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A Robust L-Comoments Covariance Matrix-Based Hotelling's T^2 Control Chart for Monitoring High-Dimensional Non-Normal Multivariate Data in the Presence of Outliers

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ABSTRACT

This article introduces a new robust multivariate Hotelling T-square (TS) control chart that incorporates an L-Comoments covariance matrix into a multivariate statistical process control (MSPC) charting scheme to enhance its robustness and detection ability. However, among the most popular, conventional Hotelling TS charts are affected by outliers and based on the so-called classical covariance matrix estimators, which in turn presuppose normality and independence. This sensitivity reduces their usefulness in complicated practical problems where skewness, heavy tails, and outliers are likely to appear. Using L-Comoments as a basis of the new chart can overcome these limitations since L-Comoments are not affected by outliers. The performance of the proposed Hotelling TS (HTS) chart is assessed using total and generalized variances. By comparing the effectiveness of the L-Comoments-based TS chart using simulated and renewable energy data, the new chart based on the proposed approach outperforms the traditional chart and robust charts based on powerful estimators such as the minimum volume ellipsoid (MVE) and the minimum covariance determinant (MCD). Hence, the new approach enables the new exploration of robust and reliable multivariate quality control analysis for high-dimension and complex datasets.

1 | Introduction

Production quality in various industries has traditionally been critical in maintaining the customer base and market share. Numerous classical control techniques, like Shewhart control charts, have been used in the past for tracking variations concerning a single quality performance characteristic for a particular period of time (Montgomery [1]). However, sophisticated manufacturing and service operations require the specification of several factors that can be interdependent. Since quality is a complex variable, commonly a combination of several interrelated variables, it is important to find better statistical aid. Such situations can be effectively managed using bivariate charts since the univariate charts do not effectively portray the interactions of the various quality variables.

Hotelling [2] pioneered the work on multivariate quality control in the middle of the twentieth century when he introduced the concept of the T-square (TS) statistic. The chart offered a means for detecting changes in mean vectors, while covariance focused on variability. However, the TS chart has two potential limitations, even though it seems useful. First, it is based on classical assumptions, such as normal distribution and the absence of outliers, which are irrelevant to most real-life data. Second, mean vectors and covariance matrices used in the described evaluation are susceptible to outliers, which can occur due to errors or exceptional circumstances. These outliers may lead to small control limits, rendering the charts practically useless when weak control limits are concerned (Farcomeni and Greco [3]).

To overcome the aforementioned limitations, multivariate statistical process control (MSPC) recommends improved and advanced statistical approaches as potential realistic solutions. In addition to Hotelling's control chart, Alfaro and Ortega [4] discussed the comparative analysis of other methods. Vargas [5] and Jensen et al. [6] proposed two robust Hotelling's TS (HTS) control charts for individual namely, the least minimum volume ellipsoid (MVE) and the minimum covariance determinant (MCD) estimators. Later on, Chenouri et al. [7] improved the HTS chart based on the MCD estimator using reweighted MCD (RMCD) estimators. Note that their extension is motivated by the robustness of the shrinkage reweighted estimator; see, for example, Cabana and Lillo [8]. In addition, Kordestani et al. [9] and Moheghi et al. [10] proposed a reliable estimator for simple linear profiles, and they reported that the resulting methods were unable to explain the functional quantitative characteristics of quality characteristics.

In statistical process control (SPC), L-moments-based methods are used to enhance the effectiveness of a process, particularly in cases of dealing with extreme values and identifying and controlling for outlying observations. L-moments have demonstrated the potential to improve the efficiency of process control and monitoring, leading to a growing appreciation and adoption in recent literature. For instance, Lee [11]' s control charts were derived from two skewness measures, the third central moment, and the third L-moment, with means represented by empirical simulations on manufacturing process differences. Domański et al. [12] proposed the application of L-moment statistics and L-moment ratio diagrams (LMRDs) to monitor the dynamics in the robustness, resilience, and sustainability of proportional-Integral-Derivative (PID)-based control systems and introduced a new measure of discordance to account for their changes. Likewise, Domański et al. [13] used LMRDs for the portrayal and evaluation of control system quality. These contributions stress the applicability of L-moments and other related instruments, including L-Comoments, for the further enhancement of statistical process monitoring. This fact alone is evidence of a trend among representatives of the academic community, proving the advisability of employing these techniques and confirming the compatibility of the proposed approaches with new methods, like the utilization of L-moments characteristics in quality control. L-Comoments are extended L-moments used to analyze the relationship and dependence structure between two random variables to get insights beyond traditional correlation measures. Analogous to L-moments for a single variable distribution, these statistics measures are calculated from the concomitants of order statistics and are very useful in multivariate analysis, particularly when dealing with outliers and heavy-tailed distributions.

It has been reported that L-moments have limitations related to their sensitivity to outliers in the tails of distributions, the lack of power when analyzing highly skewed distributions, and their reduced sensitivity compared to traditional moments; see Lee [11] and Serfling and Xiao [14]. These limitations reveal the need for the most robust and efficient moments-based statistical measures. Therefore, in this paper, we suggest using L-Comoments statistical measures because of their robustness to outliers and their power when analyzing non-normal distributions. We acknowledge the computational complexity and scalability of L-Comoments-based statistics. However, in this paper, we also propose a solution to these limitations.

