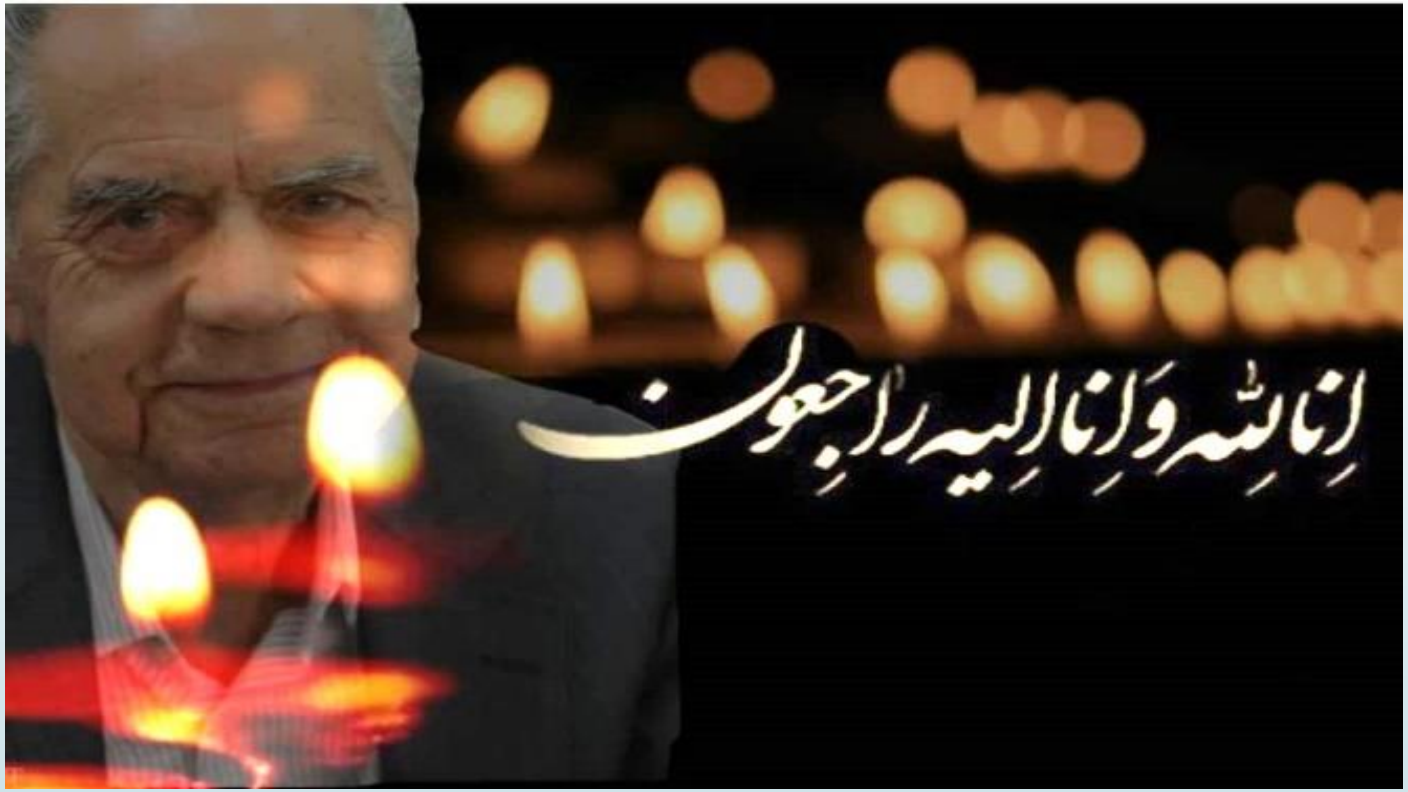
انالله و انا الیه راجعون

تقدیم به روح پاک استاد فقید: ابوالقاسم بزرگ نیا-(بنیان گذار نظریه وابستگی منفی در ایران)
و استاد فقید حسینعلی نیرومند-(متخصص سریهای زمانی)



- استاد عزیز، یاد شما همواره در دل‌های ما جاودانه خواهد بود و ما با عشق و ارادت، درس‌های شما را به نسل‌های آینده منتقل خواهیم کرد.

روحتان شاد؛ یادتان گرامی و راهتان پر رهرو باد

به نام خداوند جان و خرد

- مروری بر همگرایی کامل دنباله های تصادفی m- وابسته منفی چندگانه (m-NOD)

Review on complete convergence for *m*-negatively orthant dependent (*m*-NOD) random variables

محمد امینی و حبیب نادری

گروه آمار، دانشکده علوم ریاضی، دانشگاه فردوسی مشهد

Department of Statistics, Faculty of Mathematical Sciences,
Ferdowsi University of Mashhad

گروه آمار، دانشکده علوم ریاضی، دانشگاه سیستان و بلوچستان

Department of Statistics, Faculty of Mathematical Sciences,
University of Sistan and Baluchestan,
Zahedan, Iran

مقدمه

- در این سخنرانی مروری بر همگرایی کامل دنباله تصادفی از متغیرهای تصادفی به طور گسترده ام-وابسته منفی ارایه می شود.
-
- تعمیم نتایج "جاجت" برای تعیین نرخ همگرایی کامل، این دنباله تصادفی از اهداف دیگر می باشد.
- علاوه بر این مطالعه شبیه سازی برای تعیین نرخ همگرایی کامل از نتایج پایانی می باشد.
-

فهرست مطالب

2 Negative dependence random variables

Different forms of non-independent random variables have been of interest for many years. Lehmann (1966) investigated various concepts of positive and negative dependence in the bivariate cases. Strong concepts of bivariate positive and negative dependence were introduced by Esary and Proschen (1967). Also Esary , Proschen and Walkup (1972) introduced a concept of association which implied a strong form of positive dependence. Their concept has been very useful in reliability theory and its applications. Multivariate generalizations of these concepts of dependence were initiated by Ebrahimi and Ghosh (1981) and Bozorgnia et.al.(1996).

