_0, u_1) \in \mathcal{N}^- \times H_0^2(\Omega)$. *Then the blow-up time* $T^* = T_{\max}$ *has the following lower bound*

$$\int_{\chi(0)}^{+\infty} \frac{1}{\xi_1 y^{p+\mu-1} + y + \xi_2}\,\mathrm{d}y \le T^*,$$

where $\mu > 0$, *and*

$$\chi(0) = \frac{1}{2}\left(\|u_0\|_{H_0^2}^2 + \|u_1\|_{H_0^2}^2\right),$$
$$\xi_1 = \frac{C_{r^*}^2}{2\mu^2 e^2} C_{r(p+\mu-1)}^{2(p+\mu-1)} \left(\frac{2}{\ell}\right)^{p+\mu-1},$$
$$\xi_2 = \frac{|\Omega|}{2e^2(p-1)^2},$$

C_{r^*} *is the optimal embedding constant of* $H_0^2(\Omega) \hookrightarrow L^{r^*}(\Omega)$ *and* $C_{r(p+\mu-1)}$ *is the optimal embedding constant of* $H_0^2(\Omega) \hookrightarrow L^{r(p+\mu-1)}(\Omega)$ *for any positive constants r and* r^* *such that* $\frac{1}{r} + \frac{1}{r^*} = 1$.

Proof: Define

$$\chi(t) = \frac{1}{2}\left(\|u_t\|_{H_0^2}^2 + \|u\|_{H_0^2}^2 + \alpha(t)(g \circ \nabla u)(t) - \alpha(t)\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2\right),$$

and so using (6), we have $\chi(t) = E(t) + \frac{1}{p}\int_\Omega |u|^p \ln|u|\,\mathrm{d}x - \frac{1}{p^2}\|u\|_p^p$.

Taking the derivative of $\chi(t)$ with respect to t and using Lemma 2.2, we obtain

$$\begin{aligned}\chi'(t) &= E'(t) + \int_\Omega |u|^{p-2} u u_t \ln|u|\,\mathrm{d}x \\ &\le \int_\Omega |u|^{p-2} u u_t \ln|u|\,\mathrm{d}x. \end{aligned} \tag{60}$$

On the other hand, recalling the definition of $F(t)$ in (48), (**H**), Sobolev embedding (5), Hölder and Young inequalities, we get

$$\begin{aligned}\ell F(t) &= \ell(u, u_t) + \ell(\Delta u, \Delta u_t) - \frac{\ell p}{\gamma} E(t) \\ &\le \ell\|u\|_2\|u_t\|_2 + \ell\|\Delta u\|_2\|\Delta u_t\|_2 \\ &\le \frac{\ell}{2}\|u_t\|_{H_0^2}^2 + \frac{\ell}{2}\|u\|_{H_0^2}^2 \\ &\le \frac{1}{2}\|u_t\|_{H_0^2}^2 + \frac{1}{2}\|u\|_{H_0^2}^2 + \frac{1}{2}\left(1 - \alpha(t)\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 \\ &\le \frac{1}{2}\|u_t\|_{H_0^2}^2 + \frac{1}{2}(C_1 + 1)\|u\|_{H_0^2}^2 - \frac{1}{2}\alpha(t)\left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2\end{aligned}$$

$$+\frac{1}{2}\alpha(t)(g\circ\nabla u)(t)$$
$$\le (C_1+1)\chi(t). \tag{61}$$

The conclusion of Lemma 5.1 indicates that functional $F(t)$ exponentially grows ; therefore, (61) yields

$$\lim_{t\to T^{*-}}\chi(t)=+\infty. \tag{62}$$

Now, from (60), Hölder, Young's inequality and Sobolev embedding $H_0^2(\Omega)\hookrightarrow L^{r^*}(\Omega)$ with constant C_{r^*}, we obtain