Even though much progress has been made statistical process monitoring, many of the established methods, including the Hotelling TS chart, use traditional covariance matrix estimators that are highly susceptible to outliers. However, more complex methods such as the MVE or the MCD estimators exist, but they are more limited to only some aspects of robustness. Recently, L-moments and their alternative, L-Comoments, have been considered efficient in analyzing multivariate data with outliers and/or non-normally distributed data. Recognizing this gap, we propose a new robust HTS chart that utilizes the L-Comoments covariance matrix as a potential robust alternative for monitoring a multi-variable process. To compare the performance of the new method with that of the classical, MVE, and MCD-based methods, we will examine the relationship between total and generalized variance and assess the enhanced reliability of process control systems. Moreover, the performance comparison based on the Phase I analysis in outlier detection accuracy is conducted through Monte Carlo simulations.

The remainder of this article is divided into specific sections. Section 2 highlights the different quality control techniques, specifically the HTS chart considered in this study and its dependency on classical assumptions. It also describes strong counterparts, like MVE and MCD methods, which deal with the problem of outliers in multivariate data. Section 3 is the main contribution of this study and introduces the L-Comoments-based TS chart and its robust against outliers, skewness, and/or heavy-tailed datasets. Section 4 provides numerical simulations and real data applications to support the proposed robust chart, showing better solutions than classical, MVE, and MCD methods. Lastly, the final section of this article provides a conclusion and future research ideas.

2 | Quality Control and HTS Charts

A control chart is normally created to track one type of data at a time, such as machine productivity, product quality, delivery performance, etc. However, some cases require the simultaneous tracking of related factors, necessitating multivariate control charts. The most popular of these charts is the HTS control chart, which employs the Hotelling TS statistic as the central statistic in a multivariate control chart. It highlights the importance of the distance moved between the out-of-control mean vector and the nominal mean vector, assuming a constant covariance matrix (Ali et al. [15]). In an industrial context, it has been shown that the quality of the finished product consists of a combination of factors, and one factor alone cannot affect the quality of the final product. In the past, individual marginal univariate control charts were used to indicate the quality level of each variable separately, assuming that variables operate independently. However, this assumption is not valid in most cases. Therefore, there is a need for a method that handles the correlations and, at the same time, maintains the quality of the output. The simultaneous control of all variables and non-susceptibility to the problem of multivariate techniques in a multivariate system make multivariate control charts a viable solution to the above problem.

Departing from univariate quality control methods that work on single variables, multivariate quality control methods use correlations between variables to test whether a process is in control. They have several advantages, such as computing the probability of false alarms, capturing the dependencies among variables, and utilizing only one charting statistic to establish the overall process stability instead of using one standalone chart for each quality characteristic. These benefits have increased the application of multivariate quality control mechanisms to great heights. Note, though, that these charts are incredibly valuable for data-driven decision-making and depend on the availability of the data. (Ali et al., [16]).

2.1 | Classical Covariance Matrix-based TS Chart

Among the existing multivariate charts, the HTS control chart is the most popular in literature, and its use is highly recommended for processes with multiple quality characteristics. Let $X = [x_{jk}]$ be a data matrix from a multivariate normal distribution, where $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, p$. Then, the sample mean $\bar{x}_k$, variance s_k^2 and covariance matrix $(i = 1, 2, \ldots)$ at the $i^t h$ sampling time are defined as:

$$\bar{x}_{ki} = \frac{1}{n}\sum_{j=1}^{n} x_{jki}, \tag{1}$$

$$s_{ki}^2 = \frac{1}{n}\sum_{j=1}^{n}(x_{jki} - \bar{x}_{ki})^2 \tag{2}$$

and

$$s_{khi} = \frac{1}{n-1}\sum_{j=1}^{n}(x_{jki} - \bar{x}_{ki})(x_{jhi} - \bar{x}_{hi}), \tag{3}$$

for all $k \neq h$, where, $k = h = 1, 2, \ldots, p$.

When discussing the HTS statistic, let us describe the value of the i^{th} charting statistic. Thus, at the i^{th} sampling time, the HTS statistic is defined by:

$$T_i^2 = (\bar{\mathbf{x}} - \mu_0)' S^{-1}(\bar{\mathbf{x}} - \mu_0) \tag{4}$$

where $\bar{\mathbf{x}}$ is the sample, μ_0 is the target mean vector and S is the sample covariance matrix.

The use of the control chart, as proposed by Henning et al. [17], can be divided into two phases. In Phase I, the upper control limit (UCL) and lower control limit (LCL) are defined by:

$$UCL = \frac{p(m-1)(n-1)}{mn - m + 1 - p} F_{\beta,p,m-p-1}, \quad and LCL = 0, \tag{5}$$

respectively; where p is the number of quality characteristics, m is the number of Phase I samples, n is the size of each sample and $F_{\beta,p,m-p-1}$ refers to Snedecor's F distribution.

In Phase II, the control limits are updated as:

$$UCL = \frac{p(m+1)(n-1)}{mn - m + 1 - p} F_{\beta,p,m-p-1}, \quad and\ LCL = 0. \tag{6}$$

The LCL remains at zero in both phases of the control. In order to construct the multivariate control charts, two assumptions should be tested: (1) the normality, and (2) the independence of data. However, extreme values significantly affect the normality of the service environment. In this regard, recommendations are to build robust HTS charts to alleviate such concerns, which are explained in the subsequent section.

2.2 | MVE and MCD TS Charts

When large sets of high-dimensional multivariate data are available, the MVE and MCD can be used as efficient estimators for the mean vector and covariance matrix. The structure of the estimates is protected in the MVE method since it delivers the MVE that encompasses a sufficient number of data points. On the other hand, the MCD method aims to construct an estimator of dispersion around a location measure based on a subset of the data and minimize the determinant of the covariance matrix, with the ability to downweight up to 50% of outliers. The two methods are important and valuable approaches to Multivariate Analysis designed to deal with problems that stem from outliers and extreme values during the process of data gathering.

