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Blow-up of Solutions for a Class of Fourth-order Equation Involving Dissipative Boundary Condition and Positive Initial Energy

SHAHROUZI Mohammad *

Department of Mathematics, Jahrom University, Jahrom 74137-66171, Iran.

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Abstract. In this paper we consider a forth order nonlinear wave equation with dissipative boundary condition. We show that there are solutions under some conditions on initial data which blow up in finite time with positive initial energy.

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1 Introduction

In this article, we are concerned with the problem

$$u_{tt}+\Delta[(a_0+a|\Delta u|^{m-2})\Delta u]-b\Delta u_t=g(x,t,u,\Delta u)+|u|^{p-2}u,\quad x\in\Omega,\quad t>0, \tag{1.1}$$

$$u(x,t)=0,\quad \Delta u(x,t)=-c_0\partial_\nu u(x,t),\quad x\in\partial\Omega,\quad t>0, \tag{1.2}$$

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in\Omega, \tag{1.3}$$

where $\Omega\subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$ in order that the divergence theorem can be applied. ν is the unit normal vector pointing toward the exterior of Ω and $p>m+1>3$. Moreover, the constants a_0,a,b,c_0 are positive numbers and $g(x,t,u,\Delta u)$ is a real function that satisfies specific condition that will be enunciated later.

The one-dimension case of the fourth order wave equation is written as

$$u_{tt}+u_{xxxx}-a(u_x^2)_x=f(x),\quad x\in\Omega\subset R,\quad t>0, \tag{1.4}$$

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*Corresponding author. *Email addresses:* `mshahrouzi@shirazu.ac.ir` (M. Shahrouzi)

which was first introduced in [1] to describe the elasto-plastic-microstructure models for the longitudinal motion of an elasto-plastic bar. Chen and Yang [2] studied the Cauchy problem for the more general Eq. (1.4).

Young Zhou, in [3] studied the following nonlinear wave equation with damping and source term on the whole space:

$$u_{tt}+a|u_t|^{m-1}u_t-\phi\Delta u=f(x,u),$$

where $a,b>0,m\geq 1$ are constants and $\phi(x):R^N\to R,n\geq 2$. He has obtained the criteria to guarantee blow up of solutions with positive initial energy, both for linear and nonlinear damping cases.

In [4], the same author has been studied the following Cauchy problem

$$\begin{aligned}&u_{tt}+au_t-\Delta u=b|u|^{p-1}u, \quad x\in R^N,t>0,\\&u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), \quad x\in R^N,\end{aligned}$$

where $a,b>0$. He proved that the solution blows up in finite time even for vanishing initial energy if the initial datum (u_0,u_1) satisfies $\int_{R^N}u_0u_1\mathrm{d}x\geq 0$. (See also [5])

Recently, in [6] Bilgin and Kalantarov investigated blow up of solutions for the following initial-boundary value problem

$$\begin{aligned}&u_{tt}-\nabla[(a_0+a|\nabla u|^{m-2})\nabla u]-b\Delta u_t=g(x,t,u,\nabla u)+|u|^{p-2}u, \quad x\in\Omega, \quad t>0,\\&u(x,t)=0, \quad x\in\partial\Omega, \quad t>0,\\&u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega.\end{aligned}$$

They obtained sufficient conditions on initial functions for which there exists a finite time that some solutions blow up at this time.

Tahamtani and Shahrouzi studied the following fourth order viscoelastic equation

$$u_{tt}+\Delta^2u-\int_0^t g(t-s)\Delta^2u(s)\mathrm{d}s=|u|^pu,$$

in a bounded domain and proved the existence of weak solutions in [7]. Furthermore, they showed that there are solutions under some conditions on initial data which blow up in finite time with non-positive initial energy as well as positive initial energy. Later, the same authors investigated global behavior of solutions to some class of inverse source problems. In [8], the same authors investigated the global in time behavior of solutions for an inverse problem of determining a pair of functions $\{u,f\}$ satisfying the equation

