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On generalizations of Baer's theorem and its converses

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Abstract
A well-known theorem of Baer states that if G is a group and $G/Z_n(G)$ is finite, then $\gamma_{n+1}(G)$ is finite. Kurdachenko et al. proved that if $G/Z_n(G)$ is a locally finite group of finite exponent, then so is $\gamma_{n+1}(G)$. In this article, we extend this theorem to groups G with subgroups A of $Aut(G)$ which contain $Inn(G)$. Furthermore, some new upper bounds of the exponents of $\gamma_{n+1}(G)$ and $\gamma_{n+1}(G, A)$ are presented. Moreover we give a proof for the converse of Baer's theorem considering groups G such that $G/Z_n(G, A)$ and $A/Inn(G)$ are finitely generated or have finite special rank. Finally we conclude that the index of the subgroup $Z_n(G, A)$ is bounded by a precisely determined function in terms of the order of $\gamma_{n+1}(G, A)$.

Keywords Schur's theorem · Baer's theorem · Exponent · Hypocenter

Mathematics Subject Classification Primary 20F14 · 20F28; Secondary 20F50 · 20F19

1 Introduction and Preliminaries

A class of groups $\mathfrak{X}$ is called a Schur class, if for every group G such that $G/Z(G) \in \mathfrak{X}$, we conclude that the derived subgroup $\gamma_2(G)$ belongs to $\mathfrak{X}$. According to the well-known theorem of Schur, the class of all finite groups is a Schur class. A question related to this result that arises naturally here is the relationship between $|G/Z(G)|$ and $|\gamma_2(G)|$ in this class. In [16] Wiegold answered this question in such away that if $|G/Z(G)| = t$, then $|\gamma_2(G)| \leq t^{\frac{1}{2}(\log_p t-1)}$, where p is the smallest prime number

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dividing t. In 2007, Mann [12] proved that the class of locally finite groups with finite exponent is a Schur class. In addition to that, he obtained the following relation between the exponents of $G/Z(G)$ and $\gamma_2(G)$.

Theorem 1.1 (A. Mann)*[[12], Theorem 1] Let G be a group such that $G/Z(G)$ is locally finite of exponent e. Then $\gamma_2(G)$ is likewise locally finite with a finite exponent. Moreover the best possible bound for its exponent, $m(e)$, is dividing $|R(2,e)|$, where $|R(2,e)|$ is the order of the restricted Burnside group with two generators and exponent e.*

In [1] Baer proved that for a natural number n, if $G/Z_n(G)$ is finite, then $\gamma_{n+1}(G)$ is finite, in which $Z_n(G)$ and $\gamma_{n+1}(G)$ are the nth and $(n+1)$th terms of the upper and lower central series of G, respectively. Here we bring the next concept that helps us to represent these results more properly. A class of groups $\mathfrak{X}$ is said to be a Baer class if for every group G such that $G/Z_n(G) \in \mathfrak{X}$, the $(n+1)$th terms of lower central series $\gamma_{n+1}(G)$ also belongs to $\mathfrak{X}$. Similar to Schur class, the question of finding a relationship between the factor groups $G/Z_n(G)$ and $\gamma_{n+1}(G)$ of G in the class $\mathfrak{X}$ is very common. In 2016 Kurdachenko et al. [11] extended the Mann's result to other terms of upper central series and concluded that the class of locally finite groups of finite exponent is a Baer class. In other words, they proved:

Theorem 1.2 (L. A. Kurdachenko et al.)*[[11], Theorem A] For a given group G, suppose that $G/Z_n(G)$ is locally finite with finite exponent e, for some positive integer n. Then the subgroup $\gamma_{n+1}(G)$ is locally finite of finite exponent. Moreover there exists a function $\beta_1(e,n)$ such that the exponent of $\gamma_{n+1}(G)$ is at most $\beta_1(e,n)$ defined inductively by*

$$\beta_1(e,1) = m(e), \beta_1(e,2) = m(m(e))m(e), \beta_1(e,n) = m(\beta_1(e,n-1))\beta_1(e,n-1).$$

The first aim of this paper is to improve the bound obtained in Theorem 1.2. Our improved bound is presented in the next theorem.

Theorem A *Let G be a group such that $G/Z_n(G)$ is locally finite of exponent e. Then $\gamma_{n+1}(G)$ is locally finite and its exponent divides*

$$\gcd(e^{\lceil n+1/2\rceil}m(e), \gcd(m(e)^n, \underbrace{m(m(\ldots(m(e))\ldots)}_{n-times})))$$

for odd e and

$$\gcd((2e)^{\lceil n+1/2\rceil}m(e), \gcd(m(e)^n, \underbrace{m(m(\ldots(m(e))\ldots)}_{n-times})))$$

for even e, where $\gcd(a,b)$ denotes the greatest common divisor of a and b and $\lceil x\rceil$ denotes the least integer which is greater than or equal to x.

Furthermore, by using an example we show that the bound in Theorem A is smaller than the one obtained in [[4], Theorem B].
In [8], Hegarty proved that if $G/C_G(Aut(G))$ is finite, then both $[G, Aut(G)]$ and $Aut(G)$ are finite. Moreover he showed:

Theorem 1.3 (P. Hegarty)*[8] Let* $|G/C_G(Aut(G))| = t$*, then*

$$|\gamma_2(G, Aut(G))| \leq t^{t((t-1)^2+[\frac{t}{2}])[\log_2 t]}.$$

In 2014, Dixon et al. improved Hegarty's work in [3] by generalizing the lower central series. They actually substituted $[G, Aut(G)]$ for $[G, A]$ where A is a group with $Inn(G) \leq A \leq Aut(G)$ and then generalized it to every $\gamma_i(G, A)$ when $i \in \mathbb{N}$. Throughout this paper, G is assumed to be an arbitrary group and $Inn(G) \leq A \leq Aut(G)$. To state the next definition we need the concept of A-central subgroup.

Definition 1.4 Let G be a group and let $A \leq \mathrm{Aut}(G)$ be a subgroup of the automorphism group of G. A subgroup $H \leq G$ is called A-central if it is fixed pointwise by all automorphisms in A, i.e.,

$$\forall \alpha \in A,\ \forall h \in H, \quad \alpha(h) = h.$$

Equivalently,

$$H \leq C_G(A) = \{g \in G \mid \alpha(g) = g \text{ for all } \alpha \in A\}.$$

Here we consider the upper A-central series of G, same as the series used in [3], as follows:

$$1 = Z_0(G, A) \leq Z_1(G, A) \leq \cdots \leq Z_\alpha(G, A) \leq Z_{\alpha+1}(G, A) \leq \cdots$$

defined by the rule $Z_1(G, A) = C_G(A)$, and recursively

$$\frac{Z_{\alpha+1}(G, A)}{Z_\alpha(G, A)} = Z_1\left(\frac{G}{Z_\alpha(G, A)}, \frac{A}{C_A(\frac{G}{Z_\alpha(G,A)})}\right)$$

for all ordinals α and $Z_\lambda(G, A) = \bigcup_{\alpha<\lambda} Z_\alpha(G, A)$ for a limit ordinal λ. The last term $Z_\infty(G, A) = Z_\mu(G, A)$ of this series is called the upper A-hypercenter of G and the ordinal μ is called the upper A-central length of G and is denoted by $zl(G, A)$. The lower A-central series of a group G is the series

$$G = \gamma_1(G, A) \geq \gamma_2(G, A) \geq \cdots \geq \gamma_\upsilon(G, A) \geq \gamma_{\upsilon+1}(G, A) \geq \cdots$$

in which $\gamma_2(G, A) = [G, A]$ and $\gamma_{\nu+1}(G, A) = [\gamma_\nu(G, A), A]$ for every ordinal ν and $\gamma_\beta(G, A) = \bigcap_{\nu<\beta} \gamma_\nu(G, A)$ for a limit ordinal β. The last term $\gamma_\infty(G, A) = \gamma_\delta(G, A)$ of this series is called the lower A-hypocenter of G. Some elementary properties of the mentioned upper and lower A-central series are as follows.

Lemma 1.5 *Let G be a group and A be a group with $Inn(G) \leq A \leq Aut(G)$, then*

i) $Z_\alpha(G, A) \leq Z_\alpha(G)$.
ii) $\gamma_\beta(G, A) \geq \gamma_\beta(G)$.
iii) $\gamma_2(\gamma_n(G, A)) \leq \gamma_{n+1}(G, A)$.

By applying the last generalization, two questions then arise here.