As pointed out by Sedeeq et al. [18], the HTS control charts based on MVE and MCD estimators enhance the intensity of quality control mechanisms. These methods improve the sensitivity and accuracy of the charts in a way that the mean vector and covariance structure, with respect to the probability density function, are not affected by outliers. These techniques are applicable when it is desired to identify problems and their causes as soon as possible. They establish robust control boundaries, thus raising the system's standardization level. As a result, the quality of decisions improves, and operations become twice as effective.

Let the robust mean vector of MVE be noted as $\mu_{x.mve} = (\mu_{x_1.mve}, \mu_{x_2.mve}, \ldots, \mu_{x_p.mve})$ and that of the MCD as $\mu_{x.mcd} = (\mu_{x_1.mcd}, \mu_{x_2.mcd}, \ldots, \mu_{x_p.mcd})$. The robust covariance matrices, $\mathbf{SR}(i)$, are then defined as:

$$\mathbf{SR}(i) = \begin{pmatrix} \mathrm{SR}_{x_1x_1}^{(\omega)} & \mathrm{SR}_{x_1x_2}^{(\omega)} & \cdots & \mathrm{SR}_{x_1x_p}^{(\omega)} \\ \mathrm{SR}_{x_2x_1}^{(\omega)} & \mathrm{SR}_{x_2x_2}^{(\omega)} & \cdots & \mathrm{SR}_{x_2x_p}^{(\omega)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{SR}_{x_px_1}^{(\omega)} & \mathrm{SR}_{x_px_2}^{(\omega)} & \cdots & \mathrm{SR}_{x_px_p}^{(\omega)} \end{pmatrix}, \tag{7}$$

where $\omega \in \{1, 2\}$ denotes the type of robust covariance matrix. Thus, SR(1) represents the MVE, and SR(2) the MCD.

The robust HTS statistics for MVE and MCD are given by

$$T_{MVE}^2 = (\bar{\mathbf{x}} - \mu_{x.MVE})'(\mathbf{SR}(1))^{-1}(\bar{\mathbf{x}} - \mu_{x.MVE}), \tag{8}$$

$$T_{MCD}^2 = (\bar{\mathbf{x}} - \mu_{x.MCD})'(\mathbf{SR}(2))^{-1}(\bar{\mathbf{x}} - \mu_{x.MCD}), \tag{9}$$

respectively; where $\bar{\mathbf{x}}$ denotes the sample mean vector. Note that the control limits of the Hotelling's T^2_{MVE} and $T^2_M CD$ are determined in Phase 1 and later on used in Phase 2 for continuous process monitoring.

3 | The proposed HTS based on L-Comoments Covariance Matrix

Many other extensions of the classical framework of the HTS chart have been suggested in numerous studies. For example, Ali et al. [16] proposed a new chart utilizing the F-test in place of the TS-test. New calibration variance estimators based on L-moments (L-location, L-scale, L-CV) were introduced to enhance charts' efficiency when outliers are present, which are crucial for analyzing apple fruit, simulated, and real-world data; see, for example, Shahzad et al. [19, 20]. Later on, to overcome or minimize the impact of extreme observations, L-moments and Trimmed L-moments based CV estimators were developed by Shahzad et al. [21]. A satisfactory improvement was achieved when testing these methods on sensitive and nonsensitive variables using scrambled response models in both simulation studies and practical settings. For updates on the current state of the art in robust HTS charts, see Sedeeq et al. [18] and Ali et al., [22]. Building on this rich background, this work proposes to develop an L-Comoments-based HTS control chart.

Serfling and Xiao [14]'s L-Comoments refer to more modern developments in the study of classical covariance and correlation measures developed from Hosking [23]'s L-moments. These measures are particularly useful for data that contain outliers, are skewed toward one direction, or have heavy tails. Oyegoke et al. [24] proposed the Hotelling T^2 control chart for minimum vector variance for monitoring high–dimensional correlated multivariate process.

The r^{th} L-moment of a given random variable X is defined as:

$$\lambda_r = \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{(-1)^k}{r} \mathbb{E}[X_{(r-k)}], \tag{10}$$

where $X_{(k)}$ denotes the k-th order statistic and r represents the order of moment under consideration.

The first two L-moments are defined by:

$$\lambda_1 = \mathbb{E}[X] \quad \text{(L-location)}, \quad \lambda_2 = \frac{1}{2}(\mathbb{E}[X_{(2)}] - \mathbb{E}[X_{(1)}]) \quad \text{(L-scale)}. \tag{11}$$

The second-order L-Comoment for X and Y are

$$\lambda_{2,XY} = \frac{1}{2}\mathbb{E}[(X_{(2)} - X_{(1)})(Y_{(2)} - Y_{(1)})]. \tag{12}$$

For a multivariate dataset $\mathbf{X} = (X_1, X_2, \ldots, X_p)^\top$ specified by p variables $\mathbf{S}_L$ is an L-Comoments covariance matrix, which is configured in the form of

$$\mathbf{S}_L = \begin{bmatrix} \lambda_{2,X_1X_1} & \lambda_{2,X_1X_2} & \cdots & \lambda_{2,X_1X_p} \\ \lambda_{2,X_2X_1} & \lambda_{2,X_2X_2} & \cdots & \lambda_{2,X_2X_p} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{2,X_pX_1} & \lambda_{2,X_pX_2} & \cdots & \lambda_{2,X_pX_p} \end{bmatrix}, \tag{13}$$

where λ_2, X_iX_i refer to the L-scale diagonal elements, and λ_2, X_iX_j are the L-Comoments elements off diagonal.