Definition 3. The random variables $X_1, \cdots, X_n$ are said to be ND if we have

$$P[\bigcap_{j=1}^{n}(X_j \leq x_j)] \leq \prod_{j=1}^{n} P[X_j \leq x_j], \qquad (2)$$

and

$$P[\bigcap_{j=1}^{n}(X_j > x_j)] \leq \prod_{j=1}^{n} P[X_j > x_j], \qquad (3)$$

for all $x_1, \cdots, x_n \in R$. An infinite sequence $\{X_n \ , \ n \geq 1\}$ is said to be ND if every finite subset $X_1, \cdots, X_n$ is ND. The conditions (2) and (3) are equivalent for $n = 2$, but these do not agree for $n \geq 3$.

The sequence $\{X_n\}$ is said pair-wise negative dependence (PND)if for all $i \neq j$. X_i *and* X_j are NQD, means that for all real numbers x, y,

$$P(X_i \leq x, Y_j \leq y) \leq P(X_i \leq x)P(Y_j \leq y).$$

Some peroperties

Lemma 3. *i*) Let $X_1, \cdots, X_n$ be ND random variables and $f_1 \cdots, f_n$ be a sequence of Borel functions which all are monotone increasing (or all are monotone decreasing) , then $f_1(X_1), \cdots, f_n(X_n)$ are ND random variables.

ii) Let $X_1, \cdots, X_n$ be ND nonnegative random variables. Then

$$E[\prod_{j=1}^{n} X_j] \leq \prod_{j=1}^{n} E[X_j].$$

Modes of Convergence for Random Variables

Random variables can converge in several different ways. Here is a brief introduction, highlighting examples of the different behaviors that can occur. Many of the examples below are taken from Stoyanov (2013).

There are at least six different notions of convergence for random variables.

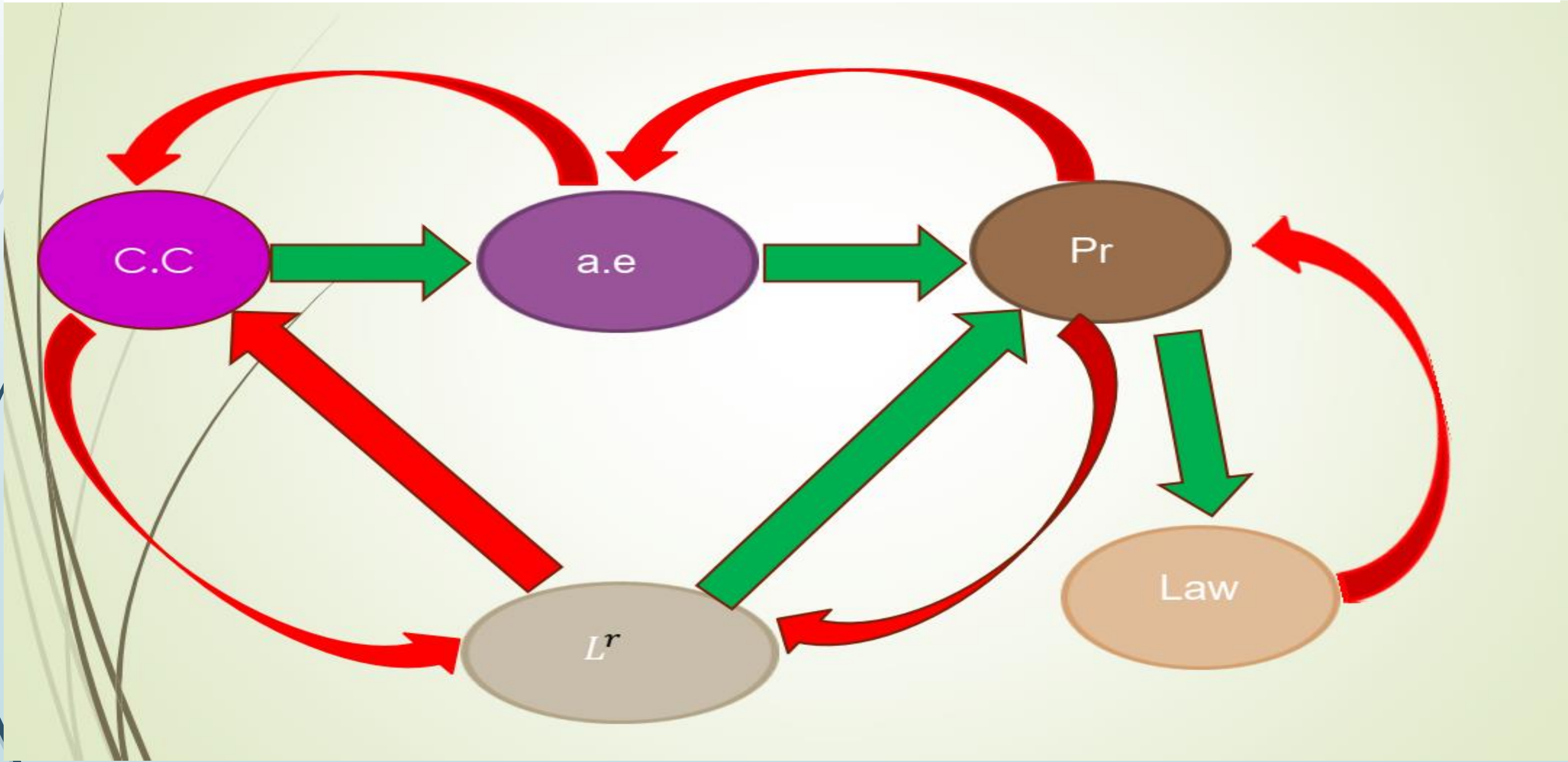
1. X_n converges to X **pointwise** if $X_n(\omega) \to X(\omega)$ for all ω.
2. X_n converges to X **almost surely** or **almost everywhere** if $X_n(\omega) \to X(\omega)$ for almost all ω, i.e., for all ω in a set of probability 1. Thus, almost sure convergence requires $\Pr(\{\omega \mid X_n(\omega) \to X(\omega)\}) = 1$.
3. X_n converges to X in **probability** if for any $\epsilon > 0$ we have $\Pr(|X_n(\omega) - X(\omega)| > \epsilon) \to 0$ as $n \to \infty$.
4. X_n converges to X in **distribution** or **law** or **weakly** or in the **weak* topology** if $F_n(x) \to F(x)$ for all x for which $F(x)$ is continuous, where F_n, F are the distribution functions of X_n, X.
5. X_n converges to X in the L^1**-norm** if $\int |X_n(\omega) - X(\omega)| \Pr(d\omega) \to 0$ as $n \to \infty$. More generally, for $p \geq 1$ there is convergence for the L^p**-norm** if $\int |X_n(\omega) - X(\omega)|^p \Pr(d\omega) \to 0$.
6. X_n converges to X in the **sup-norm** if $\sup_\Omega |X_n - X| \to 0$.