$$\begin{aligned}
\chi'(t) &\le \int_\Omega |u|^{p-2}uu_t\ln|u|\,\mathrm{d}x\\
&= \int_{\{x\in\Omega:|u|\ge1\}}|u|^{p-2}uu_t\ln|u|\,\mathrm{d}x+\int_{\{x\in\Omega:|u|<1\}}|u|^{p-2}uu_t\ln|u|\,\mathrm{d}x\\
&\le \frac{1}{e\mu}\int_{\{x\in\Omega:|u|\ge1\}}|u|^{p+\mu-1}|u_t|\,\mathrm{d}x+\frac{1}{e(p-1)}\int_{\{x\in\Omega:|u|<1\}}|u_t|\,\mathrm{d}x\\
&\le \frac{1}{e\mu}\||u|^{p+\mu-1}\|_r\|u_t\|_{r^*}+\frac{\sqrt{|\Omega|}}{e(p-1)}\|u_t\|_2\\
&\le \frac{C_{r^*}}{e\mu}\||u|^{p+\mu-1}\|_r\|\Delta u_t\|_2+\frac{|\Omega|}{2\,\mathrm{e}^2(p-1)^2}+\frac{1}{2}\|u_t\|_2^2\\
&\le \frac{C_{r^*}^2}{2\mu^2e^2}\||u|^{p+\mu-1}\|_r^2+\frac{1}{2}\|\Delta u_t\|_2^2+\frac{|\Omega|}{2e^2(p-1)^2}+\frac{1}{2}\|u_t\|_2^2.
\end{aligned} \tag{63}$$

$r(p-1)<r^*$, then we could choose $\mu>0$ such that $r(p-1+\mu)<r^*$. Using the embedding $H_0^2(\Omega)\hookrightarrow L^{r(p+\mu-1)}(\Omega)$ with constant $C_{r(p+\mu-1)}$ and (61), we deduce

$$\begin{aligned}
\chi'(t) &\le \frac{C_{r^*}^2}{2\mu^2e^2}C_{r(p+\mu-1)}^{2(p+\mu-1)}\|\Delta u\|_2^{2(p+\mu-1)}+\frac{1}{2}\|\Delta u_t\|_2^2+\frac{|\Omega|}{2e^2(p-1)^2}+\frac{1}{2}\|u_t\|_2^2\\
&\le \frac{C_{r^*}^2}{2\mu^2e^2}C_{r(p+\mu-1)}^{2(p+\mu-1)}\left(2\ell^{-1}(C_1+1)\right)^{p+\mu-1}\chi(t)^{p+\mu-1}+\frac{1}{2}\|\Delta u_t\|_2^2+\frac{|\Omega|}{2e^2(p-1)^2}\\
&\quad+\frac{1}{2}\|u_t\|_2^2\\
&\le \xi_1\chi(t)^{p+\mu-1}+\chi(t)+\xi_2,
\end{aligned} \tag{64}$$

where $\xi_1=\frac{C_{r^*}^2}{2\mu^2e^2}C_{r(p+\mu-1)}^{2(p+\mu-1)}\left(2\ell^{-1}(C_1+1)\right)^{p+\mu-1}$, $\xi_2=\frac{|\Omega|}{2e^2(p-1)^2}$.

Integrating (64) over $[0,t]$, yields

$$\int_0^t\frac{\chi'(s)}{\xi_1\chi(s)^{p+\mu-1}+\chi(s)+\xi_2}\,\mathrm{d}s\le t,$$

let $t\to T^*$ and thanks to the (62), thus we get

$$\int_{\chi(0)}^{+\infty}\frac{1}{\xi_1y^{p+\mu-1}+y+\xi_2}\,\mathrm{d}y\le T^*, \tag{65}$$

with $\chi(0) = \frac{1}{2}(\|u_0\|^2_{H_0^2} + \|u_1\|^2_{H_0^2})$. Therefore, (65) provide a lower bound for the blow-up time and proof of Theorem 5.4 is completed. ■

Statements and declarations

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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