Since the L-Comoments covariance matrix is less sensitive and computationally efficient than the classical covariance matrix, it is feasible to construct HTS charts. As a result, TS charts improve anomaly detection and monitoring in a number of ways based on the limitations encountered by traditional covariance estimation, especially when dealing with large raw datasets influenced by outliers and skewness. The L-Comoments covariance matrix $\mathbf{S}_L$-based robust HTS chart is then given by,

$$T^2_L = n(\bar{\mathbf{X}} - \hat{\boldsymbol{\mu}})^\top \mathbf{S}_L^{-1} (\bar{\mathbf{X}} - \hat{\boldsymbol{\mu}}), \tag{14}$$

where

- $\bar{\mathbf{X}} = (\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_p)^\top$: Sample mean vector.
- $\boldsymbol{\mu}$: Population mean vector.
- $\mathbf{S}_L$: L-Comoments covariance matrix.

The control limits of the proposed chart are computed using the classical F-distribution approach, thus ensuring the chart's compatibility with established SPC techniques. Despite using L-Comoments for a better covariance matrix estimation to handle non-normality and outliers, the control limits continue to rely on F-distribution to make them maintainable and interpretable. The proposed method is based on L-Comoments that enable it to perform well with outliers while using standard F-distribution-based limits as outlined by Equations (5) and (6). Thus, in Phase II, the proposed control chart gives a signal if the charting statistic defined in Equation (14) plots on or beyond the control limit defined in Equation (6), i.e., $T^2_L \geq \text{UCL}$.

It is worth noting that:

- L-Comoments are least affected by outliers within the data compared to the traditional T^2_L statistic, making them more reliable.
- They are able to handle datasets with deviations from normality.
- In light of the above two points, L-Comoments-based charts will decrease the number of false positives and enhance the identification of actual outliers.

The performance of the proposed robust HTS chart (T^2_L) can be evaluated by comparing it to traditional TS chart (T^2_c) and robust methods such as MVE (T^2_{mve}) and MCD (T^2_{mcd}) using $total\ variance(Trace(S))$ and $Generalized\ variance(Det(S))$. The chart that yields the lowest values for these variances is deemed the most effective.

4 | Numerical Illustration

The use of L-Comoments in robust HTS charts is relatively new, and to the best of the author's knowledge, this subject has not yet been investigated in the literature. Consequently, a comparative analysis was carried out to quantify the total and general variances in relation to the adapted and proposed charts. The computational complexity and scalability of the L-Comoments-based HTS control chart are important aspects, especially regarding a high-dimensional dataset. The proposed method's major computation task is the L-Comoments matrix's computation, which calls for the computation of pairwise differences for all the variables. Although this process is similar to the usual covariance matrix estimation, it offers better robustness to outliers and non-normality. In a high-dimensional setting, the computational cost may blow up significantly. However, the proposed method reduces this by using linear combinations of order statistics that reduce the influence of extreme values and optimize matrix operations. In addition, the matrix inversion step in the computation of the T_L^2 statistics is a common step in the multivariate analysis procedure. The scalability can be further improved by implementing the matrix operations in a parallelized or distributed computing environment, which enables easy processing of large datasets. The following subsections are devoted to the findings and analyses.

4.1 | Simulation Study

4.1.1 | Phase I Performance Analysis

To evaluate the performance based on Phase I analysis, we conducted Monte Carlo simulations with 1000 replications. In each replication, we generated 100 subgroups of size 12, where D% of the data were considered outliers with specified shift sizes applied to all parameters. In this study, the proposed chart's charting statistic is compared against the UCL, and a signal is considered correct if the statistic exceeds the UCL in the presence of outliers. The accuracy of correctly detecting outliers is recorded for each iteration and reported for four estimators across various shift sizes and contamination levels in Table 1. The best-performing method (i.e., highest accuracy) is highlighted in bold in each row. The average accuracy for each method is also included at the bottom of the table to provide a more comprehensive comparison. In addition, we computed the rank of each method per condition (row), with lower ranks indicating better performance. The average rank for each method is also reported. The results show that in most cases, the L-Comoments estimator achieves higher detection accuracy and consistently ranks first among the evaluated methods. Another notable finding is that the MCD estimator performs better in scenarios involving positive parameter shifts, whereas the L-Comoments estimator demonstrates superior accuracy under negative shifts. However, when considering overall performance across all scenarios, L-Comoments is regarded as the most effective and reliable method.

4.1.2 | Phase II Performance Analysis

To perform the Phase II analysis, 12 subgroups (m = 12), each of size 6 (i.e., n = 6) of multivariate normal data were initially generated in R using a mean vector (5, 10, 15) and covariance $\begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 1.5 \\ 1 & 1.5 & 2 \end{bmatrix}$. Thus, $p = 3$ of which each variate counts 72 observations. Outliers were then included in the dataset to model a practical, realistic situation. These steps helped ensure that our dataset not only contains non-normal data but also includes issues often experienced in applications. The covariance and correlation matrix were estimated using existing and proposed robust methods, as presented in Tables 2 and 3. The results of the first simulation experiment, which included Mahalanobis distance values, are illustrated in Figure 1. The control charts configured in Phase I are depicted in Figures 2–5. It is necessary to note that out-of-control points are also indicated on the charts to display the charts' ability to identify outliers. As shown in the figures, the plotted points are within the control limits; see, for example, Figures 2–5. Hence, it proves the reliability and validity of the charts in the Phase II context. In addition, the results in Table 4 clearly indicate that the proposed robust HTS chart is better than the adapted charts in terms of the total and generalized variance profiles. The proposed chart is the best one, see Table 4. Also, the experiment was repeated with $p = 2$

TABLE 1 | The accuracy of different estimators in outlier detection through Phase I analysis.