$$u_{tt}+\Delta^2u-|u|^pu+a(x,t,u,\nabla u,\Delta u)=f(t)\omega(x), \quad x\in\Omega, \quad t>0,$$

the initial conditions

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega,$$

the boundary conditions

$$u(x,t)=\partial_\nu u(x,t)=0,\quad x\in\partial\Omega,\quad t>0,$$

and the over-determination condition

$$\int_\Omega u(x,t)\omega(x)\mathrm{d}x=1,\quad t>0.$$

Also, the asymptotic stability result has been established with the opposite sign of power-type nonlinearities.

In [9], Tahamtani and Shahrouzi considered

$$\begin{aligned}&u_{tt}+\Delta^2u-\alpha_1\Delta u+\alpha_2u_t+\alpha_3|u|^pu+\mathfrak{b}(x,t,u,\nabla u,\Delta u)=f(t)\omega(x),\quad x\in\Omega,\quad t>0,\\&u(x,t)=0,\quad \Delta u=-c_0\partial_\nu u(x,t),\quad x\in\Gamma,\quad t>0,\\&u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in\Omega,\\&\int_\Omega u(x,t)\omega(x)\mathrm{d}x=\phi(t),\quad t>0.\end{aligned}$$

They showed that the solutions of this problem under some suitable conditions are stable if α_1,α_2 being large enough, $\alpha_3\geq 0$ and $\phi(t)$ tends to zero as time goes to infinity. Also, established a blow-up result, if $\alpha_3<0$ and $\phi(t)=k$ be a constant. Their approaches are based on the Lyapunov function and perturbed energy method for stability result and concavity argument for blow-up result. The interested reader is referred to the papers [10–14].

Motivated by the aforementioned works, we take b,λ and c_0 in the appropriately domain and prove that some solutions of (1.1)-(1.4) blow up in a finite time. Our approaches are based on the modified concavity argument method.

2 Preliminaries and main results

In this section, we present some material needed in the proof of our main results. We shall assume that the function $g(x,t,u,\Delta u)$ and the functions appearing in the data satisfy the following conditions

(A1)

$$u_0\in H_0^2(\Omega)\cap L^{p+2}(\Omega),\quad u_1\in L^2(\Omega),$$

(A2)

$$|g(x,t,u,\Delta u)|\leq M(|\Delta u|^{\frac{m}{2}}+|u|^{\frac{p}{2}}),$$

with some positive $M>0$.

Throughout this paper all the functions considered are real-valued. We denote by $\|\cdot\|_q$ the L^q-norm over Ω and $\|\cdot\|_{q,\partial\Omega}$ the L^q-norm over $\partial\Omega$. In particular, the L^2-norm is denoted $\|\cdot\|$ in Ω and $\|\cdot\|_{\partial\Omega}$ in $\partial\Omega$ also $(\cdot,\cdot)$ denotes the usual L^2-inner product. We use familiar function spaces H_0^2.

We sometimes use the Poincaré inequality

$$\|u\|^2 \leq \theta \|\nabla u\|^2, \tag{2.1}$$

and Young's inequality

$$ab \leq \beta a^q + C(\beta,q) b^{q'}, \quad a,b \geq 0, \quad \beta > 0, \quad \frac{1}{q} + \frac{1}{q'} = 1, \tag{2.2}$$

where $\theta = \theta(\Omega,n)$ and $C(\beta,q) = \frac{1}{q'}(\beta q)^{-\frac{q'}{q}}$ are constants.

We will use the trace inequality

$$\|\nabla u\|^2_{\partial\Omega} \leq B \|\Delta u\|^2, \tag{2.3}$$

where B is the optimal constant.

