

On Behavior of Solutions to a Class of Nonlinear Hyperbolic Inverse Source Problem

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Abstract This article is concerned with a class of hyperbolic inverse source problem with memory term and nonlinear boundary damping. Under appropriate assumptions on the initial data and parameters in the equation, we establish two results on behavior of solutions. At first we proved stability of solutions when the integral overdetermination tends to zero as time goes to infinity and finally a blow-up result is established for certain solution with positive initial energy.

Keywords Inverse problem, asymptotic stability, blow up, memory, boundary feedback

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1 Introduction

In this paper, we study the global behavior of solutions for inverse problem of determining a pair of functions $\{u(x, t), f(t)\}$ that satisfy

$$u_{tt} - \Delta u - \operatorname{div}(|\nabla u|^m \nabla u) + \int_0^t g_1(t - \tau) \Delta u(\tau) d\tau + u_t + a|u|^p u = h(x, t, u, \nabla u) + f(t)\omega(x), \quad x \in \Omega, t > 0, \quad (1.1)$$

$$\begin{cases} u(x, t) = 0, & x \in \Gamma_0, t > 0, \\ \frac{\partial u}{\partial n}(x, t) = \int_0^t g_1(t - \tau) \frac{\partial u}{\partial n}(x, \tau) d\tau - |\nabla u|^m \nabla u + \alpha u, & x \in \Gamma_1, t > 0, \end{cases} \quad (1.2)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad (1.3)$$

$$\int_{\Omega} u(x, t)\omega(x)dx = \phi(t), \quad t > 0, \quad (1.4)$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) with smooth boundary $\Gamma_0 \cup \Gamma_1 = \partial\Omega$. Here a is a real number and α is a nonnegative constant. Also p, m are positive real numbers and moreover, $h(x, t, u, \nabla u)$, $\phi(t)$, $g_1(t)$ and $\omega(x)$ are functions that satisfy specific conditions that will be enunciated later.

The problem of proving asymptotic stability and blow up of solutions for the hyperbolic equations with boundary conditions has recently attracted a lot of attention and various results are available (see [4, 5, 10, 14, 17, 27] and references therein).

In the absence of the m -Laplacian operator in direct problems ($\omega(x) = 0$) with homogeneous Dirichlet boundary conditions, Chen and Xiong [9] studied

$$u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau + u_t = |u|^p u.$$

The authors established blow-up results for this equation, even for vanishing initial energy. Also they proved that the solutions decay under suitable conditions. See also [2, 7, 8, 13, 18, 25, 26].

Recently, in [3] Bilgin and Kalantarov investigated blow up of solutions for the following initial-boundary value problem

$$\begin{aligned} u_{tt} - \nabla[(a_0 + a|\nabla u|^{m-2})\nabla u] - b\Delta u_t &= g(x, t, u, \nabla u) + |u|^{p-2}u, \quad x \in \Omega, t > 0, \\ u(x, t) &= 0, \quad x \in \partial\Omega, t > 0, \\ u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega. \end{aligned}$$

They obtained sufficient conditions on initial functions for which there exists a finite time that some solutions blow up at this time.

Shahrouzi and Tahamtani investigated global behavior of solutions to some class of inverse source problems. In [22], they investigated the global in time behavior of solutions for an inverse problem of determining a pair of functions $\{u, f\}$ satisfying the equation

$$u_{tt} + \Delta^2 u - |u|^p u + a(x, t, u, \nabla u, \Delta u) = f(t)\omega(x), \quad x \in \Omega, t > 0,$$

with the initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,$$

the boundary conditions

$$u(x, t) = \partial_\nu u(x, t) = 0, \quad x \in \partial\Omega, t > 0,$$

and the over-determination condition

$$\int_{\Omega} u(x, t)\omega(x)dx = 1, \quad t > 0,$$

also, the asymptotic stability result has been established with the opposite sign of power-type nonlinearities.

Later, in [24], Tahamtani and Shahrouzi considered the following problem:

$$\begin{aligned} u_{tt} + \Delta^2 u - \alpha_1 \Delta u + \alpha_2 u_t + \alpha_3 |u|^p u + \mathfrak{b}(x, t, u, \nabla u, \Delta u) &= f(t)\omega(x), \quad x \in \Omega, t > 0, \\ u(x, t) &= 0, \quad \Delta u = -c_0 \partial_\nu u(x, t), \quad x \in \Gamma, t > 0, \\ u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\ \int_{\Omega} u(x, t)\omega(x)dx &= \phi(t), \quad t > 0. \end{aligned}$$

They showed that the solutions of this problem under some suitable conditions are stable if α_1, α_2 are large enough, $\alpha_3 \geq 0$ and $\phi(t)$ tends to zero as time goes to infinity. They also established a blow-up result, if $\alpha_3 < 0$ and $\phi(t) = k$ is a constant. Their approaches are based on the Lyapunov function and perturbed energy method for stability result and concavity argument for blow-up result.