1. For which classes of groups $\mathfrak{X}$, if we consider $G/Z_n(G, A) \in \mathfrak{X}$ for some subgroup $Inn(G) \leq A \leq Aut(G)$ and positive integer n, we can conclude that $\gamma_{n+1}(G, A) \in \mathfrak{X}$?
2. What relationship does appear between the factor group $G/Z_n(G, A)$ and the subgroup $\gamma_{n+1}(G, A)$ of G in the class $\mathfrak{X}$?

With the assumption $|A/Inn(G)| < \infty$, Dixon et al. illustrated that the class of finite groups satisfies the condition of the first question, when $zl(G, A) = n$ is finite. They also found a relationship between the orders of the factor group $G/Z_n(G, A)$ and the subgroup $\gamma_{n+1}(G, A)$ of G. In 2011 Dietrich and Moravec proved:

Theorem 1.6 (H. Dietrich and P. Moravec)*[[2], Theorem 2.1] Suppose that M is an H-group. If $M/Z_H(M)$ is locally finite of exponent e, then $[M, H]$ is likewise locally finite of exponent $m(e)$ in which $m(e)$ is introduced in Theorem 1.1.*

Recall that a group M is an H-group whenever there exists a homomorphism $\varphi : H \to Aut(M)$, such that the image of φ contains $Inn(M)$. Suppose that $Inn(G) \leq A \leq Aut(G)$. Then by substitutions $M = G$ and $H = A$, it is concluded that G can be an A-group. In addition, with this interpretation, the quotient group $M/Z_H(M)$ is equal to $G/Z(G, A)$. Therefore, by Moravec's theorem, if $G/Z(G, A)$ is locally finite of finite exponent e, then $\gamma_2(G, A)$ is locally finite of finite exponent dividing $m(e)$. Our next goal is to extend this recent statement to groups G such that $G/Z_n(G, A)$ is locally finite of finite exponent. More precisely, we have the next theorem.

Theorem B *Let G be an arbitrary group and A be a subgroup of $Aut(G)$ containing $Inn(G)$ such that $G/Z_n(G, A)$ is locally finite of finite exponent e. Then $\gamma_{n+1}(G, A)$ is locally finite and its exponent divides*

$$\gcd(e^{\lceil n+1/2 \rceil} m(e), \gcd(m(e)^n, \underbrace{m(m(\ldots(m(e))\ldots)}_{n-times})))$$

when e is odd and divides

$$\gcd((2e)^{\lceil n+1/2 \rceil} m(e), \gcd(m(e)^n, \underbrace{m(m(\ldots(m(e))\ldots)}_{n-times})))$$

when e is even, where $\gcd(a, b)$ *denotes the greatest common divisor of a, b.*

With the assumptions of Theorem B, we can obtain a bound for the exponent of $\gamma_{2n}(G, A)$ which depends only on e, but not on n. In particular, the exponent of $\gamma_\infty(G, A)$ is independent of m, where $m = zl(G, A)$ is finite. (See Corollaries 2.8 and 2.9)

The last purpose of this paper is to consider the converse of Baer's theorem for groups G such that $G/Z_n(G, A)$ and $A/Inn(G)$ are finitely generated or have finite special rank. It is well-known that the converse of Baer's theorem is not true in general. It is clear that when $A = Inn(G)$, the groups $G/Z_n(G, A)$ and $\gamma_{n+1}(G, A)$ do not satisfy the converse of Baer's theorem. In [13] Niroomand gave a proof for the converse of Schur's theorem for finitely generated group $G/Z(G)$. He showed that if $G/Z(G)$ is finitely generated and $\gamma_2(G)$ is finite, then $G/Z(G)$ is also finite and its order is bounded by a function of $|\gamma_2(G)|$. He also proved that if G is nilpotent, then the order of $G/Z(G)$ divides this bound.

Theorem 1.7 (P. Niroomand)*[13, Corollary 2.1] Let G be a nilpotent group such that $d(G/Z(G))$ and $\gamma_2(G)$ are both finite, then $|G/Z(G)|$ divides $|\gamma_2(G)|^{d(G/Z(G))}$.*

Recently in [7] Hatamian et al. generalized the result obtained by Niroomand when $G/Z_n(G)$ is finitely generated and $\gamma_{n+1}(G)$ is finite. It should be noticed that their proofs are based on the isoclinic theory (see [9]). In fact they proved:

Theorem 1.8 (R. Hatamian et al.)*[7, Main Theorem] For a given group G, suppose that $\gamma_{n+1}(G)$ is finite and $G/Z_n(G)$ is finitely generated. Then*

$$|\frac{G}{Z_n(G)}| \leq |\gamma_{n+1}(G)|^{d(\frac{G}{Z_n(G)})^n}. \tag{1.1}$$

Theorem 1.9 (R. Hatamian et al.)*[7, Corollary 1] Let G be a nilpotent group such that $G/Z_n(G)$ is finitely generated. If $\gamma_{n+1}(G)$ is finite, then*

$$|\frac{G}{Z_n(G)}|\,\Big|\,|\gamma_{n+1}(G)|^{d(\frac{G}{Z_n(G)})^n}.$$

Here with a different proof, we generalize Theorems 1.8 and 1.9 for groups $G/Z_n(G, A)$ where $A/Inn(G)$ and $G/Z_n(G, A)$ are finitely generated or have finite special rank. A group G is said to have finite special rank $r(G) = r$ if every finitely generated subgroup of G can be generated by r elements and r is the smallest positive integer with this property.

Theorem C *For a group G, let $\gamma_{n+1}(G, A)$ be finite. Then*

$$|\frac{G}{Z_n(G, A)}| \leq |\gamma_{n+1}(G, A)|^{(d+k)^n}, \tag{1.2}$$

if one of the following conditions holds:

1. *$A/Inn(G)$ and $G/Z_n(G, A)$ are generated by k and d elements, respectively.*
2. *One of the two groups $A/Inn(G)$ and $G/Z_n(G, A)$ has finite special rank k and the other one is generated by d elements.*
3. *$A/Inn(G)$ and $G/Z_n(G, A)$ have finite special ranks equal to k and d, respectively.*

Theorem D *Let G be a nilpotent group such that $\gamma_{n+1}(G, A)$ is finite. Then*

$$|\frac{G}{Z_n(G, A)}|\,\big|\,|\gamma_{n+1}(G, A)|^{(d+k)^n},$$

if one of the following conditions holds:

1. *$A/Inn(G)$ and $G/Z_n(G, A)$ are generated by k and d elements, respectively.*
2. *One of the two groups $A/Inn(G)$ and $G/Z_n(G, A)$ has finite special rank k and the other one is generated by d elements.*
3. *$A/Inn(G)$ and $G/Z_n(G, A)$ have finite special ranks equal to k and d, respectively.*

In [17] Yadav propounded two following questions:

Question 1 Let $G/Z(G)$ be finitely generated and $\gamma_2(G)$ be finite. Can we conclude from the equality

$$|\frac{G}{Z(G)}| = |\gamma_2(G)|^{d(\frac{G}{Z(G)})}$$

that G is nilpotent?

Question 2 Does there exist a non-nilpotent group G which is not isomorphic to $X \times H$, where X is a finite group and H is a nilpotent group, in such a way that $\gamma_2(G)$ is finite and $G/Z(G)$ is infinite?

In the next theorem, we give an answer to a general version of Question 1 for the group $G/Z_n(G, A)$. More precisely, we prove:

Theorem E *Suppose that $G/Z_n(G, A)$ and $A/Inn(G)$ are generated by d and k elements, respectively. If Inequality 1.2 is sharp, then G is nilpotent.*

In Section 2 we consider the class of locally finite groups of finite exponent. Then, we find a better bound for the exponent of $\gamma_{n+1}(G)$ and a new upper bound for the exponent of $\gamma_{n+1}(G, A)$. In Section 3 we focus on the generalization of the converse of Schur's and Baer's theorems. Then we answer Question 2 proposed by Yadav.

2 Bounds for the exponents of $\gamma_{n+1}(G, A)$ in the class of locally finite groups of finite exponent

Throughout this section, G is an arbitrary group and $L\mathfrak{F}$ denotes the class of locally finite groups of finite exponent, $Inn(G) \leq A \leq Aut(G)$ and p is an odd prime.