Shift size	D	Classical	MVE	MCD	L-Comoments
−3, −3, −2	0.33	0	**0.001**	0	**0.001**
−3, −3, −2	0.40	0.003	0.010	0	**0.017**
−3, −3, −2	0.47	0.062	0.127	0.009	**0.148**
−3, −3, −2	0.54	0.356	0.500	0.096	**0.563**
−3, −3, −2	0.61	0.760	0.850	0.414	**0.892**
−3, −3, −2	0.68	0.967	0.983	0.811	**0.993**
−3, −3, −2	0.75	0.997	0.998	0.983	**0.999**
3, 1, 2	0.33	0	0.001	**0.006**	0.005
3, 1, 2	0.40	0.001	0.005	**0.095**	0.039
3, 1, 2	0.47	0.025	0.105	**0.435**	0.222
3, 1, 2	0.54	0.180	0.405	**0.848**	0.603
3, 1, 2	0.61	0.607	0.831	**0.986**	0.887
3, 1, 2	0.68	0.919	0.982	**0.999**	0.991
3, 1, 2	0.75	0.994	0.996	**1.000**	0.999
Average of accuracy		0.419	0.485	0.477	**0.526**
Average of rank		3.536	2.464	2.464	**1.536**

TABLE 2 | Covariance matrices for $p = 3$.

Classical ∈	$\begin{pmatrix} 3.706334 & 1.549703 & 1.186819 \\ 1.549703 & 2.463255 & 1.242680 \\ 1.186819 & 1.242680 & 1.868866 \end{pmatrix}$
MVE ∈	$\begin{pmatrix} 3.729679 & 1.547495 & 1.178412 \\ 1.547495 & 2.744551 & 1.482215 \\ 1.178412 & 1.482215 & 1.829621 \end{pmatrix}$
MCD ∈	$\begin{pmatrix} 3.941915 & 1.761626 & 1.072898 \\ 1.761626 & 2.764842 & 1.358384 \\ 1.072898 & 1.358384 & 1.727217 \end{pmatrix}$
L − Comoments ∈	$\begin{pmatrix} 2.512185 & 1.282087 & 1.056093 \\ 1.282087 & 2.405270 & 1.039499 \\ 1.056093 & 1.039499 & 2.188231 \end{pmatrix}$

TABLE 3 | Correlation matrices for $p = 3$.

$$\textbf{Classical} \in \begin{pmatrix} 1.0000000 & 0.5128869 & 0.4509446 \\ 0.5128869 & 1.0000000 & 0.5791826 \\ 0.4509446 & 0.5791826 & 1.0000000 \end{pmatrix}$$

$$\textbf{MVE} \in \begin{pmatrix} 1.0000000 & 0.4836797 & 0.4511082 \\ 0.4836797 & 1.0000000 & 0.6614466 \\ 0.4511082 & 0.6614466 & 1.0000000 \end{pmatrix}$$

$$\textbf{MCD} \in \begin{pmatrix} 1.0000000 & 0.5336112 & 0.4111795 \\ 0.5336112 & 1.0000000 & 0.6216044 \\ 0.4111795 & 0.6216044 & 1.0000000 \end{pmatrix}$$

$$\textbf{L} - \textbf{Comoments} \in \begin{pmatrix} 1.0000000 & 0.5215669 & 0.4504325 \\ 0.5215669 & 1.0000000 & 0.4531015 \\ 0.4504325 & 0.4531015 & 1.0000000 \end{pmatrix}$$

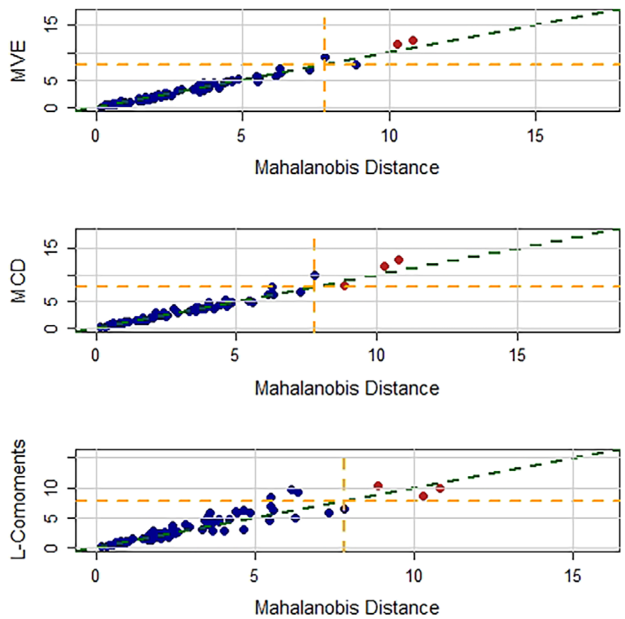

FIGURE 1 | Mahalanobis distance for the **Classical**, **MVE**, **MCD** and **L – Comoments** methods.