Very recently, Shahrouzi [21] investigated the asymptotic behavior of solutions for the following inverse problem:

$$\begin{aligned} u_{tt} + \Delta^2 u - \int_0^t g(t-\tau) \Delta^2 u(\tau) d\tau - a_1 \Delta u + a_2 u_t &= f(t) \omega(x), \quad x \in \Omega, t > 0, \\ \begin{cases} u(x, t) = 0, & x \in \Gamma_0, t > 0, \\ \Delta u(x, t) = \int_0^t g(t-\tau) \Delta u(\tau) d\tau - a_3 |\nabla u|^p \nabla u, & x \in \Gamma_1, t > 0, \end{cases} \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\ \int_{\Omega} u(x, t) \omega(x) dx = \phi(t), \quad t > 0, \end{aligned}$$

He obtained sufficient conditions on relaxation function and initial data for which the solutions of problem are asymptotically stable when the integral overdetermination tends to zero as time goes to infinity. For more information about inverse problems, the interested reader is referred to the papers [1, 6, 11, 12, 15, 20, 23]

Our objective in the present work is to extend the results of [3, 24] by considering problem (1.1)–(1.4) in the presence of m -Laplacian operator and boundary conditions.

Motivated by the aforementioned works, our result here is twofold: First, we consider $a = 1$ and show that if we take initial data and parameters in the appropriately domain, then solutions of (1.1)–(1.4) are asymptotically stable when $\phi(t)$ tends to zero as time goes to infinity. Our approaches are based on the Lyapunov function and perturbed energy method. Second, if we take $a = -1$, $\phi(t) \equiv k$ (k is a constant), then by using the modified concavity argument, we prove the blow up of solutions for the problem (1.1)–(1.4) with positive initial data and appropriate domain for parameters.

2 Preliminaries and Main Results

In this section, we present some material needed in the proof of our main results. We shall assume that the functions $\omega(x)$, $h(x, t, u, \nabla u)$ and the functions appearing in the data satisfy the following conditions:

(A1)

$$u_0 \in H_0^1(\Omega) \cap L^{p+2}(\Omega) \cap L^{m+2}(\Omega), \quad u_1 \in L^2(\Omega), \quad \int_{\Omega} u_0(x) \omega(x) dx = 1;$$

(A2)

$$\omega \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{p+2}(\Omega) \cap L^{m+2}(\Omega), \quad \int_{\Omega} \omega^2(x) dx = 1;$$

(A3)

$$|h(x, t, u, \nabla u)| \leq L(|\nabla u|^{\frac{m}{2}} + |u|^{\frac{p}{2}}),$$

where $L > 0$.

Throughout this paper, all the functions considered are real-valued. We denote by $\|\cdot\|_q$ the L^q -norm over Ω . In particular, the L^2 -norm is denoted by $\|\cdot\|$ in Ω and $\|\cdot\|_{\Gamma_i}$ in Γ_i . Also $(\cdot, \cdot)$ denotes the usual L^2 -inner product. We use familiar function spaces H_0^1, H^2 .

We recall the trace Sobolev embedding

$$H_{\Gamma_0}^1(\Omega) \hookrightarrow L^q(\Gamma_1) \quad \text{for } 2 \leq q < \frac{2(n-1)}{n-2},$$

where

$$H_{\Gamma_0}^1(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$$

and the embedding inequality

$$\|u\|_{q,\Gamma_1} \leq B_q \|\nabla u\|_2, \quad (2.1)$$

where B_q is the optimal constant.

We sometimes use the Poincaré inequality

$$\|u\|^2 \leq \theta^2 \|\nabla u\|^2, \quad (2.2)$$

and the Young inequality

$$a_1 b \leq \beta^q + C(\beta, q) b^{q'}, \quad a_1, b \geq 0, \quad \beta > 0, \quad \frac{1}{q} + \frac{1}{q'} = 1, \quad (2.3)$$

where $C(\beta, q) = \frac{1}{q'}(\beta q)^{-\frac{q'}{q}}$ are constants.