Theorem 2.1 *Let G be a group such that $G/Z_n(G, A)$ is of exponent e and belongs to $L\mathfrak{F}$. Then $\gamma_{n+1}(G, A) \in L\mathfrak{F}$ and its exponent divides $m(e)^n$.*

Proof We proceed by induction on n. For $n = 1$ the result follows by Theorem 1.6. Suppose $G/Z_n(G, A)$ is of exponent e and assume that it belongs to $L\mathfrak{F}$. Then by induction hypothesis on n for $G/Z(G, A)/Z_{n-1}(G/Z(G, A), A)$ it is concluded that $\gamma_n(G/Z(G, A), A)$ belongs to $L\mathfrak{F}$ and is of exponent dividing $m(e)^{n-1}$. Now

$\gamma_n(G, A)/\gamma_n(G, A) \cap Z_n(G, A)$ is isomorphic to a subgroup of $G/Z_n(G, A)$ and hence belongs to $L\mathfrak{F}$ and has exponent e, but $\gamma_n(G, A) \cap Z_n(G, A) \subseteq Z(\gamma_n(G, A))$ therefore Theorem 1.1 implies $\gamma_2(\gamma_n(G, A))$ belongs to $L\mathfrak{F}$ and its exponent divides $m(e)$.

Now for each $\alpha \in A$ define:

$$f_\alpha : \frac{\gamma_n(G, A)}{\gamma_n(G, A) \cap Z(G, A)} \longrightarrow \frac{\gamma_{n+1}(G, A)}{\gamma_2(\gamma_n(G, A))}$$
$$\tilde{x} \longmapsto \overline{[x, \alpha]}$$

in which $x \in \gamma_n(G, A)$. It is easy to see that, f_α is a homomorphism. Since

$$\gamma_{n+1}(G, A) = \langle [x, \alpha] | x \in \gamma_n(G, A), \alpha \in A \rangle,$$

the order of each generator of $\gamma_{n+1}(G, A)/\gamma_2(\gamma_n(G, A))$ is finite and divides $m(e)^{n-1}$. Moreover the exponent of $\gamma_{n+1}(G, A)/\gamma_2(\gamma_n(G, A))$ divides $m(e)^{n-1}$, since it is abelian. Therefore $\gamma_{n+1}(G, A) \in L\mathfrak{F}$ and its exponent divides $m(e)^n$. □

Definition 2.2 Let G be a group and let A be a subgroup of $Aut(G)$ which contains $Inn(G)$. Then G is said to be A-nilpotent of class c if $Z_c(G, A) = G$ and $Z_{c-1}(G, A) \neq G$. Similarly an A-invariant subgroup M of G is called A-nilpotent of class c if $M \subseteq Z_c(G, A)$ and $M \nsubseteq Z_{c-1}(G, A)$.

It is clear that conditions of Definition 2.2 are equivalent to $\gamma_{c+1}(G, A) = 1$ and $\gamma_c(G, A) \neq 1$. Similarly, $\gamma_{c+1}(M, A) = 1$ and $\gamma_c(M, A) \neq 1$. On the other hand, it is easy to see that each A-nilpotent group is in fact nilpotent. Since a nilpotent group of finite exponent is locally finite, so it is the direct product of p_i-components. Hence, in Lemma 2.4 we give an upper bound for the exponent of $\gamma_{n+1}(M, A)$, which is independent of n, where $M/Z_n(G, A)$ is A-invariant A-nilpotent subgroup of class c of exponent p^m. For achieving this, the following lemma is required.

Lemma 2.3 *[10] Let G be a group and let A be a subgroup of $Aut(G)$ such that $Inn(G) \leq A$. Consider that G has a series of A-invariant subgroups*

$$1 = Z_0 \leq Z_1 \leq \cdots \leq Z_m$$

whose factors are A-central. Then $\gamma_m(G, A) \leq C_G(Z_m)$.

Lemma 2.4 *Assume that for a group G, M is an A-invariant subgroup such that $M/Z_n(G, A)$ is A-nilpotent of class c of exponent p^m. Then $\gamma_{n+1}(M, A)$ is of finite exponent which divides $p^{m\lceil c/2 \rceil}$.*

Proof The proof is done by induction on c. Let $c = 1$, then $g \in Z_{n+1}(G, A)$ for all $g \in M$. As a result, $[g^{p^m}, \alpha_1, \ldots, \alpha_n] = [g, \alpha_1, \ldots, \alpha_n]^{p^m} = 1$, because $[g, \alpha_1, \ldots, \alpha_{n+1}] = 1$ for all $\alpha_i \in A$. Hence the order of every autocommutator of weight $n + 1$ divides p^m. Since $\gamma_{n+1}(M, A) \subseteq Z(G, A)$, it is concluded that the exponent of $\gamma_{n+1}(M, A)$ divides p^m. If $c = 2$, then $g \in Z_{n+2}(G, A)$ for all $g \in M$.

Similar to the case $c = 1$, the order of every autocommutator of weight $n + 2$ divides p^m. Now by induction on n we have

$$[g^i, \alpha_1, \ldots, \alpha_n] = [g, \alpha_1, \ldots, \alpha_n]^i [g, \alpha_1, g, \alpha_2, \ldots, \alpha_n]^{\binom{i}{2}} \qquad (mod \gamma_{n+3}(M, A)),$$

for every $i \in \mathbb{N}$. So

$$[g^{p^m}, \alpha_1, \ldots, \alpha_n] = [g, \alpha_1, \ldots, \alpha_n]^{p^m} [g, \alpha_1, g, \alpha_2, \ldots, \alpha_n]^{\binom{p^m}{2}} = 1,$$

since p odd implies $p^m \Big| \binom{p^m}{2}$. Thus the order of every autocommutator of weight $n+1$ divides p^m. By Lemma 2.3, $\gamma_{n+1}(M, A) \subseteq C_G(Z_{n+1}(G, A))$ and since

$$\gamma_{n+1}(M, A) \subseteq Z_2(G, A) \subseteq Z_{n+1}(G, A),$$

it is concluded that $\gamma_{n+1}(M, A)$ is abelian. Therefore the result follows in this case. Let $c > 2$ and assume that for any group G the result holds for any A-nilpotent subgroup of class less than $n + c$. We consider the subgroup

$$\gamma_3(M, A)Z_n(G, A)/Z_n(G, A)$$

of $G/Z_n(G, A)$. It is an A-nilpotent subgroup of class $c - 2$. Thus by induction hypothesis $\gamma_{n+1}(\gamma_3(M, A), A)$ is of finite exponent dividing $p^{m\lceil (c-2)/2 \rceil}$. Now we consider the subgroup $(M/\gamma_{n+1}(\gamma_3(M, A), A))/Z_n(G/\gamma_{n+1}(\gamma_3(M, A), A), A)$ of the group

$$(G/\gamma_{n+1}(\gamma_3(M, A), A))/Z_n(G/\gamma_{n+1}(\gamma_3(M, A), A), A)$$

which is of finite exponent dividing p^m and is an A-nilpotent subgroup of class at most 2. Finally by induction hypothesis $\gamma_{n+1}(M, A)/\gamma_{n+1}(\gamma_3(M, A), A)$ divides p^m and the proof is completed. □

In Lemma 2.5 we obtain the exponent of $\gamma_{n+1}(M, A)$, where $M/Z_n(G, A)$ is of exponent 2^m.

Lemma 2.5 *Given a group G with an A-invariant subgroup M, suppose that $M/Z_n(G, A)$ is an A-nilpotent group of class c which is of exponent 2^m. Then $\gamma_{n+1}(M, A)$ has finite exponent which divides $2^{m\lceil c/2 \rceil + \lfloor c/2 \rfloor}$.*

Proof It is proved similar to Lemma 2.4 by considering $2^m \Big| \binom{2^{m+1}}{2}$ instead of $p^m \Big| \binom{p^m}{2}$ for the case $c = 2$. □

In the sequel, we give an upper bound for the exponent of $\gamma_{n+1}(M, A)$ where M is an A-invariant subgroup of G and $M/Z_n(G, A)$ is A-nilpotent of class c with exponent equal to $p_1^{m_1} \ldots p_k^{m_k}$.

Theorem 2.6 *Assume that $p_1, \ldots, p_k$ are prime numbers such that $p_1 < \cdots < p_k$. Let M be an A-invariant subgroup of G such that $M/Z_n(G, A)$ is an A-nilpotent* *group of class c with exponent* $p_1^{m_1}\dots p_k^{m_k}$. *Then the exponent of* $\gamma_{n+1}(M, A)$ *is finite and divides* $p_1^{\lceil c/2\rceil s_1}\dots p_k^{\lceil c/2\rceil s_k}$, *where*

$$s_i=\begin{cases} m_i & p_i\neq 2,\\ m_i+1 & p_i=2,\end{cases}$$

for all $1\le i\le k$.