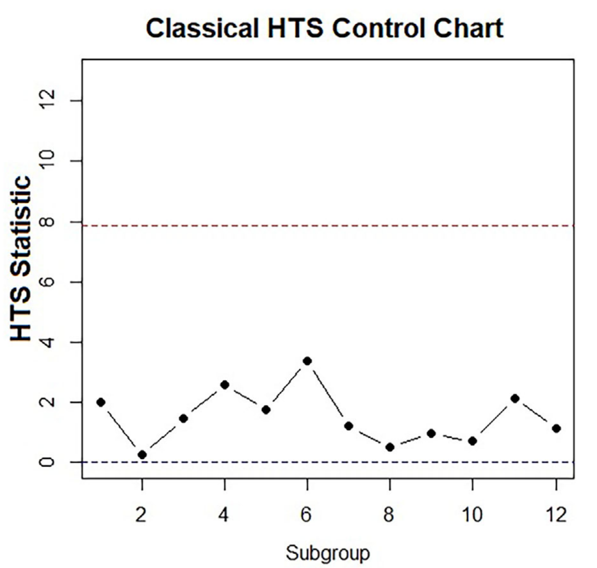

FIGURE 2 | **Classical** HTS control chart.

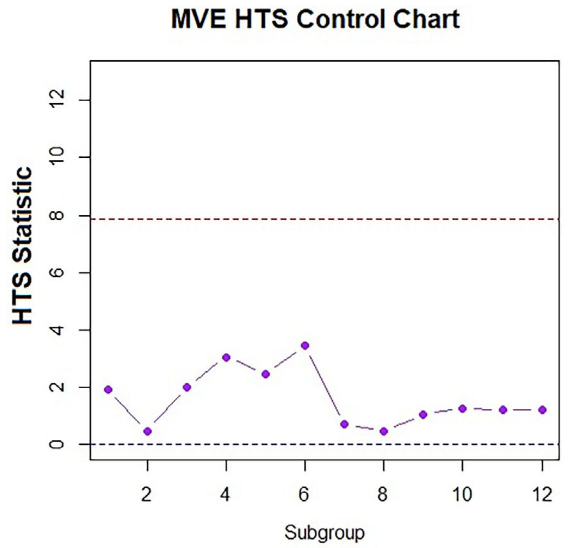

FIGURE 3 | **MVE** HTS control chart.

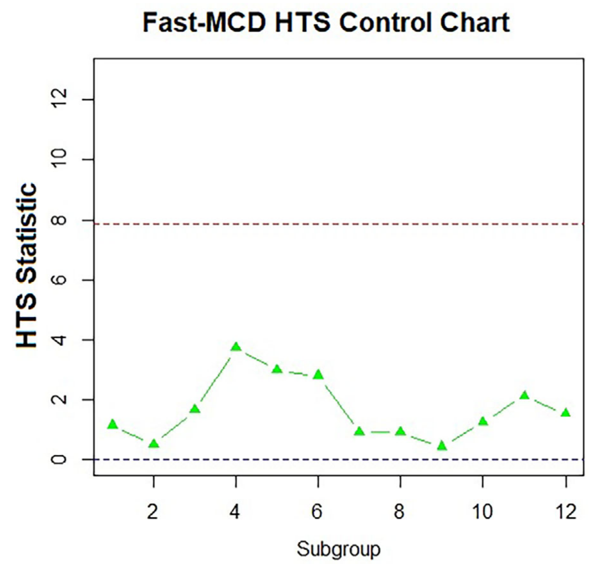

FIGURE 4 | **MCD** HTS control chart.

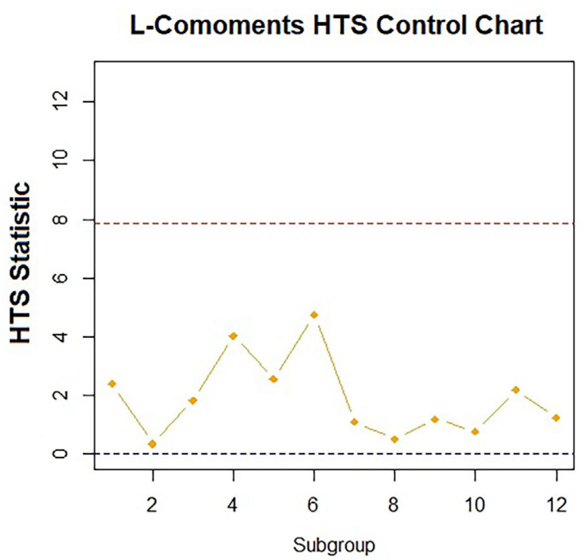

FIGURE 5 | **L – Comoments** HTS control chart.

TABLE 4 | Estimated variances for single experiment with $p = 3$.

Methods	Total variance	General variance
CLASSICAL	8.038455	7.951858
MVE	8.023851	7.847778
MCD	8.033974	7.902949
L – Comoments	7.105685	7.04318

TABLE 5 | Estimated variances for single experiment with $p = 2$.

Methods	Total variance	General variance
CLASSICAL	6.897234	6.88774
MVE	6.369495	6.769141
MCD	6.262833	6.817549
L – Comoments	4.369212	3.415774

TABLE 6 | Estimated variances for thousand experiments with $p = 3$.

Methods	Total variance	General variance
CLASSICAL	9.890883	11.73827
MVE	9.696847	10.89521
MCD	9.740129	11.24723
L – Comoments	7.174125	6.220549

and showed similar behavior, and the results are summarized in Table 5.

Moreover, the experiment was conducted ten times more, i.e., 1000 runs for (p,n,m) = (3, 6, 12). The results of the generalized and total variances were calculated, and summarized in Tables 6 and 7.