The following lemma was introduced in [16]; it will be used in Section 4 in order to prove the blow-up result.

Lemma 2.1 *Let $\kappa > 0, c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. Assume that $\chi(t)$ is a twice differentiable positive function such that*

$$\chi''\chi - (1 + \kappa)[\chi']^2 \geq -2c_1\chi\chi' - c_2[\chi]^2 \quad (2.4)$$

for all $t \geq 0$. If

$$\chi(0) > 0 \quad \text{and} \quad \chi'(0) + \rho_2\kappa^{-1}\chi(0) > 0, \quad (2.5)$$

then

$$\chi(t) \rightarrow +\infty \quad \text{as} \quad t \rightarrow t_1 \leq t_2 = \frac{1}{2\sqrt{c_1^2 + \kappa c_2}} \log \frac{\rho_1\chi(0) + \kappa\chi'(0)}{\rho_2\chi(0) + \kappa\chi'(0)}. \quad (2.6)$$

Here

$$\rho_1 = -c_1 + \sqrt{c_1^2 + \kappa c_2} \quad \text{and} \quad \rho_2 = -c_1 - \sqrt{c_1^2 + \kappa c_2}.$$

Adapting the idea of Prilepko et al. [19], the key observation is that the problem (1.1)–(1.4) is equivalent to the following direct problem:

$$\begin{aligned} u_{tt} - \Delta u - \operatorname{div}(|\nabla u|^m \nabla u) + \int_0^t g_1(t - \tau) \Delta u(\tau) d\tau + u_t + a|u|^p u \\ = h(x, t, u, \nabla u) + f(t)\omega(x), \quad x \in \Omega, t > 0, \end{aligned} \quad (2.7)$$

$$\begin{cases} u(x, t) = 0, & x \in \Gamma_0, t > 0, \\ \frac{\partial u}{\partial n}(x, t) = \int_0^t g_1(t - \tau) \frac{\partial u}{\partial n}(x, \tau) d\tau - |\nabla u|^m \nabla u + \alpha u, & x \in \Gamma_1, t > 0, \end{cases} \quad (2.8)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad (2.9)$$

in which the unknown function $f(t)$ is replaced by

$$f(t) = \phi''(t) + \phi'(t) + (\nabla u, \nabla \omega) + (|\nabla u|^m \nabla u, \nabla \omega) - \int_0^t g_1(t - \tau) (\nabla u(\tau), \nabla \omega) d\tau$$

$$+ a(|u|^p u, \omega) - (\widehat{h}(t, u), \omega), \quad (2.10)$$

where

$$\widehat{h}(t, u) := h(x, t, u, \nabla u).$$

We use the following additional conditions on $g_1(s) = e^{\lambda s} g(s)$:

(B1)

$$g(s) \geq 0, \quad g'(s) \leq -\lambda g(s);$$

(B2)

$$1 - \int_0^\infty g_1(t) dt = l > 0.$$

For the sake of simplicity, to prove the asymptotic stability result we suppose that $a = 1$.

In order to formulate our results, it is convenient to introduce the energy of the problem (2.7)–(2.9):

$$\begin{aligned} E(t) = & \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \left(1 - \int_0^t g_1(s) ds \right) \|\nabla u\|^2 + \frac{1}{2} (g_1 * \nabla u)(t) + \frac{1}{m+2} \|\nabla u\|_{m+2}^{m+2} \\ & + \frac{1}{p+2} \|u\|_{p+2}^{p+2} - \frac{\alpha}{2} \|u\|_{\Gamma_1}^2, \end{aligned} \quad (2.11)$$

where $(g_1 * v)(t) = \int_0^t g_1(t-s) \|v(t) - v(s)\|^2 ds$.

Now, we are in a position to state our asymptotic stability result:

Theorem 2.2 *Let the conditions (A1)–(A3) and (B1), (B2) are satisfied and suppose that ϕ, ϕ', ϕ'' are continuous functions defined on $[0, \infty)$, such that ϕ'' is a bounded function and ϕ, ϕ' tend to zero as $t \rightarrow \infty$. Also for sufficiently large M, N , we take α, ξ and δ such that*

$$\delta < \min \left\{ 2, \frac{l}{3\theta^2}, \frac{2N}{M} \right\}, \quad \alpha < \frac{l - 3\delta\theta^2}{2B^2}, \quad \int_0^\infty g_1(s) ds \leq \frac{2\xi}{1 + 2\xi}.$$