ارتباط همگراییها

$$X_n \xrightarrow{c.c} X \Rightarrow X_n \xrightarrow{a.e} X \Rightarrow X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X,$$

$$\Uparrow$$

$$X_n \xrightarrow{r} X.$$



Arrays of rowwise *m*-negatively associated random variables -2009

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On complete convergence for arrays of rowwise *m*-negatively associated random variables

Tien-Chung Hu [a], Chen-Yu Chiang [a], Robert L. Taylor [b,*]

[a] *Department of Mathematics, National Tsing Hua University, Hsinchu 300, Taiwan, ROC*

[b] *Department of Mathematical Science, Clemson University, Clemson SC 29634-0975, USA*

A R T I C L E I N F O

Keywords:
Complete convergence
m-negatively associated random variables
Array of rowwise *m*-negatively associated random variables

A B S T R A C T

We establish complete convergence results for arrays of rowwise *m*-negatively associated random variables. Some results for negatively associated and independent random variables in the literature can be obtained as corollaries.

M-negatively orthant random variables(m-NOD)-2018

MOMENT INEQUALITIES FOR m-NOD RANDOM VARIABLES AND THEIR APPLICATIONS*

X. J. WANG†, S. H. HU†, AND A. I. VOLODIN‡

Abstract. The concept of m-negatively orthant dependent (m-NOD) random variables is introduced, and the moment inequalities for m-NOD random variables, especially the Marcinkiewicz–Zygmund-type inequality and Rosenthal-type inequality, are established. As one application of the moment inequalities, we study the L_r convergence and strong convergence for m-NOD random variables under some uniformly integrable conditions. On the other hand, the asymptotic approximation of inverse moments for nonnegative m-NOD random variables with finite first moments is established. The results obtained in the paper generalize or improve some known ones for independent sequences and some dependent sequences.

Stat Papers (2017) 58:911–928
DOI 10.1007/s00362-015-0731-x

REGULAR ARTICLE

Moment inequalities for m-negatively associated random variables and their applications

Aiting Shen[1] · Yu Zhang[1] · Benqiong Xiao[1] · Andrei Volodin[2]

Inspired by the definition of NA random variables, Hu et al. (2009) introduced the concept of m-negatively associated random variables as follows.