The following lemma was introduced in [15]; it will be used in Section 3 in order to prove the blow-up result.(see also [16,17])

Lemma 2.1. *Let $\mu > 0, c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. Assume that $\psi(t)$ is a twice differentiable positive function such that*

$$\psi'' \psi - (1+\mu)[\psi']^2 \geq -2c_1 \psi \psi' - c_2 [\psi]^2, \tag{2.4}$$

for all $t \geq 0$. If

$$\psi(0) > 0 \quad \text{and} \quad \psi'(0) + \gamma_2 \mu^{-1} \psi(0) > 0, \tag{2.5}$$

then

$$\psi(t) \to +\infty \quad \text{as} \quad t \to t_1 \leq t_2 = \frac{1}{2\sqrt{c_1^2 + \mu c_2}} \log \frac{\gamma_1 \psi(0) + \mu \psi'(0)}{\gamma_2 \psi(0) + \mu \psi'(0)}. \tag{2.6}$$

Here

$$\gamma_1 = -c_1 + \sqrt{c_1^2 + \mu c_2} \quad \text{and} \quad \gamma_2 = -c_1 - \sqrt{c_1^2 + \mu c_2}.$$

We consider the following problem that is obtained from (1.1)-(1.3) by substituting $v(x,t) = e^{-\lambda t} u(x,t)$:

$$\begin{aligned} & v_{tt} + 2\lambda v_t + \lambda^2 v + \Delta\big[(a_0 + a e^{\lambda(m-2)t} |\Delta v|^{m-2}) \Delta v\big] - \lambda b \Delta v - b \Delta v_t \\ & \qquad = e^{-\lambda t} g(x,t,e^{\lambda t} v, e^{\lambda t} \Delta v) + e^{\lambda(p-2)t} |v|^{p-2} v, \quad x \in \Omega, \quad t > 0, && (2.7) \\ & v(x,t) = 0, \quad \Delta v(x,t) = -c_0 \partial_\nu v(x,t), \quad x \in \partial\Omega, \quad t > 0, && (2.8) \\ & v(x,0) = u_0(x), \quad v_t(x,0) = u_1(x) - \lambda u_0(x), \quad x \in \Omega. && (2.9) \end{aligned}$$

The energy associated with problem (2.7)-(2.9) is given by

$$E_\lambda(t)=\frac{e^{\lambda(p-2)t}}{p}\|v\|_p^p-\frac{1}{2}I\Big(v(t)\Big), \tag{2.10}$$

where

$$\begin{aligned}I(v(t)):=&\|v_t\|^2+\lambda^2\|v\|^2+a_0\|\Delta v\|^2+a_0c_0\|\nabla v\|_{\partial\Omega}^2\\&+\frac{2ac_0^{m-1}}{m}e^{\lambda(m-2)t}\|\nabla v\|_{m,\partial\Omega}^m+\frac{2a}{m}e^{\lambda(m-2)t}\|\Delta v\|_m^m.\end{aligned} \tag{2.11}$$

Now we are in a position to state our blow-up result as follows.

Theorem 2.1. *Let the conditions* (A1) *and* (A2) *are satisfied. Assume that* $E_\lambda(0)>0$ *and*

$$\begin{aligned}&c_0\in(0,\frac{1}{B}],\ \ b\in\left(0,\frac{a_0}{4\theta M_1}\sqrt{\frac{a(p-m)(p+m-4)^3}{m+ap}}\right],\\&\lambda\geq\max\left\{M\sqrt{\frac{m+ap}{a(p-m)(p+m-4)}},M\sqrt{\frac{ap+m(p-m-1)}{2a(m-1)(p-m-1)}}\right\}.\end{aligned}$$

then there exists a finite time t_1 *such that the solution of the problem* (1.1)-(1.3) *blows up in a finite time, that is*

$$\|u(t)\|\to+\infty\ \text{ as }\ t\to t_1. \tag{2.12}$$

3 Blow up

In this section we are going to prove that for sufficiently large initial data some of the solutions blow up in a finite time. To prove the blow-up result for certain solutions with positive initial energy, we need the following Lemma.