Proof We prove by induction on k. For $k=1$, the result is obtained by Lemmas 2.4 and 2.5. Let $k>1$ and suppose that $M/Z_n(G, A)$ is a group of exponent $p_1^{m_1}\dots p_k^{m_k}$ and of class c. Then $M/Z_n(G, A)$ is the direct product of p_i-components, where $1\le i\le k$. Each of these components is an A-invariant subgroup. First we consider the p_1-component. The induction hypothesis for the p_1-component M_{p_1} implies that the exponent of $\gamma_{n+1}(M_{p_1}, A)$ divides $p_1^{\lceil c/2\rceil s_1}$. By considering

$$(M/\gamma_{n+1}(M_{p_1}, A))/Z_n(G/\gamma_{n+1}(M_{p_1}, A), A)$$

which is of exponent $p_2^{m_2}\dots p_k^{m_k}$ and applying the induction hypothesis, it is concluded that the exponent of $\gamma_{n+1}(M, A)/\gamma_{n+1}(M_{p_1}, A)$ divides $p_2^{\lceil c/2\rceil s_2}\dots p_k^{\lceil c/2\rceil s_k}$. Thus the exponent of $\gamma_{n+1}(M, A)$ divides $p_1^{\lceil c/2\rceil s_1}\dots p_k^{\lceil c/2\rceil s_k}$, which completes the proof. □

Now, by an example we show that there exists a group G with subgroups M of G and A of $Aut(G)$ such that the exponent of $\gamma_{n+1}(M, A)$ is coincided the exponent obtained in Theorem 2.6. In other words the exponent is given in Theorem 2.6 is sharp. We know that for natural numbers $n, m_1,\dots,m_k$ and prime numbers $p_1,\dots,p_k$ there exists a group G such that $G/Z_n(G)$ is abelian of exponent $p_1^{m_1}\dots p_k^{m_k}$ and $exp(\gamma_{n+1}(G)) = p_1^{m_1}\dots p_k^{m_k}$(see [4, Theorem B]). If G is a nilpotent group of class $(n+1)$ such that $G/Z_n(G)$ is of odd exponent, then for $A = Inn(G)$ and $M = G$ the bound in Theorem 2.6 holds.

It is well-known that if $G/Z_n(G)$ is a locally finite π-group, then so is $\gamma_{n+1}(G)$ (for instance see [15, page 113]). In the next theorem we deduce that if $G/Z_n(G, A)$ is a locally finite π-group, then $\gamma_{n+1}(G, A)$ is also a locally finite π-group. Furthermore, based on the proof of Theorem 2.7, we obtain an upper bound for the exponent of $\gamma_{2n}(G, A)$ where $G/Z_n(G, A)$ is a locally finite group of exponent e.

Theorem 2.7 *For a given group G, let* $G/Z_n(G, A)$ *be a locally finite* π*-group. Then* $\gamma_{n+1}(G, A)$ *is a locally finite* π*-group.*

Proof Let $G/Z_n(G, A)$ be a locally finite π-group, then

$$\gamma_n(G, A)/\gamma_n(G, A)\cap Z_n(G, A)$$

is also a locally finite π-group. Now for each $\alpha_1,\dots,\alpha_n\in Aut(G)$ consider the map

$$f_{\alpha_1,\dots,\alpha_n}:\frac{\gamma_n(G, A)}{\gamma_n(G, A)\cap Z_n(G, A)}\longrightarrow\frac{\gamma_{2n}(G, A)}{\gamma_2(\gamma_n(G, A))}$$
$$\tilde{x}\longmapsto\overline{[x,\alpha_1,\dots,\alpha_n]}$$

in which $x \in \gamma_n(G, A)$. By a similar method used in the proof of Theorem 2.1 and considering the corollary of [15, Theorem 4.12], $\gamma_2(\gamma_n(G, A))$ is a locally finite π-group. It is clear that $f_{\alpha_1,\ldots,\alpha_n}$ is a homomorphism. Since

$$\gamma_{2n}(G, A) = \langle [x, \alpha_1, \ldots, \alpha_n] \,|\, x \in \gamma_n(G, A), \forall i, 1 \leq i \leq n, \alpha_i \in A \rangle,$$

then every generator of $\gamma_{2n}(G, A)/\gamma_2(\gamma_n(G, A))$ is a π-element. Moreover $\gamma_{2n}(G, A)/\gamma_2(\gamma_n(G, A))$ is abelian and so it is a π-group. Hence $\gamma_{2n}(G, A)$ is a locally finite π-group. In particular if $G/Z_n(G, A) \in L\mathfrak{F}$ is of exponent e, then $\gamma_{2n}(G, A) \in L\mathfrak{F}$ is of exponent dividing $em(e)$ by Theorem 1.1. Now for each i, $1 \leq i \leq n-1$, we define

$$h_i : \frac{\gamma_i(G, A)}{(Z_{n+1-i}(G, A) \cap \gamma_i(G, A))\gamma_{i+1}(G, A)} \times A \longrightarrow \frac{\gamma_{i+1}(G, A)}{(Z_{n-i}(G, A) \cap \gamma_{i+1}(G, A))\gamma_{i+2}(G, A)}.$$
$$(x_i, \alpha) \longmapsto [x_i, \alpha]$$

It is clear that for all i, $1 \leq i \leq n-1$, h_i is a homomorphism on each component and so $\gamma_{i+1}(G, A)/(\gamma_{i+1}(G, A) \cap Z_{n-i}(G, A))\gamma_{i+2}(G, A)$ is a locally finite π-group. A similar statement holds for h_n and all h_j $(n+1 \leq j \leq 2n-2)$ which are defined as follows:

$$h_n : \frac{\gamma_n(G, A)}{(Z(G, A) \cap \gamma_n(G, A))\gamma_{n+1}(G, A)} \times A \longrightarrow \frac{\gamma_{n+1}(G, A)}{\gamma_{n+2}(G, A)},$$
$$(x_n, \alpha) \longmapsto [x_n, \alpha]$$

$$h_j : \frac{\gamma_j(G, A)}{\gamma_{j+1}(G, A)} \times A \longrightarrow \frac{\gamma_{j+1}(G, A)}{\gamma_{j+2}(G, A)}.$$
$$(x_j, \alpha) \longmapsto \left[x_j, \alpha\right]$$

Thus the factor group $\gamma_{n+1}(G, A)/\gamma_{2n}(G, A)$ is a locally finite π-group. Therefore $\gamma_{n+1}(G, A)$ is also a locally finite π-group. □

The proof of Theorem 2.7 results in the following two interesting corollaries.

Corollary 2.8 *Suppose that the exponent of $G/Z_n(G, A) \in L\mathfrak{F}$ equals e. Then $\gamma_{2n}(G, A) \in L\mathfrak{F}$ and its exponent divides $em(e)$. In particular, if the exponent of $G/Z_n(G) \in L\mathfrak{F}$ equals e, then $\gamma_{2n}(G) \in L\mathfrak{F}$ and its exponent divides $em(e)$.*

Corollary 2.9 *Let A be a subgroup of automorphisms of an arbitrary group G such that $Inn(G) \leq A \leq Aut(G)$ and let Z be the upper A-hypercenter of G. Suppose that $zl(G, A) = m$ and $G/Z \in L\mathfrak{F}$ is of exponent e. Then $\gamma_\infty(G, A) \in L\mathfrak{F}$ and its exponent divides $em(e)$.*

Now, we are in a position to state and prove the other bound of Theorem A. More precisely, Theorem 2.10 proves that the upper bound obtained for the exponent of $\gamma_{n+1}(G, A)$ is $e^{\lceil (n+1)/2 \rceil} m(e)$ or $(2e)^{\lceil (n+1)/2 \rceil} m(e)$.