Definition 1.2 Let $m \geq 1$ be a fixed integer. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be m-negatively associated (m-NA, in short) if for any $n \geq 2$ and any $i_1, i_2, \ldots, i_n$ such that $|i_k - i_j| \geq m$ for all $1 \leq k \neq j \leq n$, we have that $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$ are NA.

m-WOD random variables-2025

- Complete convergence and complete moment convergence for maximal randomly weighted sums of *m*-WOD random variables with applications
- Z Zhang, X Zheng, S Wang, Y Li… - … in Statistics-Theory and …, 2025 - Taylor & Francis

- L_r convergence for arrays of rowwise *m*-extended negatively dependent random variables
- Z Wang, Y Wu, Y Du, X Wang - Communications in Statistics …, 2024 - Taylor & Francis

- Limiting Behaviors for Weighted Sums of m-WOD Random Variables under Integrability Assumptions
- Y Wu, MM Xi, XJ Wang - Theory of Probability & Its **Applications**, 2025 – SIAM

- Limiting behaviours for weighted sums of **m-WOD random variables** under integrability assumptions
- Y Wu, M Xi, XJ Wang - Теория вероятностей и ее применения, 2024

- Some results on **m-linearly extended negative quadrant dependent random variables**
- M Kaber El Alem - Communications in Statistics-Theory and …, 2025 - Taylor & Francis

- Complete Convergence for Moving Average Processes under **m-WOD Random Variables**
- S Mingzhu, WU Yongfeng - Wuhan University Journal of …, 2024 - wujns.edpsciences.org

- Strong convergence properties for weighted sums of *m*-asymptotic negatively associated random variables and statistical applications
- Y Wu, X Wang, A Shen - Statistical Papers, 2021 – Springer
- Complete moment convergence for moving average process based on *m*-WOD random variables
- R Wang, A Shen - Communications in Statistics-Theory and …, 2023 - Taylor & Francis
- Complete moment convergence of moving average processes for m-WOD sequence
- L Guan, Y Xiao, Y Zhao - Journal of Inequalities and **Applications**, 2021 – Springer
- Complete Convergence for M-Pairwise Negatively Dependent Random Variables
- TVA Vo - Journal of Technical Education Science, 2022 - jte.edu.vn

M-Pairwise Negatively Dependent Random Variables

- **Convergence in Mean for Double Arrays of M-Pairwise Negatively Dependent Random Variables**
- VT Van Anh, NTN Anh, NTT Hien, NN Tu - Bulletin of **the** Malaysian …, 2022 – Springer
- **Complete Convergence for M-Pairwise Negatively Dependent Random Variables**
- VT Van Anh - scholar.archive.org
- **Complete qth-Moment Convergence of Moving Average Process for m-WOD Random Variable**
- S Mingzhu, WU Yongfeng… - Wuhan University Journal …, 2022 - wujns.edpsciences.org

DOI 10.1007/s10986-025-09687-x
Lithuanian Mathematical Journal

A note on complete convergence for m-NOD random variables

Habib Naderi [a], **Mehdi Jafari** [a], **Przemysław Matuła** [b], **and Mohammad Amini** [c]

[a] Department of Statistics, Faculty of Mathematics, Statistics and Computer Science, University of Sistan and Baluchestan, Zahedan, Iran
[b] Institute of Computer Science and Mathematics, Marie Curie-Skłodowska University, pl. M.C.-Skłodowskiej 1, 20-031 Lublin, Poland
[c] Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran
(e-mail: h.h.naderi@gmail.com; mejafari1361@gmail.com; przemyslaw.matula@mail.umcs.pl; m-amini@um.ac.ir)

Received January 12, 2025; revised April 24, 2025

Abstract. In this paper, we extend Jajte's technique to study the rate of complete convergence of a weighted sequence of m-NOD random variables. In addition, we make a simulation study to illustrate the asymptotic behavior in the sense of the rate of complete convergence.

MSC: 60F15, 60E15, 62H20

Keywords: dependent random variables, complete convergence, weighted sums

Definition of m-NOD random sequencs

DEFINITION 2. Let $m \geqslant 1$ be a fixed integer. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be m-negatively orthant dependent (m-NOD) if for all $n \geqslant 2$ and $i_1, \ldots, i_n$ such that $|i_k - i_j| \geqslant m$ for all $1 \leqslant k \neq j \leqslant n$, we have that $X_{i_1}, \ldots, X_{i_n}$ are NOD.

An array $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ of random variables is said to be rowwise m-NOD if for every $n \geqslant 1$, $\{X_{ni}, i \geqslant 1\}$ is a sequence of m-NOD random variables. For $m = 1$, the concept m-NOD random variables reduces to the NOD random variables. Hence the concept of m-NOD random variables is a natural extension of NOD random variables, which includes independent random variables and negatively associated (NA) random variables.

The m-NOD property is preserved under monotonic functions:

Lemma 1. (*See* [18].) *Let $\{X_n, n \geqslant 1\}$ be a sequence of m-NOD random variables. If $\{g_n, n \geqslant 1\}$ are all nondecreasing (or nonincreasing) functions, then a sequence of random random variables $\{g_n(X_n), n \geqslant 1\}$ is also an m-NOD random sequence.*

Jajte-Method

Jajte [8] studied a large class of summability methods defined as follows: a sequence $\{X_n,\ n \geqslant 1\}$ of random variables is said to be almost surely summable to a random variable X by the method (h, g) if

$$\frac{1}{g(n)} \sum_{k=1}^{n} \frac{1}{h(k)} X_k \xrightarrow{\text{a.s.}} X, \quad n \to \infty.$$

For a sequence $\{X_n,\ n \geqslant 1\}$ of i.i.d. random variables, Jajte proved that $\{X_n - \mathbf{E}X_n\mathbf{I}_{\{|X_n| \leqslant \psi(n)\}},\ n \geqslant 1\}$ is almost surely summable to 0 by the method (h, g) iff $\mathbf{E}\psi^{-1}(|X_1|) < \infty$ (ψ^{-1} is the inverse of ψ), where g, h, and $\psi(y) = g(y)h(y)$ are functions satisfying some additional conditions. The most up-to-date survey on this matter may be found in Fazakas et al. [5], Wang [19], Matuła and Seweryn [11], Shen [14], Tang [17], Son et al. [16], and Naderi et al. [12, 13].