The results presented in Tables 4–7, illustrate the performances of the Classical, MVE, MCD, and L-Comoments methods for variance estimation in various experiments. For single experiments with $p = 3$ in Table 4, the Classical method gives total variances of 8.038455 and total generalized variance of 7.951858. However, it is very sensitive to outliers. MVE (8.023851 and 7.847778) and MCD (8.033974 and 7.902949) show better results compared to the Classical method. Nevertheless, the L-Comoments-based chart has the lowest variances of 7.105685 and 7.04318, indicating

TABLE 7 | Estimated variances for thousand experiments with $p = 2$.

Methods	Total variance	General variance
CLASSICAL	7.566753	8.682598
MVE	7.340383	8.427893
MCD	7.326569	8.49546
L – Comoments	4.610207	3.062352

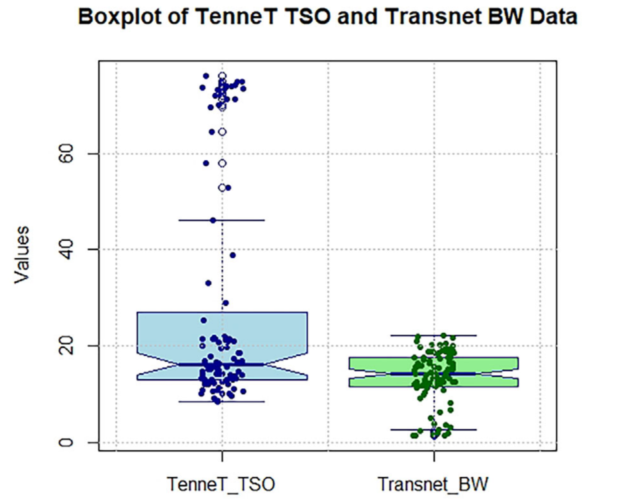

FIGURE 6 | Power data boxplot.

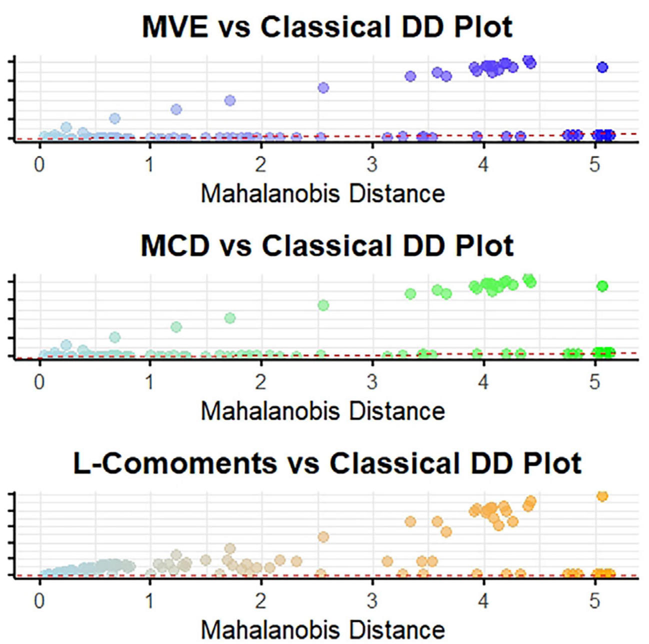

FIGURE 7 | Power data Mahalanobis distance for the **Classical**, **MVE**, **MCD** and **L – Comoments** methods.

TABLE 8 | Estimated variances for power data.

Methods	Total variance	General variance
CLASSICAL	566.064	11465.12
MVE	82.04595	1661.264
MCD	68.09585	1156.679
L – Comoments	8.003718	9.89456

the inherent variability of the data and stabilizing outliers or extreme impacts with a relatively high level of accuracy. For single experiments with $p = 2$ in Table 5, this trend persists. Once again, the L-Comoments method provides minimum variances

TABLE 9 | Power data for TenneT TSO and Transnet BW.

TenneT TSO	Transnet BW	TenneT TSO	Transnet BW	TenneT TSO	Transnet BW	TenneT TSO	Transnet BW
9.68	1.44	10.16	1.38	10.94	1.38	11.39	1.50
12.09	1.77	12.79	1.87	13.33	1.95	13.88	2.47
14.73	2.68	14.95	3.02	15.29	3.64	16.01	3.94
16.60	5.03	17.82	6.16	18.59	6.69	19.50	8.08
20.15	9.07	20.90	9.81	21.43	11.57	21.89	14.28
21.47	15.76	21.59	17.60	21.45	18.76	20.71	19.08
19.71	18.71	16.97	16.45	14.11	14.43	12.47	13.24
10.13	12.27	8.56	11.23	8.54	10.40	9.25	10.73
10.20	11.45	10.54	12.05	10.94	11.99	10.58	15.77
11.06	16.34	12.28	17.56	13.69	15.80	15.31	19.68
14.31	18.25	15.08	17.64	16.11	17.61	16.55	20.32
16.85	20.49	16.44	16.82	15.79	15.29	14.73	16.36
13.32	14.48	14.25	12.92	13.30	15.42	12.98	14.08
14.09	14.89	14.24	13.81	13.11	16.03	12.47	12.01
13.10	14.47	13.11	12.93	11.99	13.08	12.02	12.90
13.46	11.91	13.02	13.42	15.68	12.64	15.32	11.62
16.34	14.05	15.71	11.98	17.33	11.54	18.57	12.58
20.90	12.48	21.27	12.54	21.84	13.18	25.36	12.43
28.86	11.93	33.16	16.82	38.89	14.87	46.23	15.10
52.98	15.86	57.96	17.47	64.50	18.84	71.26	19.88
74.08	19.16	73.61	21.08	74.94	20.34	76.18	20.03
74.02	19.58	72.44	22.09	73.52	21.99	74.03	18.33
74.77	18.56	73.32	18.56	74.83	17.11	73.10	14.19
71.26	15.69	71.95	16.71	69.97	18.88	69.58	21.67

of 4.369212 and 3.415774, demonstrating accuracy and stability, especially in the lower dimensionality of data.