Theorem 2.10 *For a given group G, suppose that $G/Z_n(G, A) \in L\mathfrak{F}$ and it has finite exponent $e = p_1^{m_1} \dots p_k^{m_k}$. Then $\gamma_{n+1}(G, A) \in L\mathfrak{F}$ and its exponent divides $p_1^{\lceil (n-1)/2 \rceil s_1} \dots p_k^{\lceil (n-1)/2 \rceil s_k} em(e)$, where*

$$s_i = \begin{cases} m_i & p_i \neq 2, \\ m_i + 1 & p_i = 2. \end{cases}$$

Proof When $n = 1$, the result follows by Corollary 2.8. Let $n \geq 2$. Then by Corollary 2.8, the exponent of $\gamma_{2n}(G, A)$ divides $em(e)$. Now we consider

$$(G/\gamma_{2n}(G, A))/Z_n(G/\gamma_{2n}(G, A), A)$$

of class $n-1$ and exponent dividing e. The exponent of $\gamma_{n+1}(G, A)/\gamma_{2n}(G, A)$ divides $p_1^{\lceil (n-1)/2 \rceil s_1} \dots p_k^{\lceil (n-1)/2 \rceil s_k}$ by Theorem 2.6, where

$$s_i = \begin{cases} m_i & p_i \neq 2, \\ m_i + 1 & p_i = 2, \end{cases}$$

as required. □

Theorem 2.11 *Let G be an arbitrary group and A be a subgroup of $Aut(G)$ containing $Inn(G)$ such that $G/Z_n(G, A)$ is locally finite of finite exponent e. Then $\gamma_{n+1}(G, A)$ is locally finite and its exponent divides $\underbrace{m(m(\dots(m(e))\dots))}_{n-times}$.*

Proof The proof is done by induction on n. For $n = 1$ the result holds by [2, Theorem 2.1]. Assume that for $n - 1$ the result is in hand and $G/Z_n(G, A)$ is locally finite of exponent e. Then $\gamma_n(G, A)Z(G, A)/Z(G, A)$ is locally finite and its exponent divides $\underbrace{m(m(\dots(m(e))\dots))}_{n-1-times}$. Define $\phi : A \to Aut(\gamma_n(G, A))$ by the rule $\phi(f) = \bar{f}$. It is clear that $\gamma_n(G, A)$ is an A-group. Since $Z(\gamma_n(G, A), A) \supseteq \gamma_n(G, A) \cap Z(G, A)$ and $\gamma_n(G, A)/\gamma_n(G, A) \cap Z(G, A)$ is of exponent $\underbrace{m(m(\dots(m(e))\dots))}_{n-1-times}$, by [2, Theorem 2.1], it is concluded that the exponent $\gamma_{n+1}(G, A)$ divides $\underbrace{m(m(\dots(m(e))\dots))}_{n-times}$, which completes the proof. □

Proof of Theorem B Using Theorems 2.11, 2.10 and 2.1, the result is attained. □

Finally, by an example we show that the bound obtained in Theorem B is sharp.

Example Let $G/Z_n(G, A)$ be a locally finite group of exponent 2 or 3, then exponent of $\gamma_{n+1}(G, A)$ is 2 or 3. Hence the bound obtained in Theorem B is sharp.

Proof of Theorem A By putting $A = Inn(G)$, the result follows. □

It is concluded by induction on n that $(m(e))^{2^{n-1}}$ divides the bound obtained by Kurdachenko et al. (Theorem 1.2), that is $(m(e))^{2^{n-1}} | \beta_1(e, n)$, while the bound of Theorem A divides at most $m(e)^n$. Now, by comparing the obtained bound in Theorem A with the one obtained by Kurdachenko et al. (Theorem 1.2), it is easy to see that the given bound in this article is smaller than the one obtained by Kurdachenko et al.

3 A generalization of the converse of Schur's and Baer's theorems

The converse of Schur's theorem is not true in general. For instance, the infinite extra-special p-groups are not satisfying in the converse of Schur's theorem. Moreover, the groups constructed by Hall in [6] are not satisfying in the converse of Baer's theorem. But in the same paper [6, Theorem 2], a partial converse of Baer's theorem is proved which says if $\gamma_{n+1}(G)$ is finite, then $G/Z_{2n}(G)$ is finite, too. Recently Dixon et al. [3, Theorem 3] proved that $G/Z_{2n}(G, A)$ is finite provided that both $\gamma_{n+1}(G, A)$ and $A/Inn(G)$ are finite, where $Inn(G) \leq A \leq Aut(G)$.
In this section we generalize another form of the converse of Schur's and Baer's theorems, which are stated in Section 1 as Theorems C and D. To do this, first we present some lemmas which are needed in the proofs of Theorems C and D. Then, two Questions 1 and 2 will be answered. In this section G is assumed to be an arbitrary group such that $Inn(G) \leq A \leq Aut(G)$ and $d(X)$ denotes the minimum number of generators of a given group X. In the next lemma $\Phi(G)$ denotes the Frattini subgroup of G

Lemma 3.1 *[5, Theorem 1.1] Let G be a group such that $\gamma_2(G)$ is finite and $\Phi(G) = 1$. Then*

$$|G/Z(G)| \leq |\gamma_2(G)|^2. \tag{3.1}$$

The equality holds if and only if G is abelian.

Lemma 3.2 *[13, Main Theorem] Given a group G, let $d(G/Z(G))$ and $\gamma_2(G)$ be finite, then*

$$|G/Z(G)| \leq |\gamma_2(G)|^{d(G/Z(G))}. \tag{3.2}$$

Lemma 3.3 *[[14], Theorem 7] If G is a finite capable group and $\gamma_2(G)$ is cyclic, then $|G/Z(G)| \leq |\gamma_2(G)|^2$.*

Now, we prove Theorem C via proving many statements. The following theorem proves the first part of Theorem C.

Proof of Theorem C (1) The proof is done by induction on n. Let

$$\langle \alpha_1 Inn(G), \ldots, \alpha_k Inn(G) \rangle = A/Inn(G)$$

for some $\alpha_1, \ldots, \alpha_k \in A$. Let $n = 1$ and assume that $\{y_1 Z(G, A), \ldots, y_d Z(G, A)\}$ is a generating set for the group $G/Z(G, A)$. We define the following map

$$\begin{aligned} f_1 : \frac{G}{Z(G, A)} &\longrightarrow (\gamma_2(G, A))^{d(\frac{G}{Z(G,A)})+d(\frac{A}{Inn(G)})} \\ x_1 Z(G, A) &\longmapsto ([x_1, y_1], \ldots, [x_1, y_d], [x_1, \alpha_1], \ldots, [x_1, \alpha_k]). \end{aligned}$$

It is easy to see that f_1 is a one-to-one function. Suppose that the statement holds for $n-1$. Assume that $\{y_1 Z_n(G, A), \dots, y_d Z_n(G, A)\}$ is a generating set for $G/Z_n(G, A)$ and $\gamma_{n+1}(G, A)$ is a finite group. Define

$$\begin{aligned} f_n : \frac{\gamma_n(G, A)}{\gamma_n(G, A) \cap Z(G)} &\longrightarrow (\gamma_{n+1}(G, A))^{d(\frac{G}{Z_n(G,A)})} \\ x_n(\gamma_n(G, A) \cap Z(G)) &\longmapsto ([x_n, y_1], \dots, [x_n, y_d]). \end{aligned}$$

We claim that f_n is one-to-one as well. Suppose that

$$x_n(\gamma_n(G, A) \cap Z(G)) = x'_n(\gamma_n(G, A) \cap Z(G)).$$

Then $x_n x_n'^{-1} \in Z(G)$ and hence

$$\begin{aligned} ([x'_n, y_1], \dots, [x'_n, y_d]) &= ([x_n x_n'^{-1} x'_n, y_1], \dots, [x_n x_n'^{-1} x'_n, y_d]) \\ &= ([x_n, y_1], \dots, [x_n, y_d]). \end{aligned}$$

Conversely suppose that $([x'_n, y_1], \dots, [x'_n, y_d]) = ([x_n, y_1], \dots, [x_n, y_d])$. Then

$$\begin{aligned} ([x_n x_n'^{-1}, y_1], \dots, [x_n x_n'^{-1}, y_d]) &= ([x_n, y_1]^{x_n'^{-1}}, \dots, [x_n, y_d]^{x_n'^{-1}}) \\ &\quad ([x_n'^{-1}, y_1], \dots, [x_n'^{-1}, y_d]) \\ &= ([x_n, y_1]^{x_n'^{-1}}, \dots, [x_n, y_d]^{x_n'^{-1}}) \\ &\quad (([x'_n, y_1]^{-1})^{x_n'^{-1}}, \dots, ([x'_n, y_d]^{-1})^{x_n'^{-1}}) \\ &= 1. \end{aligned}$$

On the other hand, $[x_n x_n'^{-1}, z] = 1$ for all $z \in Z_n(G, A)$ by Lemma 2.3. As a result, we have proved the claim. Consider the map

$$h_n : \gamma_n(G, A) \cap Z(G) \longrightarrow (\gamma_{n+1}(G, A))^{d(A/Inn(G))},$$

such that $h_n(z_n) = ([z_n, \alpha_1], \dots, [z_n, \alpha_k])$. Then h_n is a homomorphism with $\ker h_n = \gamma_n(G, A) \cap Z(G, A)$. Consequently

$$|\gamma_n(G, A)/\gamma_n(G, A) \cap Z(G, A)| \leq (\gamma_{n+1}(G, A))^{d(A/Inn(G))+d(G/Z_n(G,A))}.$$

Now by induction hypothesis we have

$$|G/Z_n(G, A)| \leq |\gamma_n(G, A)/(\gamma_n(G, A) \cap Z(G, A))|^{(d(A/Inn(G))+d(G/Z_n(G,A)))^{n-1}},$$

which completes the proof. $\square$

In the next corollary, we show that in Theorem C(1) the condition of being finitely generated for $A/Inn(G)$, when $n = 1$, can be omitted.