Now we recall the concept of stochastic domination.

DEFINITION 3. A sequence $\{X_n,\ n \geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$\mathbf{P}\left(|X_n| > x\right) \leqslant C\mathbf{P}\left(|X| > x\right)$$

for all $x \geqslant 0$ and $n \geqslant 1$.

The next section will be devoted to the study of the rate of complete convergence for weighted m-NOD random variables in format of method (h, g).

We further denote by C positive constants, not depending on n, which may be different in various places by $\lfloor \cdot \rfloor$ the integer part of a number, and by $\mathbf{I}_A$ the indicator function of a set A.

Jajite-Method

We begin with the assumptions that will be imposed on our weights. Let $g : [0, \infty) \to \mathbb{R}$ and $h : [0, \infty) \to \mathbb{R}$ be nonnegative functions, and let $\psi(y) = g(y)h(y)$. We assume that the functions g, h, and ψ satisfy the following conditions:

(A1) h is nondecreasing, and ψ is strictly increasing with $\psi([0, \infty)) = [0, \infty)$,

(A2) there exist constants $a, b > 0$, $r \geqslant 1$, and $\alpha \in (1, 2]$ and a strictly increasing function H such that

$$\psi^{\alpha}(s) \int_{s}^{\infty} \frac{x^{r-1}}{\psi^{\alpha}(x)} \, \mathrm{d}x \leqslant aH(s) + b \quad \forall s > 0,$$

(A3) there exists a constant $C > 0$ such that for some $\alpha \in (1, 2]$,

$$\sum_{i=1}^{n} \frac{1}{h^{\alpha}(i)} \leqslant C \frac{n}{h^{\alpha}(n)}.$$

Some Results

Lemma 2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables stochastically dominated by a random variable X. For all $\beta > 0$ and $b > 0$, we have the following inequalities:*

$$\mathbf{E}|X_n|^{\beta}\mathbf{I}_{\{|X_n|\leqslant b\}} \leqslant C_1\left[\mathbf{E}|X|^{\beta}\mathbf{I}_{\{|X|\leqslant b\}} + b^{\beta}\mathbf{P}(|X| > b)\right],$$
$$\mathbf{E}|X_n|^{\beta}\mathbf{I}_{\{|X_n|>b\}} \leqslant C_2\mathbf{E}|X|^{\beta}\mathbf{I}_{\{|X|>b\}},$$

where C_1 and C_2 are positive constants. It is also obvious that $\mathbf{E}(|X_n|^{\beta}) \leqslant C\mathbf{E}(|X|^{\beta})$.

Lemma 3 [Rosenthal-type inequality]. *Let $\{X_n, n \geqslant 1\}$ be a sequence of m-NOD random variables with $\mathbf{E}X_n = 0$ and $\mathbf{E}|X_n|^p < \infty$ for some $p \geqslant 1$ and every $n \geqslant 1$. Then there exist positive constants $C_{m,p}$ and $D_{m,p}$, depending only on m and p, such that for all $n \geqslant m$,*

$$\mathbf{E}\left|\sum_{i=1}^{n} X_i\right|^p \leqslant \begin{cases} C_{m,p}\sum_{i=1}^{n}\mathbf{E}|X_i|^p & \text{for } 1 \leqslant p \leqslant 2, \\ D_{m,p}\{\sum_{i=1}^{n}\mathbf{E}|X_i|^p + (\sum_{i=1}^{n}\mathbf{E}X_i^2)^{p/2}\} & \text{for } p > 2. \end{cases} \tag{2.1}$$

We further need a basic inequality for the m-NOD random variables.

Lemma 4. *Let $\{X_n, n \geqslant 1\}$ be a sequence of m-NOD random variables. Then there exists a positive constant C such that for all $x \geqslant 0$ and $n \geqslant 1$,*

$$\left(1 - \mathbf{P}\left(\max_{1\leqslant i\leqslant n}|X_k| > x\right)\right)^2 \sum_{i=1}^{n}\mathbf{P}(|X_i| > x) \leqslant C\mathbf{P}\left(\max_{1\leqslant i\leqslant n}|X_i| > x\right).$$

Some Results

In the following, $\{X_n,\ n \geqslant 1\}$ is a sequence of m-NOD random variables dominated by a random variable Y. We will also use the notations $\hat{X}_i = -\psi(n)\mathbf{I}_{\{X_i<-\psi(n)\}} + X_i\mathbf{I}_{\{|X_i|\leqslant\psi(n)\}} + \psi(n)\mathbf{I}_{\{X_i>\psi(n)\}}$ and $m(n,i) = \mathbf{E}X_i\mathbf{I}_{\{|X_i|\leqslant\psi(n)\}}$ for $i, n \geqslant 1$.