Then the solution of the problem (1.1)–(1.4) asymptotically stable and

$$\lim_{t \rightarrow +\infty} E(t) = 0. \quad (2.12)$$

Finally, we consider the following problem that is obtained from (1.1)–(1.4) when $\phi(t) = k$ and $a = -1$, by substituting (similar as in [11]) $v(x, t) = e^{-\lambda t} u(x, t)$:

$$\begin{aligned} v_{tt} + (2\lambda + 1)v_t + \lambda(\lambda + 1)v - \Delta v - e^{\lambda mt} \operatorname{div}(|\nabla v|^m \nabla v) + \int_0^t g(t-\tau) \Delta v(\tau) d\tau \\ = e^{-\lambda t} \widehat{h}(t, e^{\lambda t} v) + e^{\lambda pt} |v|^p v + e^{-\lambda t} f(t) \omega(x), \quad x \in \Omega, \quad t > 0, \end{aligned} \quad (2.13)$$

$$\begin{cases} v(x, t) = 0, & x \in \Gamma_0, t > 0, \\ \frac{\partial v}{\partial n} = \int_0^t g(t-\tau) \frac{\partial v}{\partial n}(x, \tau) d\tau - e^{\lambda mt} |\nabla v|^m \nabla v + \alpha v, & x \in \Gamma_1, t > 0, \end{cases} \quad (2.14)$$

$$v(x, 0) = u_0(x), \quad v_t(x, 0) = u_1(x) - \lambda u_0(x), \quad x \in \Omega, \quad (2.15)$$

$$\int_\Omega v(x, t) \omega(x) dx = k e^{-\lambda t}, \quad t > 0. \quad (2.16)$$

The value of the parameter λ will be prescribed later. Similar to (2.10), we multiply (2.13) by $\omega(x)$ and use (2.16) to express

$$e^{-\lambda t} f(t) = (\nabla v, \nabla \omega) + e^{\lambda mt} (\nabla v|^m \nabla v, \nabla \omega) - \int_0^t g(t-\tau) (\nabla v(\tau), \nabla \omega) d\tau - e^{\lambda pt} (|v|^p v, \omega)$$

Lemma 3.2 Under the conditions of Theorem 2.2, the function $f(t)$, defined by (2.10), for some $M, N > 0$ and $\gamma_0, \gamma_1 > 0$ satisfies

$$|M\phi'(t) + N\phi(t)|f(t) \leq \left(\frac{\delta N\theta^2}{2} + \frac{M\gamma_0}{2} + \frac{1-l}{2} \right) \|\nabla u\|^2 + \frac{M\gamma_0}{2} \|\nabla u\|_{m+2}^{m+2} + \frac{M\gamma_1}{2} \|u\|_{p+2}^{p+2} + \frac{1-l}{2l} (g_1 * \nabla u)(t) + H(t), \quad (3.2)$$

where $\delta > 0$ and

$$H(t) = |M\phi'(t) + N\phi(t)|\phi''(t) + \phi'(t) + \frac{|M\phi'(t) + N\phi(t)|^2}{2} \left(1 + \frac{1}{\delta N\theta^2} \right) \|\nabla \omega\|^2 + \frac{|M\phi'(t) + N\phi(t)|^{m+2}}{(m+2) \left[\frac{M\gamma_0(m+2)}{4m+4} \right]^{m+1}} \|\nabla \omega\|_{m+2}^{m+2} + \frac{|M\phi'(t) + N\phi(t)|^{p+2}}{(p+2) \left[\frac{M\gamma_1(p+2)}{4p+4} \right]^{p+1}} \|\omega\|_{p+2}^{p+2} + \frac{L^2 |M\phi'(t) + N\phi(t)|^2}{M} \left(\frac{1}{\gamma_0} + \frac{1}{\gamma_1} \right) \|\omega\|^2. \quad (3.3)$$

Proof We have from the definition of $f(t)$ that

$$\begin{aligned} |M\phi'(t) + N\phi(t)|f(t) &= |M\phi'(t) + N\phi(t)|(\phi''(t) + \phi'(t)) + |M\phi'(t) + N\phi(t)|(|\nabla u|^m \nabla u, \nabla \omega) \\ &\quad - |M\phi'(t) + N\phi(t)| \int_0^t g_1(t-\tau)(\nabla u(\tau), \nabla \omega) d\tau - |M\phi'(t) + N\phi(t)|(\widehat{h}(t, u), \omega) \\ &\quad + |M\phi'(t) + N\phi(t)|(|u|^p u, \omega) + |M\phi'(t) + N\phi(t)|(\nabla u, \nabla \omega). \end{aligned} \quad (3.4)$$