In the context of aggregated results of 1000 experiments, as shown in Tables 6 and 7, the L-Comoments method is highly resilient and precise. For $p = 3$ in Table 6, the Classical method exhibits very high variances, indicating a strong influence of outliers. Comparing MVE and MCD with Classical variances, we can conclude that these methods work better for further reductions of variances, with MCD > MVE having values of 9.696847, 10.89521; 9.740129, and 11.24723. However, the variance that belongs to L-Comoments is the lowest at 7.174125 and 6.220549, amplifying important data trends while reducing noise and outlier impact. Similarly, for $p = 2$ in Table 7, the L-Comoments method shows better variance computations than other methods, with variances of 4.610207 and 3.062352. In this analysis, L-Comoments provides a fairly acceptable level of quantification accuracy that is relatively immune to data distortion, making it suitable for large and complex datasets. The new usage of L-Comoments introduced makes it a novel analyzing tool for datasets with outliers, noise, skewness, and high variability, such as in renewable energy systems and other real-world systems that will be explained in the upcoming subsection.

4.2 | Wind Energy Power Data

Wind energy offers a clean, renewable alternative to fossil fuels, reducing greenhouse gas emissions and aligning with global sustainability goals. By applying quality and reliability engineering principles, the wind energy sector can enhance efficiency, reduce costs, and accelerate the transition to sustainable energy. In this article, the wind energy dataset used in this analysis contains power generation data from two German energy companies, TenneT TSO and Transnet BW. The dataset comprises non-normalized power generation values recorded at 15-min intervals, resulting in a total of 96 observations per day. Specifically, the data analyzed corresponds to a single day, dated August 23, 2019, with the first recorded value at the time 00:00:00 and the last at 23:45:00. The dataset is publicly available on the Kaggle platform.

Finally, through the boxplot in Figure 6, it can be seen that TenneT actually has relatively high variability, with many points allocated beyond the whiskers, which may contain some abnormal and extraordinary values in the dataset. In contrast, the Transnet BW dataset density is more concentrated, and there are no extreme values dominating, making it more desirable. In such situations, the Hotelling TS control chart developed from the covariance matrix of L-Comoments is more optimal than the one developed from conventional methods, as L-Comoments act as a better estimator of distribution characteristics for skewed and heavy-tailed data. Real outliers are revealed through L-Comoments, enhancing the monitoring and interpretation of the Hotelling TS chart and assisting in monitoring powerful datasets for anomaly detection.

The depth-depth (DD) plots in Figure 7 demonstrate the effectiveness of robust methods in comparing Mahalanobis distance computation with the classical approach. Among these methods, L-Comoments show high efficiency in reconstructing the identity line with minor corrections to Mahalanobis distances and the ability to preserve the shape of the data. Furthermore, the estimates of L-Comoments also show that they are robust against outliers and non-Gaussian data. The L-Comoments method results in lower total and generalized variance compared to Classical, MVE, and MCD methods.

The values of variance presented in Table 8 show the significant advantage of L-Comoments, proving that it is a statistically accurate and effective approach. The classical method, lacking robustness, showed high total variance (566.064) and general variance (11465.12) due to sensitivity to outliers and normality assumptions not suitable for wind power data. However, MVE (82.04595, 1661.264) and MCD (68.09585, 1156.679), which are known for their restrictive nature in handling outliers, had high variance estimates. Thus, when using the L-Comoments method, the variances were very small, 8.003718 and 9.89456, confirming its high trustworthiness and precise data manipulation in the case of outliers, positive skewness, and high kurtosis. These results indicate the efficiency and reliability of L-Comoments for use in TS charts, paving the way for improved modeling and decision-making in renewable energy systems. The data used is provided in Table 9. Overall, L-Comoments prove to be the best-suited robust TS chart method, opening up possibilities for other control charts in the field.

5 | Conclusion

This comprehensive study suggests the new proposed control chart, i.e., L-Comoments-based robust HTS control chart, that improves the efficiency and reliability of multivariable quality control scenarios. As compared to other available methods, the newly proposed L-Comoments HTS Control chart is not overrepresented in the case of outliers and does not need any normality assumption. Therefore, it is more robust against outliers that can distort skewed data or data sets that contain a lot of outlying values. Real-world usage assessment based on renewable energy data and simulated comparative examination proved that the proposed HTS control chart is superior than the available classical, MVE, and MCD covariance-based TS charts with significantly having smaller variances (both total and generalized). This improvement is not only beneficial for enhancing outlier detection performance but also useful for ensuring stable monitoring in large and high-dimensional data.

Future studies can deal with extending L-Comoments-based methods to several multivariate quality monitoring settings, especially in high-dimensional or big data scenarios. Combining these methods with modern high-performance computing frameworks may increase scalability and computational efficiency in manufacturing and other data-intensive industries. In addition, the scope can be broadened to integrate broader performance metrics such as false alarm rates, ARL, and breakdown robustness that may be ideally facilitated using L-Comoments and their extensions. This programmatic union would offer a more stringent and sturdy bias for recording multivariate processes under demanding data environments.

Data Availability Statement

The data that supports the findings of this study are available in the supplementary material of this article.

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