Lemma 5. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of m-NOD random variables stochastically dominated by a random variable Y. Assume that the functions g, h, and ψ satisfy conditions* (A1) *and* (A3) *and* $\lim_{n\to\infty} n\mathbf{P}(|Y| > \psi(n)) = 0$. *Then*

$$\lim_{n\to\infty} \frac{1}{g(n)} \left| \sum_{i=1}^{n} \frac{\mathbf{E}(\hat{X}_i) - m(n,i)}{h(i)} \right| = 0.$$

Let us state our main result.

Theorem 1. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of m-NOD random variables stochastically dominated by a random variable Y. Moreover, assume that the functions g, h, and ψ satisfy conditions* (A1), (A2), *and* (A3) *and* $\mathbf{E}(H(\psi^{-1}(|Y|))) < \infty$. *If for some $r \geqslant 1$,*

$$\mathbf{E}\left(\psi^{-1}(|Y|)\right)^r < \infty, \tag{2.3}$$

then

$$\sum_{n=1}^{\infty} n^{r-2}\mathbf{P}\left(\left| \sum_{i=1}^{n} \frac{X_i - m(n,i)}{h(i)} \right| > \varepsilon g(n) \right) < \infty \quad \forall \varepsilon > 0.$$

Some results

DEFINITION 4. A measurable function $U : [a, \infty) \to (0, \infty)$, $a \in \mathbb{R}$, is called regularly varying at infinity with exponent ρ, denoted $U \in \mathcal{RV}(\rho)$, if $\lim_{x\to\infty} U(tx)/U(x) = t^\rho$ for all $t > 0$. If $\rho = 0$, then we say that U is slowly varying at infinity and write $U \in \mathcal{SV}$.

Theorem 2. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of identically distributed m-NOD random variables such that $\mathbf{P}(|X_k| > x) \in \mathcal{RV}(\rho)$. Moreover, assume that the functions g, h, and ψ satisfy condition* (A1). *If*

$$\sum_{n=1}^{\infty} n^{r-2}\mathbf{P}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}\frac{X_i - m(n,i)}{h(i)}\right| > \varepsilon g(n)\right) < \infty \quad \forall\varepsilon > 0, \tag{2.4}$$

then

$$\mathbf{E}\left[\psi^{-1}(|Y|)\right]^r < \infty, \quad r \geqslant 2. \tag{2.5}$$

Now following Theorem 11.2 of [6], we can restate the Hsu–Robbins-type theorem for m-NOD sequences.

Corollary 2. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of identically distributed m-NOD random variables. If $\mathbf{E}X = 0$ and $\mathbf{E}X^2 < \infty$, then*

$$\sum_{n=1}^{\infty}\mathbf{P}\left(\left|\sum_{k=1}^{n} X_k\right| \geqslant \varepsilon n\right) < \infty \quad \forall\varepsilon > 0.$$

Simulation Study

In this section, we illustrate the efficiency and rate of complete convergence in Theorem 1 through two numerical examples. According to the Remark 1, we set $h(n) = n^p$, $g(n) = n^q$, and $H(x) = x^r$ (where $p = 0.5$, $q = 1$, $r \geqslant 1$, $\alpha = 2$) in Theorem 1, and for each $r = 1, 2, 3$, we take the sample size $n = 3(1)200$. For each n, we simulate m-NOD random variables $X_1 = x_1, \ldots, X_n = x_n$ for $m = 1$ in Example 1 and $m = 2$ in Example 2. We then compute $s_n = (1/n^q)|\sum_{i=1}^{n} x_i/i^p|$. By repeating this procedure $B = 20000$ times we observe the vector $\{S_n^1, \ldots, S_n^{B=20000}\}$ and finally compute $P_n = \sum_{i=1}^{B} \mathbf{I}_{\{S_n^i > \varepsilon\}}/B$ as an estimate of $\mathbf{P}(|\sum_{i=1}^{n} x_i/i^p|/n^q > \varepsilon)$. Now taking the cumulative sum of $n^{r-2}P_n$ and plotting the scatter plots of $(n, \sum_{j=1}^{n} n^{r-2}\mathbf{P}(|\sum_{i=1}^{j} x_i/i^p|/j^q > \varepsilon))$, we can analyze the behavior of complete convergence.

Example1

Example 1. In this example, to create an m-NOD sequence of random variables with $m = 1$, we use a multivariate normal distribution. For any fixed $n \geqslant 3$, we take an n-dimensional random vector $(X_1, \ldots, X_n)^{\mathrm{T}} \sim N_n(\underline{0}, \Sigma)$, where $\underline{0} = (0, \ldots, 0)^{\mathrm{T}}_{n \times 1}$ represents the zero vector, and the covariance matrix is

$$\sum = \begin{pmatrix} 1+\theta^2 & -\theta & 0 & \cdots & 0 & 0 & 0 \\ -\theta & 1+\theta^2 & -\theta & \cdots & 0 & 0 & 0 \\ 0 & -\theta & 1+\theta^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\theta^2 & -\theta & \\ 0 & 0 & 0 & \cdots & -\theta & 1+\theta^2 & -\theta \\ 0 & 0 & 0 & \cdots & 0 & -\theta & 1+\theta^2 \end{pmatrix}_{n \times n},$$

where $0 < \theta < 1$ (we take $\theta = 0.5$). From [9] it is obvious that $\{X_n,\ n \geqslant 1\}$ is a NOD sequence (m-NOD with $m = 1$), and we can see that this sequence is stochastically dominated by a random variable$Y \sim N(0, 1+\theta^2)$. It is clear that $\mathbf{E}(H[\psi^{-1}(|Y|)]) = \mathbf{E}(\psi^{-1}(|Y|))^r = \mathbf{E}(|Y|^{r/(p+q)}) < \infty$. Now all the conditions of Theorem 1 are satisfied, and we can easily show that $m(n, i) = 0$ for all $n \geqslant 3$ and $1 \leqslant i \leqslant n$. The results of this example are shown in Fig. 1(a).