Corollary 3.4 *Let G be a group such that $G/Z(G, A)$ is finitely generated and $\gamma_2(G, A)$ is finite. Then both $G/Z(G, A)$ and A are finite.*

Proof Since $G/Z(G, A)$ is finitely generated and $\gamma_2(G, A)$ is finite, by Lemma 3.2, $G/Z(G)$ is finite. Moreover, since

$$C_A(G/Z(G)) \leq Hom(G/Z(G, A), \gamma_2(G, A) \cap Z(G)),$$

it is concluded that $C_A(G/Z(G))$ is finite. Therefore A is finite and the result holds by Theorem C(1). □

It is easy to see that Part (2) of Theorem C is a generalization of its Part (1). Therefore to prove the generalization we need the following definition.

Definition 3.5 A set L of subgroups of the group G is called a local system of G, if $G = \bigcup_{S\in L} S$ and for every pair S, T in L, there exists a subgroup $U \in L$ of G such that $S, T \subseteq U$.

Proof of Theorem C (2) Let $G/Z_n(G, A)$ be finitely generated and assume that $A/Inn(G)$ is of finite special rank k. Suppose that L is a local system of all finitely generated subgroups of $A/Inn(G)$ and $A_1/Inn(G)$ is an arbitrary element of L. Since the finite special rank of $A/Inn(G)$ equals k, then $A_1/Inn(G)$ can be generated by at most k elements. Thus by Theorem C(1) we have

$$|\frac{G}{Z_n(G, A_1)}| \leq |\gamma_{n+1}(G, A_1)|^{(d+k)^n} \leq |\gamma_{n+1}(G, A)|^{(d+k)^n} = c.$$

Now we claim that $|G/Z_n(G, A)| \leq c$. By contradiction, if $|G/Z_n(G, A)| > c$, then for each $1 \leq i \leq c+1$, there exists $g_i Z_n(G, A) \in G/Z_n(G, A)$. Moreover, for each $1 \leq i \leq c$ there exist elements $\alpha_{1_{j_i}}, \ldots, \alpha_{n_{j_i}}$ in A, such that $\left[g_i g_j^{-1}, \alpha_{1_{j_i}}, \ldots, \alpha_{n_{j_i}}\right] \neq 1$ for $1 \leq i < j \leq c+1$. Since L is a local system, so for each $1 \leq i < j \leq c+1$ there exists a subgroup $A_{j_i}/Inn(G) \in L$, such that all $\alpha_{1_{j_i}} Inn(G), \ldots, \alpha_{n_{j_i}} Inn(G)$ belong to $A_{j_i}/Inn(G)$ and hence there exists a subgroup $A_s/Inn(G) \in L$ such that $A_{j_i}/Inn(G) \subseteq A_s/Inn(G)$, for all i and j. Based on Theorem C(1) we have

$$|\frac{G}{Z_n(G, A_s)}| \leq |\gamma_{n+1}(G, A_s)|^{(d+k)^n} \leq |\gamma_{n+1}(G, A)|^{(d+k)^n} = c,$$

which contradicts the existence of g_i's. Therefore the result holds in this case.
Now assume that $G/Z_n(G, A)$ has finite special rank k and $A/Inn(G)$ is a finitely generated group. In this case we proceed by induction on n. We also assume that L is a local system of all finitely generated subgroups of $G/Z_n(G, A)$ and $H/Z_n(G, A)$ is an arbitrary element of L. Since $G/Z_n(G, A)$ has finite special rank k, then $H/Z_n(G, A)$

may be generated by k elements. For $n = 1$, by Lemma 3.2, we have

$$|\frac{H}{Z(H)}| \leq |\gamma_2(H)|^k \leq |\gamma_2(G, A)|^k = c.$$

We claim that $|G/Z(G)| \leq |\gamma_2(G, A)|^k = c$. By contradiction, if $|G/Z(G)| > c$, then we can choose elements $g_i Z(G) \in G/Z(G)$, $1 \leq i \leq c+1$. For each g_i, g_j, there exists x_{j_i} such that $\left[g_i g_j^{-1}, x_{j_i}\right] \neq 1$. Since there exists $H_1/Z(G, A) \in L$ and $\langle g_i, x_{j_i} \mid 1 \leq i \leq j \leq c+1\rangle \subseteq H_1$, we have

$$|\frac{H_1}{Z(H_1)}| \leq |\gamma_2(H_1)|^k \leq |\gamma_2(G, A)|^k = c,$$

which contradicts the choice of g_i's. As a result, $|G/Z(G)| \leq |\gamma_2(G, A)|^k = c$. Now similar to the proof of Theorem C(1), it is easy to see that

$$\begin{aligned} f_1 : \frac{Z(G)}{Z(G, A)} &\longrightarrow (\gamma_2(G, A))^d \\ x_1 Z(G, A) &\longmapsto ([x_1, \alpha_1], \ldots, [x_1, \alpha_d]) \end{aligned}$$

is a monomorphism. Hence the result is completed in this case. Let

$$\frac{H}{Z_n(G, A)} = \langle h_1 Z_n(G, A), \ldots, h_k Z_n(G, A)\rangle$$

be an arbitrary element of L. Then we have $H = \langle h_1, \ldots, h_k, Z_n(G, A)\rangle$. Similar to the proof of Theorem C(1) we have

$$|\frac{H \cap \gamma_n(G, A)}{Z(H) \cap \gamma_n(G, A)}| \leq |\left[H \cap \gamma_n(G, A), H\right]|^k \leq |\gamma_{n+1}(G, A)|^k = c.$$

Here again we claim that $|\gamma_n(G, A)/(\gamma_n(G, A) \cap Z(G))| \leq |\gamma_{n+1}(G, A)|^k = c$. If $|\gamma_n(G, A)/(\gamma_n(G, A) \cap Z(G))| > c$, then we can take the elements $g_i(Z(G) \cap \gamma_n(G, A))$ from $\gamma_n(G, A)/(Z(G) \cap \gamma_n(G, A))$, $1 \leq i \leq c+1$. Therefore for every g_i, g_j, there exists $x_{j_i} \in G$ with $[g_i g_j^{-1}, x_{j_i}] \neq 1$. Since L is a local system, there exists $H_2 \in L$ such that

$$\langle g_i, x_{j_i}, Z_n(G, A) | 1 \leq i \leq j \leq c+1\rangle \subseteq H_2.$$

Thus

$$|\frac{H_2 \cap \gamma_n(G, A)}{Z(H_2) \cap \gamma_n(G, A)}| \leq |\left[H_2 \cap \gamma_n(G, A), H_2\right]|^k \leq |\gamma_{n+1}(G, A)|^k = c,$$

which contradicts the existence of g_i's. Similar to the proof of Theorem C(1),

$$|\frac{\gamma_n(G,A)\cap Z(G)}{\gamma_n(G,A)\cap Z(G,A)}| \leq |\gamma_{n+1}(G,A)|^d.$$

Finally by induction hypothesis we have

$$|\frac{G}{Z_n(G,A)}| \leq |\frac{\gamma_n(G,A)}{\gamma_n(G,A)\cap Z(G,A)}|^{(d+k)^{n-1}}.$$

Hence the proof is completed. □

The third part of Theorem C is proved similar to the first part of Theorem C(2). If we assume the group G to be nilpotent in Theorem C, then the result would improve as indicated in Theorem D.