Lemma 3.1. *Under the conditions of Theorem* 2.2*, the energy functional* $E_\lambda(t)$*, defined by* (2.10)*, satisfies*

$$E_\lambda(t)\geq 2\lambda\int_0^t\|v_\tau(\tau)\|^2\mathrm{d}\tau+b\int_0^t\|\nabla v_\tau(\tau)\|^2\mathrm{d}\tau. \tag{3.1}$$

Proof. A multiplication of Eq. (2.7) by v_t and integrating over Ω gives

$$\begin{aligned}E_\lambda'(t)=&\frac{\lambda(p-2)}{p}e^{\lambda(p-2)t}\|v\|_p^p+2\lambda\|v_t\|^2+\frac{b\lambda}{2}\|\nabla v\|^2+b\|\nabla v_t\|^2\\&-\frac{ac_0^{m-1}}{m}\lambda(m-2)e^{\lambda(m-2)t}\|\nabla v\|_{m,\partial\Omega}^m-\frac{a\lambda(m-2)}{m}e^{\lambda(m-2)t}\|\Delta v\|_m^m\\&-e^{-\lambda t}(\widehat{g}(t,v),v_t),\end{aligned} \tag{3.2}$$

where

$$\widehat{g}(t,v):=g(x,t,e^{\lambda t}v,e^{\lambda t}\Delta v).$$

It is easy to verify that

$$\begin{aligned} e^{-\lambda t}|(\widehat{g}(t,v),v_t)| \le & \varepsilon_0 e^{\lambda(m-2)t}\|\Delta v\|_m^m \\ & +\varepsilon_1 e^{\lambda(p-2)t}\|v\|_p^p+\frac{M^2}{4}\left(\frac{1}{\varepsilon_0}+\frac{1}{\varepsilon_1}\right)\|v_t\|^2, \end{aligned} \tag{3.3}$$

where (A2) and Young's inequality (2.2) have been used.

Taking into account estimate (3.3) in relation with (3.2), we get

$$\begin{aligned} E'_\lambda(t) \ge & \left[\frac{\lambda(p-2)}{p}-\varepsilon_1\right]e^{\lambda(p-2)t}\|v\|_p^p+\left[2\lambda-\frac{M^2}{4}\left(\frac{1}{\varepsilon_0}+\frac{1}{\varepsilon_1}\right)\right]\|v_t\|^2+\frac{b\lambda}{2}\|\nabla v\|^2+b\|\nabla v_t\|^2 \\ & -\left[\varepsilon_0+\frac{a\lambda(m-2)}{m}\right]e^{\lambda(m-2)t}\|\Delta v\|_m^m-\frac{ac_0^{m-1}}{m}\lambda(m-2)e^{\lambda(m-2)t}\|\nabla v\|_{m,\partial\Omega}^m. \end{aligned} \tag{3.4}$$

Employing the last inequality, we obtain from (2.10) the following inequality

$$\begin{aligned} & E'_\lambda(t)-[\lambda(p-2)-\varepsilon_1 p]E_\lambda(t) \\ \ge & \left[\frac{\lambda(p-2)-\varepsilon_1 p}{2}+2\lambda-\frac{M^2}{4}\left(\frac{1}{\varepsilon_0}+\frac{1}{\varepsilon_1}\right)\right]\|v_t\|^2 \\ & +\frac{\lambda^2}{2}[\lambda(p-2)-\varepsilon_1 p]\|v\|^2+\frac{a_0}{2}[\lambda(p-2)-\varepsilon_1 p]\|\Delta v\|^2+\frac{a_0c_0}{2}[\lambda(p-2)-\varepsilon_1 p]\|\nabla v\|_{\partial\Omega}^2 \\ & +\frac{ac_0^{m-1}}{m}[\lambda(p-m)-\varepsilon_1 p]e^{\lambda(m-2)t}\|\nabla v\|_{m,\partial\Omega}^m+\frac{\lambda b}{2}\|\nabla v\|^2+b\|\nabla v_t\|^2 \\ & +\left[\frac{a\lambda}{m}(p-m)-\frac{a\varepsilon_1 p}{m}-\varepsilon_0\right]e^{\lambda(m-2)t}\|\Delta v\|_m^m. \end{aligned} \tag{3.5}$$