Example1

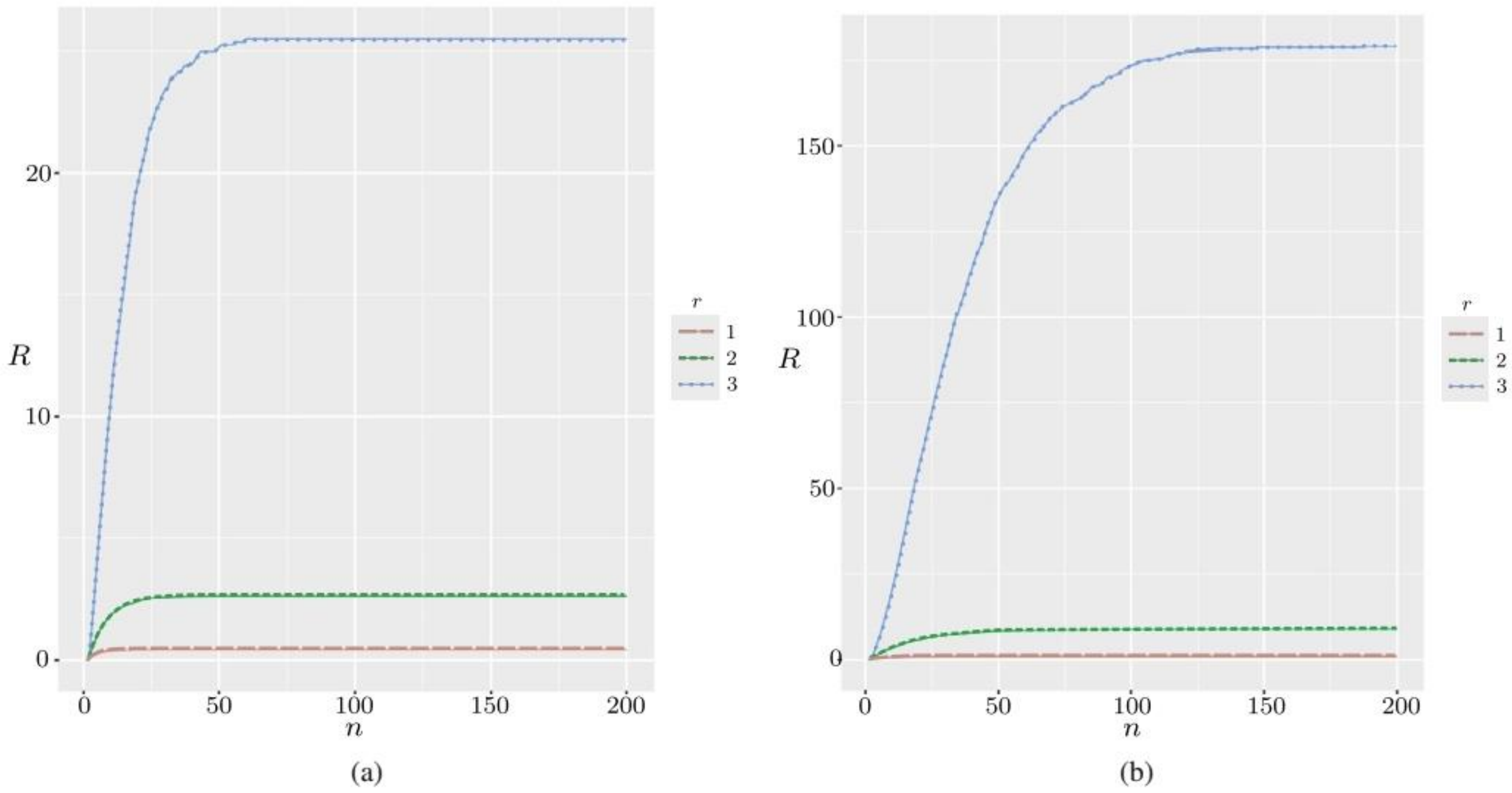


Figure 1.

Example-2

Example 2. In this example, we proceed exactly as in Example 1, with the difference that to create an m-NOD sequence of random variables with $m = 2$, we take the covariance matrix of the multivariate normal distribution

$$\sum = \begin{pmatrix} 1-\theta^2 & 0 & -\theta & 0 & 0 & \dots & 0 \\ 0 & 1-\theta^2 & 0 & -\theta & 0 & \dots & 0 \\ -\theta & 0 & 1-\theta^2 & 0 & -\theta & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & -\theta & 0 & 1-\theta^2 & 0 & -\theta \\ 0 & \dots & 0 & -\theta & 0 & 1-\theta^2 & 0 \\ 0 & \dots & 0 & 0 & -\theta & 0 & 1-\theta^2 \end{pmatrix}_{n\times n}.$$

The results of this example are shown in Fig. 1(b).