Proof of Theorem D (1) (1) Define a function

$$\begin{aligned} h_1 : Z(G) &\to \gamma_2(G,A)^{d(A/Inn(G))} \\ x &\mapsto ([x,\alpha_1],\ldots,[x,\alpha_{d(A/Inn(G))}]) \end{aligned}$$

where $\{\alpha_1 Inn(G),\ldots,\alpha_{d(A/Inn(G))}Inn(G)\}$ is a generating set of $A/Inn(G)$. Clearly h_1 is a homomorphism with $\ker h_1 = Z(G,A)$, which implies that $Z(G)/Z(G,A)$ is isomorphic to the subgroup $\prod_{i=1}^{d(A/Inn(G))}\gamma_2(G,A)$. Hence

$$|\frac{Z(G)}{Z(G,A)}|\,\big|\,|\gamma_2(G,A)|^{d(\frac{A}{Inn(G)})}.$$

By Theorem 1.7, we have

$$|\frac{G}{Z(G)}|\,\big|\,|\gamma_2(G)|^{d(\frac{G}{Z(G)})}\,\big|\,|\gamma_2(G,A)|^{d(\frac{G}{Z(G,A)})},$$

since G is nilpotent. Therefore

$$|\frac{G}{Z(G,A)}|\,\big|\,|\gamma_2(G,A)|^{(d(\frac{G}{Z(G,A)})+d(\frac{A}{Inn(G)}))},$$

and this proves the case $n = 1$. Now suppose that the statement holds for $n-1$. By Theorem C(1) and its proof, $G/Z_n(G,A)$ and $\gamma_n(G/Z(G,A),A)$ are finite. Using induction hypothesis we have

$$|\frac{G}{Z_n(G,A)}|\,\big|\,|\gamma_n(\frac{G}{Z(G,A)},A)|^{(d(\frac{G}{Z_n(G,A)})+d(\frac{A}{Inn(G)}))^{n-1}}.$$

Take an A-invariant finitely generated subgroup H of $\gamma_n(G,A)$ such that $\gamma_n(G,A) = HZ(G,A)$. It should be noticed that $H\cap Z(G,A)$ is finitely generated because

$\gamma_n(G,A)/\gamma_n(G,A) \cap Z(G,A) \cong H/H \cap Z(G,A)$ is finite. Moreover since $H \cap Z(G,A)$ is a finitely generated abelian subgroup of $\gamma_n(G,A)$, then there exists a torsion free subgroup N of $H \cap Z(G,A)$ of finite index. It is easy to see that N is an A-invariant subgroup of $\gamma_n(G,A)$. Define the following function

$$f : A \longrightarrow Aut(\frac{H}{N}) \quad \alpha \mapsto \tilde{\alpha}; \quad \forall h \in H, (hN)\tilde{\alpha} = (h)\alpha N.$$

f is a homomorphism with $\ker f = C_A(\gamma_n(G,A))$. To show this, let $\beta \in \ker f$. Then for each $hN \in \frac{H}{N}$, $(hN)\tilde{\beta} = hN$ and so $h^{-1}(h)\beta \in N$. On the other hand, since N is torsion free, $h^{-1}(h)\beta \in \gamma_{n+1}(G,A)$, which implies $(h)\beta = h$. Since $\gamma_n(G,A) = HZ(G,A)$, we conclude that for every $h \in \gamma_n(G,A)$, $(h)\beta = h$. Suppose $B = Imf$ and

$$Z(H/N, B) = (Z(G,A) \cap H)/N.$$

Thus $(H/N)/Z(H/N,B) \cong \gamma_n(G,A)/\gamma_n(G,A) \cap Z(G,A)$. Moreover,

$$\begin{aligned}\gamma_{n+1}(G,A) &= \langle [hz,\alpha]\,|h \in H, z \in \gamma_n(G,A) \cap Z(G,A), \alpha \in A\rangle \\ &= \langle [h,\alpha]\,|h \in H, \alpha \in A\rangle \\ &= \gamma_2(H,A) \cong \gamma_2(\frac{H}{N}, B).\end{aligned}$$

Since H/N is a finite nilpotent group, then $H/N \cong P_1/N \times \cdots \times P_k/N$, in which P_i/N is the p_i-Sylow subgroup of H/N. Note that P_i/N is a characteristic subgroup of H/N and H is an A-invariant subgroup of G. This means that $P_i \trianglelefteq G$. Let $\{y_1Z_n(G,A), \ldots, y_dZ_n(G,A)\}$ is a generating set for $G/Z_n(G,A)$. Define $l_i : P_i/(Z(G) \cap P_i) \to (\gamma_2(H,A) \cap P_i)^{d(G/Z_n(G,A))}$, such that l_i carries every $x(Z(G) \cap P_i) \in P_i/(Z(G) \cap P_i)$ to $([x,y_1], \ldots, [x,y_d])$. Obviously l_i is a one-to-one map. Since $N \subseteq Z(G)$, then $P_i/(Z(G) \cap P_i)$ is finite. Hence

$$\begin{aligned}|\frac{P_i/N}{Z(G) \cap P_i/N}| &\le |\frac{\gamma_2(H,A) \cap P_i}{\gamma_2(H,A) \cap P_i \cap N}|^{d(\frac{G}{Z_n(G,A)})} \\ &= |\gamma_2(H,B) \cap P_i|^{d(\frac{G}{Z_n(G,A)})}.\end{aligned}$$

Since $(P_i/N)/((Z(G) \cap P_i)/N)$ and $(\gamma_2(H,A) \cap P_i)/(\gamma_2(H,A) \cap P_i \cap N)$ are both p_i-groups, then

$$|\frac{P_i}{Z(G) \cap P_i}|\,\big|\,|\gamma_2(H,A) \cap P_i|^{d(\frac{G}{Z_n(G,A)})}.$$

It implies that $|(H/N)/((Z(G) \cap H)/N)|\,\big|\,|\gamma_2(H,A)|^{d(G/Z_n(G,A))}$. On the other hand

$$\frac{H}{Z(G) \cap H} \cong \frac{HZ(G)}{Z(G)} = \frac{HZ(G,A)Z(G)}{Z(G)} \cong \frac{\gamma_n(G,A)}{\gamma_n(G,A) \cap Z(G)}.$$

Consequently,

$$|\frac{\gamma_n(G,A)}{\gamma_n(G,A) \cap Z(G)}|\,\big|\,|\gamma_{n+1}(G,A)|^{d(\frac{G}{Z_n(G,A)})}.$$

Define homomorphism $h_n : \gamma_n(G, A) \cap Z(G) \to \gamma_{n+1}(G, A)$ by

$$h_n(x) = ([x, \alpha_1], \ldots, [x, \alpha_{d(A/Inn(G))}]).$$

Since $\ker h_n = Z(G, A) \cap \gamma_n(G, A)$ then

$$\frac{Z(G) \cap \gamma_n(G, A)}{Z(G, A) \cap \gamma_n(G, A)} \cong \prod_{i=1}^{d(A/Inn(G))} \gamma_{n+1}(G, A).$$

Hence

$$|\frac{\gamma_n(G, A) \cap Z(G)}{\gamma_n(G, A) \cap Z(G, A)}|\big||\gamma_{n+1}(G, A)|^{d(\frac{A}{Inn(G)})}.$$

Finally by induction hypothesis the result is obtained.
(2) The proof is easily done similar to the proof of Theorem C(2) by considering Theorem D(1) and Theorem 1.7 instead of Theorem C and Lemma 3.2, respectively.
(3) The proof is similar to Part (2). □

Now we are ready to give an answer to Question 1 of Yadav.

Lemma 3.6 *There is no non-nilpotent group G for which the Equality 3.2 is satisfied. In other words if* $|G/Z(G)| = |\gamma_2(G)|^{d(G/Z(G))}$, *then G is nilpotent.*

Proof Assume that there exists a group G for which $|G/Z(G)| = |\gamma_2(G)|^{d(G/Z(G))}$. We show that G is nilpotent. Define

$$F : \frac{G/\Phi(G)}{Z(G/\Phi(G))} \longrightarrow (\frac{\gamma_2(G)\Phi(G)}{\Phi(G)})^{d(G/Z(G))},$$

with $F(g\Phi(G)Z(G/\Phi(G))) = ([g, x_1]\Phi(G), \ldots, [g, x_{d(G/Z(G))}]\Phi(G))$, in which $\{x_i Z(G)\}_{1 \leq i \leq d(G/Z(G))}$ is a generating set of $G/Z(G)$ and $\Phi(G)$ is Frattini subgroup. We claim that F is bijection. If $g_1\Phi(G)Z(G/\Phi(G)) = g_2\Phi(G)Z(G/\Phi(G))$, then $g_1 g_2^{-1}\Phi(G) \in Z(G/\Phi(G))$. As a result,

$$\begin{aligned}
[g_1 g_2^{-1}\Phi(G), x_i\Phi(G)] &= [g_1\Phi(G), x_i\Phi(G)]^{g_2^{-1}\Phi(G)}[g_2^{-1}\Phi(G), x_i\Phi(G)] \\
&= [g_1\Phi(G), x_i\Phi(G)]^{g_2^{-1}\Phi(G)}([g_2\Phi(G), x_i\Phi(G)]^{-1})^{g_2^{-1}\Phi(G)} \\
&= ([g_1\Phi(G), x_i\Phi(G)][g_2\Phi(G), x_i\Phi(G)]^{-1})^{g_2^{-1}\Phi(G)} \\
&= \Phi(G).
\end{aligned}$$

Hence F is well-defined. The converse of the above statement is also true. This means F is one-to-one. To show that F is onto, we define

$$\begin{aligned}
f : G/Z(G) &\to (\gamma_2(G))^{d(G/Z(G))} \\
gZ(G) &\mapsto ([g, x_1], \ldots, [g, x_{d(G/Z(G))}])
\end{aligned}$$

which is a one-to-one function. By the equality 3.2, we conclude that f is onto. Hence F is onto and based on Lemma 3.1 we have $|(G/\Phi(G))/Z(G/\Phi(G))| \leq |(G'\Phi(G))/\Phi(G)|^2$, which implies that $d(G/Z(G)) \leq 2$. If $d = 1$, then $G/Z(G)$ is cyclic and the result holds. If $d(G/Z(G)) = 2$, then the Equality 3.1 holds that means $G/\Phi(G)$ is abelian and $G' \subseteq \Phi(G)$. This proof is now complete. □

Pertaining to the Yadav's problem, a natural question is the following.