At this point, let us recall the Poincaré and trace inequalities to estimate the terms on the right side of (3.5), we obtain

$$\begin{aligned} & E'_\lambda(t)-[\lambda(p-2)-\varepsilon_1 p]E_\lambda(t) \\ \ge & \left[\frac{\lambda(p-2)-\varepsilon_1 p}{2}-\frac{M^2}{4}\left(\frac{1}{\varepsilon_0}+\frac{1}{\varepsilon_1}\right)\right]\|v_t\|^2+2\lambda\|v_t\|^2+b\|\nabla v_t\|^2 \\ & +\frac{ac_0^{m-1}}{m}[\lambda(p-m)-\varepsilon_1 p]e^{\lambda(m-2)t}\|\nabla v\|_{m,\partial\Omega}^m+\left[\frac{a\lambda}{m}(p-m)-\frac{a\varepsilon_1 p}{m}-\varepsilon_0\right]e^{\lambda(m-2)t}\|\Delta v\|_m^m \\ & +\frac{\lambda^2}{2}\left[\lambda(p-2)+\frac{b}{\lambda\theta}-\varepsilon_1 p\right]\|v\|^2+\frac{a_0}{2}[\lambda(p-2)-\varepsilon_1 p](1-c_0B)\|\Delta v\|^2. \end{aligned} \tag{3.6}$$

Now, if we choose $\varepsilon_0 = a\lambda/m$ and $\varepsilon_1 = \lambda(p-m-1)/p$ then we obtain from (3.6) the inequality

$$\begin{aligned}
&E'_\lambda(t) - \lambda(m-1)E_\lambda(t)\\
\geq &\left[\frac{\lambda(m-1)}{2} - \frac{M^2}{4}\left(\frac{ap+m(p-m-1)}{a\lambda(p-m-1)}\right)\right]\|v_t\|^2 + 2\lambda\|v_t\|^2 + b\|\nabla v_t\|^2\\
&+\frac{\lambda^2}{2}\left[\lambda(m-1)+\frac{b}{\lambda\theta}\right]\|v\|^2 + \frac{a_0\lambda(m-1)}{2}(1-c_0B)\|\Delta v\|^2\\
&+\frac{a\lambda c_0^{m-1}}{m}(m-1)e^{\lambda(m-2)t}\|\nabla v\|^m_{m,\partial\Omega}.
\end{aligned} \tag{3.7}$$

Thanks to the hypothesises of Theorem 2.2, if we choose $\lambda \geq M\sqrt{\frac{ap+m(p-m-1)}{2a(m-1)(p-m-1)}}$ and $c_0 \in (0, B^{-1}]$ then we get

$$E'_\lambda(t) \geq \lambda(m-1)E_\lambda(t) + 2\lambda\|v_t\|^2 + b\|\nabla v_t\|^2. \tag{3.8}$$

Since $E_\lambda(0) > 0$, we obtain from (3.8) that $E_\lambda(t) \geq e^{\lambda(m-1)t}E_\lambda(0) \geq 0$. Therefore integrating (3.8) yields

$$E_\lambda(t) \geq 2\lambda\int_0^t \|v_\tau(\tau)\|^2 \mathrm{d}\tau + b\int_0^t \|\nabla v_\tau(\tau)\|^2\mathrm{d}\tau,$$

and proof of Lemma is competed. □

Proof of Theorem 2.2. For obtain the blow-up result, we consider the following functional

$$\psi(t) = \|v(t)\|^2 + 2\lambda\int_0^t \|v(\tau)\|^2\mathrm{d}\tau + b\int_0^t \|\nabla v(\tau)\|^2\mathrm{d}\tau + C, \tag{3.9}$$

where C is a positive constant that will be chosen appropriately.