Figure 1 exhibits the scatter plots of $(n,\, R = \sum_{j=1}^{n} n^{r-2}\mathbf{P}(|\sum_{i=1}^{j} x_i/i^p|/j^q > \varepsilon))$ for $r = 1, 2, 3$. Observed that R is a increasing function of n that tends to a fixed value and is dominated by it for each $r = 1, 2, 3$.

References

1. A. Adler and A. Rosalsky, Some general strong laws for weighted sums of stochastically dominated random variables, *Stochastic Anal. Appl.*, **5**(1):1–16, 1987, https://doi.org/10.1080/07362998708809104.

2. A. Adler, A. Rosalsky, and R.L. Taylor, Strong laws of large numbers for weighted sums of random elements in normed linear spaces, *Int. J. Math. Math. Sci.*, **12**(3):507–530, 1989, https://doi.org/10.1155/S0161171289000657.

3. N.H. Bingham, C.M. Goldie, and J.L. Teugels, *Regular Variation*, Encycl. Math. Appl., Vol. 27, Cambridge Univ. Press, Cambridge, 1987.

4. P. Erdős, On a theorem of Hsu and Robbins, *Ann. Math. Stat.*, **20**:286–291, 1949, https://doi.org/10.1214/aoms/1177730037.

5. I. Fazekas, P. Matuła, and M. Ziemba, A note on the weighted strong law of large numbers under general conditions, *Publ. Math. Debr.*, **90**(3–4):373–386, 2017, https://doi.org/10.5486/PMD.2017.7566.

References

6. A. Gut, *Probability: A Graduate Course*, 2nd ed., Springer, New York, 2013, https://doi.org/10.1007/978-1-4614-4708-5.

7. P.L. Hsu and H. Robbins, Complete convergence and the law of large numbers, *Proc. Natl. Acad. Sci. USA*, **33**(2):25–31, 1947, http://www.jstor.org/stable/87477.

8. R. Jajte, On the strong law of large numbers, *Ann. Probab.*, **31**(1):409–412, 2003, https://doi.org/10.1214/aop/1046294315.

9. K. Joag-Dev and F. Proschan, Negative association of random variables with applications, *Ann. Stat.*, **11**:286–295, 1983, https://doi.org/10.1214/aos/1176346079.

10. A. Kuczmaszewska and Z.A. Lagodowski, Convergence rates in the SLLN for some classes of dependent random fields, *J. Math. Anal. Appl.*, **380**(2):571–584, 2011, https://doi.org/10.1016/j.jmaa.2011.03.042.

11. P. Matuła and M. Seweryn, Weighted strong law of large numbers for random variables indexed by a sector, *J. Probab. Stat.*, **2011**:701952, 2011, https://doi.org/10.1155/2011/701952.

References

12. H. Naderi, P. Matuła, M. Amini, and A. Bozorgnia, On stochastic dominance and the strong law of large numbers for dependent random variables, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, **110**(2):771–782, 2016, https://doi.org/10.1007/s13398-015-0263-1.

13. H. Naderi, P. Matuła, M. Salehi, and M. Amini, On weak law of large numbers for sums of negatively superadditive dependent random variables, *C. R., Math., Acad. Sci. Paris*, **357**(1):13–21, 2020, https://doi.org/10.5802/crmath.7.

14. A. Shen, On strong law of large numbers for weighted sums of negatively superaddative dependent random variables, *J. Korean Math. Soc.*, **53**(1):45–55, 2016, https://doi.org/10.4134/JKMS.2016.53.1.45.

15. A. Shen, M. Xue, and A. Volodin, Complete moment convergence for arrays of rowwise NSD random variables, *Stochastics*, **88**(4):606–621, 2016, https://doi.org/10.1080/17442508.2015.1110153.

16. T.C. Son, T.M. Cuong, B.K. Hang, and L.V. Dung, On the Baum–Katz theorem for randomly weighted sums of negatively associated random variables with general normalizing sequences and applications in some random design regression models, *Stat. Pap.*, **65**(3):1869–1900, 2024, https://doi.org/10.1007/s00362-023-01483-4.

17. X. Tang, Some strong laws of large numbers for weighted sums of asymptotically almost negatively associated random variables, *J. Inequal. Appl.*, **2013**:4, 2013, https://doi.org/10.1186/1029-242X-2013-4.

18. X.J. Wang, S.H. Hu, and A.I. Volodin, Moment inequalities for m-NOD random variables and their applications, *Theory Probab. Appl.*, **62**(3):471–490, 2018, https://doi.org/10.1137/S0040585X97T988745.

19. Z. Wang, On strong law of large numbers for dependent random variables, *J. Inequal. Appl.*, **2011**:279754, 2011, https://doi.org/10.1155/2011/279754.

20. Q. Wu, Complete convergence for weighted sums of sequences of negatively dependent random variables, *J. Probab. Stat.*, **2011**:202015, 2011, https://doi.org/10.1155/2011/202015.


خدایا چنان کن سرانجام کار
تو خوشنود باشی و ما رستگار