Question 3 Suppose that $d(G/Z_n(G, A)), d(A/Inn(G))$ and $\gamma_{n+1}(G, A)$ are all finite and

$$|\frac{G}{Z_n(G, A)}| = |\gamma_{n+1}(G, A)|^{(d(\frac{G}{Z_n(G)})+d(\frac{A}{Inn(G)}))^n}.$$

Is it possible for G to be nilpotent?

In order to respond to this question, we need the following lemma.

Lemma 3.7 *Assume that G is a group in which $\gamma_{n+1}(G)$ is finite and $d(G/Z_n(G)) = d$. Then*

$$|G/Z_n(G)| \leq |\gamma_2(G/Z_{n-1}(G))|^d \leq \cdots \leq |\gamma_{n+1}(G)|^{d^n}. \tag{3.3}$$

Proof Define the map

$$\begin{aligned} \theta : \frac{\gamma_i(G)}{Z_{n-i+1}(G) \cap \gamma_i(G)} &\longrightarrow (\frac{\gamma_{i+1}(G)}{Z_{n-i}(G) \cap \gamma_{i+1}(G)})^d \\ hZ_{n-i+1}(G) \cap \gamma_i(G) &\longmapsto ([h, x_1]Z_{n-i}(G) \cap \gamma_{i+1}(G), \ldots, \\ &\qquad [h, x_d]Z_{n-i}(G) \cap \gamma_{i+1}(G)). \end{aligned}$$

Since θ is clearly well-defined, it is enough to prove that θ is one-to-one. Let

$$\begin{aligned} &([h_1, x_1]Z_{n-i}(G) \cap \gamma_{i+1}(G), \ldots, [h_1, x_d]Z_{n-i}(G) \cap \gamma_{i+1}(G)) \\ =&([h_2, x_1]Z_{n-i}(G) \cap \gamma_{i+1}(G), \ldots, [h_2, x_d]Z_{n-i}(G) \cap \gamma_{i+1}(G)). \end{aligned}$$

Then for each $x_1^{m_1} \ldots x_d^{m_d} z_n = g \in G$ we have

$$\begin{aligned} [h_1, x_1^{m_1} \ldots x_d^{m_d} z_n]Z_{n-i}(G) \cap \gamma_{i+1}(G) &= [h_1, z_n] \ldots [h_1, x_1]^{\beta_m} Z_{n-i}(G) \cap \gamma_{i+1}(G) \\ &= [h_2, z_n] \ldots [h_2, x_1]^{\beta_m} Z_{n-i}(G) \cap \gamma_{i+1}(G) \\ &= [h_2, x_1^{m_1} \ldots x_d^{m_d} z_n]Z_{n-i}(G) \cap \gamma_{i+1}(G). \end{aligned}$$

Therefore

$$\begin{aligned} [h_1 h_2^{-1}, g]Z_{n-i}(G) \cap \gamma_{i+1}(G) &= [h_1, g]^{h_2^{-1}} [h_2^{-1}, g]Z_{n-i}(G) \cap \gamma_{i+1}(G) \\ &= [h_1, g]^{h_2^{-1}} ([h_2, g]^{-1})^{h_2^{-1}} Z_{n-i}(G) \cap \gamma_{i+1}(G) \\ &= ([h_1, g][h_2, g]^{-1})^{h_2^{-1}} Z_{n-i}(G) \cap \gamma_{i+1}(G) = 1. \end{aligned}$$

It implies that $h_1h_2^{-1} \in Z(G/Z_{n-i}(G))$. Hence

$$h_1Z_{n-i+1}(G) \cap \gamma_i(G) = h_2Z_{n-i+1}(G) \cap \gamma_i(G)$$

which means θ is one-to-one. □

We are now ready to give an answer to Question 3, when $A = Inn(G)$. The next theorem shows that there is no non-nilpotent group can substitute the inequality used in Theorem 1.8 for the equality.

Theorem 3.8 *Let G be a finite group. If $|G/Z_n(G)| = |\gamma_{n+1}(G)|^{d^n}$, then G is nilpotent.*

Proof For $n \neq 1$, the first and the last terms of Inequality 3.3 are equal. Thus the proof is done by Lemma 3.6. □

The following theorem gives an answer to Question 3.

Proof of Theorem E According to the proof of Theorem C(1)

$$\begin{aligned}|\frac{G}{Z_n(G,A)}| &\le |\gamma_2(\frac{G}{Z_{n-1}(G,A)}, A)|^{d(\frac{G}{Z_n(G,A)})+d(\frac{A}{Inn(G)})} \\ &\le \cdots \le |\gamma_{n+1}(G,A)|^{(d(\frac{G}{Z_n(G,A)})+d(\frac{A}{Inn(G)}))^n}. \end{aligned} \tag{3.4}$$

Since

$$|G/Z(G)| \le |\gamma_2(G)|^{d(G/Z(G))} \le |\gamma_2(G,A)|^{d(G/Z(G,A))}$$

and

$$|Z(G)/Z(G,A)| \le |\gamma_2(G,A)|^{d(A/Inn(G))},$$

we have

$$|G/Z(G)| = |\gamma_2(G)|^{d(G/Z(G))} = |\gamma_2(G,A)|^{d(G/Z(G,A))} \tag{3.5}$$

and

$$|Z(G)/Z(G,A)| = |\gamma_2(G,A)|^{d(A/Inn(G))}.$$

Hence by the Equality 3.5 and based on Lemma 3.6 we conclude that there is no non-nilpotent group for which the equality of Inequality 1.2 holds for $n = 1$. Now for $n \neq 1$ the first and the last terms in Inequality 3.4 are equal to each other and so the proof is done using the case $n = 1$. □

In what follows, we present an interesting answer to Question 2.

Theorem 3.9 *There exists a non-nilpotent group G, not isomorphic to $X \times H$, such that $\gamma_2(G)$ is finite but $G/Z(G)$ is infinite, where X is a finite group and H is a nilpotent group.*

Proof Let K be a non-nilpotent group isomorphic to $K = \mathbb{Z}_q \ltimes \mathbb{Z}_{pq}$ such that $q^2 > p > q$ (one should notice that such a group exists, for example D_{12}). Then $\gamma_2(K) = \mathbb{Z}_p$, since $\mathbb{Z}_p \trianglelefteq^c \mathbb{Z}_{pq}$, $\mathbb{Z}_{pq} \trianglelefteq K$ and $|(\mathbb{Z}_q \ltimes \mathbb{Z}_{pq})/\mathbb{Z}_p| = q^2$. By Lemma 3.3, we have

$$|\frac{K/Z(K)}{Z(K/Z(K))}| \leq |\frac{\gamma_2(K)Z(K)}{Z(K)}|^2.$$

Note that $|\gamma_2(K) \cap Z(K)| \neq p$, which means $\gamma_2(K) \cap Z(K) = 1$. Thus $Z(K)$ is a non-trivial q-group because

$$|\frac{K/Z(K)}{Z(K/Z(K))}| \leq |\frac{\gamma_2(K)Z(K)}{Z(K)}|^2.$$

This shows that $|Z(K)| = q$. Now define G to be a central product of an infinite number of extra-special q-groups, each of order q^{2n+1}, and the non-nilpotent group K. Since G is not finitely generated and $|Z(G)| = q$, then we conclude that $G/Z(G)$ is infinite. By the definition of the central product, $G' \cong Z_p \oplus Z_q$, as required. □

Declarations

Competing interests The authors have no competing interests to declare that are relevant to the content of this article.

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