It is easy to see that

$$\psi'(t) = 2(v, v_t) + 4\lambda\int_0^t (v_\tau, v)\mathrm{d}\tau + 2b\int_0^t (\nabla v_\tau, \nabla v)\mathrm{d}\tau + 2\lambda\|u_0\|^2 + b\|\nabla u_0\|^2, \tag{3.10}$$

and consequently

$$\psi''(t) = 2\|v_t\|^2 + 2(v, v_{tt} + 2\lambda v_t - b\Delta v_t). \tag{3.11}$$

A multiplication of Eq. (2.7) by v and integrating over Ω gives

$$\begin{aligned}
(v, v_{tt} + 2\lambda v_t - b\Delta v_t) = &-\lambda^2\|v\|^2 - a_0(\Delta^2 v, v) - ae^{\lambda(m-2)t}\Big(\Delta(|\Delta v|^{m-2}\Delta v), v\Big)\\
&+\lambda b(\Delta v, v) + e^{-\lambda t}\Big(\widehat{g}(t,v), v\Big) + e^{\lambda(p-2)t}(|v|^{p-2}v, v).
\end{aligned} \tag{3.12}$$

By using boundary conditions in terms of (3.12) and combining with (3.11) we obtain

$$\begin{aligned}\psi''(t)=&2\|v_t\|^2-2\lambda^2\|v\|^2-2a_0\|\Delta v\|^2-2a_0c_0\|\nabla v\|^2_{\partial\Omega}\\&-2ae^{\lambda(m-2)t}\|\Delta v\|^m_m-2\lambda b\|\nabla v\|^2-2ac_0^{m-1}e^{\lambda(m-2)t}\|\nabla v\|^m_{m,\partial\Omega}\\&+2e^{-\lambda t}(\widehat{g}(t,v),v)+2e^{\lambda(p-2)t}\|v\|^p_p.\end{aligned}\tag{3.13}$$

Due to the condition (A2) we have

$$2e^{-\lambda t}|(\widehat{g}(t,v),v)|\le\delta_0e^{\lambda(m-2)t}\|\Delta v\|^m_m+\delta_1e^{\lambda(p-2)t}\|v\|^p_p+M^2\left(\frac{1}{\delta_0}+\frac{1}{\delta_1}\right)\|v\|^2,\tag{3.14}$$

and so by inserting (3.14) into (3.13), we get

$$\begin{aligned}\psi''(t)\ge&2\|v_t\|^2-\left[2\lambda^2+M^2\left(\frac{1}{\delta_0}+\frac{1}{\delta_1}\right)\right]\|v\|^2-2a_0\|\Delta v\|^2-2a_0c_0\|\nabla v\|^2_{\partial\Omega}\\&-(2a+\delta_0)e^{\lambda(m-2)t}\|\Delta v\|^m_m-2ac_0^{m-1}e^{\lambda(m-2)t}\|\nabla v\|^m_{m,\partial\Omega}\\&-2\lambda b\|\nabla v\|^2+(2-\delta_1)e^{\lambda(p-2)t}\|v\|^p_p,\end{aligned}\tag{3.15}$$

by using definition of $E_\lambda(t)$ in (3.15), we have

$$\begin{aligned}\psi''(t)\ge&(p+m)E_\lambda(t)+\left(2+\frac{p+m}{2}\right)\|v_t\|^2+\left(2-\frac{p+m}{p}-\delta_1\right)e^{\lambda(p-2)t}\|v\|^p_p\\&+a_0c_0\left(\frac{p+m}{2}-2\right)\|\nabla v\|^2_{\partial\Omega}+\left[\frac{\lambda^2(p+m)}{2}-2\lambda^2-M^2\left(\frac{1}{\delta_0}+\frac{1}{\delta_1}\right)\right]\|v\|^2\\&+[\frac{a_0(p+m)}{2}-2a_0-2b\theta\lambda]\|\Delta v\|^2+ac_0^{m-1}\left(\frac{p+m}{m}-2\right)e^{\lambda(m-2)t}\|\nabla v\|^m_{m,\partial\Omega}\\&+\left[\frac{a(p+m)}{m}-2a-\delta_0\right]e^{\lambda(m-2)t}\|\Delta v\|^m_m,\end{aligned}\tag{3.16}$$

where Poincaré inequality (2.1) has been used. Since $p+m\ge4$ and by choosing $\delta_0=a(p-m)/m$ and $\delta_1=(p-m)/p$, we get

$$\begin{aligned}\psi''(t)\ge&(p+m)E_\lambda(t)+\left(2+\frac{p+m}{2}\right)\|v_t\|^2+\left[\frac{a\lambda^2(p-m)(p+m-4)-M^2(m+ap)}{2a(p-m)}\right]\|v\|^2\\&+\left[\frac{a_0(p+m-4)}{2}-2b\theta\lambda\right]\|\Delta v\|^2.\end{aligned}\tag{3.17}$$

Finally, thanks to the assumptions of Theorem 2.2 about λ and b, we deduce

$$\psi''(t)\ge(p+m)E_\lambda(t)+\left(2+\frac{p+m}{2}\right)\|v_t\|^2.\tag{3.18}$$

Combining estimation of Lemma 3.1 with (3.18) yields

$$\begin{aligned}\psi''(t) \geq & 2\lambda(p+m)\int_0^t \|v_\tau(\tau)\|^2 \mathrm{d}\tau + b(p+m)\int_0^t \|\nabla v_\tau(\tau)\|^2 \mathrm{d}\tau + \left(2+\frac{p+m}{2}\right)\|v_t\|^2 \\ \geq & \left(2+\frac{p+m}{2}\right)\left[\|v_t\|^2 + 2\lambda\int_0^t \|v_\tau(\tau)\|^2 \mathrm{d}\tau + b\int_0^t \|\nabla v_\tau(\tau)\|^2 \mathrm{d}\tau\right], \end{aligned} \tag{3.19}$$

where the inequality $2(p+m)/(p+m+4) \geq 1$ has been used.

Taking into account the estimation (3.19), it is easy to verify

$$\begin{aligned} &\psi''(t)\psi(t) - \left(\frac{p+m+4}{8}\right)[\psi'(t)]^2 \\ \geq & \left(2+\frac{p+m}{2}\right)\left[\|v_t\|^2 + 2\lambda\int_0^t \|v_\tau(\tau)\|^2 \mathrm{d}\tau + b\int_0^t \|\nabla v_\tau(\tau)\|^2 \mathrm{d}\tau + C\right]\psi(t) \\ & -(1+\frac{p+m}{4})\left[(v,v_t) + 2\lambda\int_0^t (v_\tau,v)\mathrm{d}\tau + b\int_0^t (\nabla v_\tau,\nabla v)\mathrm{d}\tau + \lambda\|u_0\|^2 + \frac{b}{2}\|\nabla u_0\|^2\right]^2 \\ & -(2+\frac{p+m}{2})C\psi(t). \end{aligned} \tag{3.20}$$

We choose now $C = \lambda\|u_0\|^2 + \frac{b}{2}\|\nabla u_0\|^2$. By the Cauchy-Schwarz inequality and since $\psi(t) \geq 0$, we deduce the following inequality

$$\psi''(t)\psi(t) - \frac{p+m+4}{8}[\psi'(t)]^2 \geq -\left(2+\frac{p+m}{2}\right)C\psi(t) \geq -\left(2+\frac{p+m}{2}\right)[\psi(t)]^2. \tag{3.21}$$

Hence we see that the hypotheses of Lemma 2.1 are fulfilled with

$$\mu = \frac{p+m-4}{8}, \qquad c_1 = 0, \qquad c_2 = \frac{p+m+4}{2},$$

and the conclusion of Lemma 2.1 gives us that some solutions of problem (2.7)-(2.9) blow up in a finite time t_1. Since this system is equivalent to (1.1)-(1.3), the proof is complete.

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