By using the Young inequality (2.3) and (A3), the last five terms on the right-hand side of (3.4) can be estimated as follows, for any $\delta > 0$ and $\gamma_0, \gamma_1 > 0$, taking $a_1 = \|\nabla u\|$, $b = |M\phi'(t) + N\phi(t)|\|\nabla \omega\|$, $q = q' = 2$ and $\beta = \frac{\delta N\theta^2}{2}$, we deduce that

$$|M\phi'(t) + N\phi(t)|(\nabla u, \nabla \omega) \leq \frac{\delta N\theta^2}{2} \|\nabla u\|^2 + \frac{|M\phi'(t) + N\phi(t)|^2}{2\delta N\theta^2} \|\nabla \omega\|^2, \quad (3.5)$$

and taking

$$a_1 = \|u\|_{p+2}^{p+1}, \quad b = |M\phi'(t) + N\phi(t)|\|\omega\|_{p+2}, \quad q = \frac{p+2}{p+1}, \quad q' = p+2,$$

with $\beta = \frac{M\gamma_1}{4}$, we deduce that

$$|M\phi'(t) + N\phi(t)|(|u|^p u, \omega) \leq \frac{M\gamma_1}{4} \|u\|_{p+2}^{p+2} + \frac{|M\phi'(t) + N\phi(t)|^{p+2}}{(p+2) \left[\frac{M\gamma_1(p+2)}{4p+4} \right]^{p+1}} \|\omega\|_{p+2}^{p+2}. \quad (3.6)$$

Similarly, we can derive

$$\begin{aligned} &|M\phi'(t) + N\phi(t)|(|\nabla u|^m \nabla u, \nabla \omega) \\ &\leq \frac{M\gamma_0}{4} \|\nabla u\|_{m+2}^{m+2} + \frac{|M\phi'(t) + N\phi(t)|^{m+2}}{(m+2) \left[\frac{M\gamma_0(m+2)}{4m+4} \right]^{m+1}} \|\nabla \omega\|_{m+2}^{m+2}, \end{aligned} \quad (3.7)$$

$$\begin{aligned} &|M\phi'(t) + N\phi(t)|(\widehat{h}(t, u), \omega) \\ &\leq \frac{M\gamma_0}{4} \|\nabla u\|_{m+2}^{m+2} + \frac{M\gamma_1}{4} \|u\|_{p+2}^{p+2} + \frac{L^2 |M\phi'(t) + N\phi(t)|^2}{M} \left(\frac{1}{\gamma_0} + \frac{1}{\gamma_1} \right) \|\omega\|^2, \end{aligned} \quad (3.8)$$

where the additional condition (A3) has been used in (3.8).

We now estimate the integral term on the right-hand side of (3.4) as follows:

$$\begin{aligned}
 & |M\phi'(t) + N\phi(t)| \int_0^t g_1(t-\tau)(\nabla u(\tau), \nabla \omega) d\tau \\
 & \leq \frac{|M\phi'(t) + N\phi(t)|^2}{2} \|\nabla \omega\|^2 + \frac{1}{2} \int_{\Omega} \left(\int_0^t g_1(t-\tau) |\nabla u(\tau)| d\tau \right)^2 dx \\
 & \leq \frac{M|\phi'(t) + N\phi(t)|^2}{2} \|\nabla \omega\|^2 + \frac{1}{2} \int_{\Omega} \left(\int_0^t g_1(t-\tau) (|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) d\tau \right)^2 dx.
 \end{aligned}$$

Using Schwarz and Young's inequality, and $\int_0^t g_1(s) ds \leq \int_0^\infty g_1(s) ds = 1 - l$, we obtain

$$\begin{aligned}
 & \int_{\Omega} \left(\int_0^t g_1(t-\tau) (|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) d\tau \right)^2 dx \\
 & \leq \int_{\Omega} \left(\int_0^t g_1(t-\tau) |\nabla u(\tau) - \nabla u(t)| d\tau \right)^2 dx + \int_{\Omega} \left(\int_0^t g_1(t-\tau) |\nabla u(t)| d\tau \right)^2 dx \\
 & \quad + 2 \int_{\Omega} \left(\int_0^t g_1(t-\tau) |\nabla u(\tau) - \nabla u(t)| d\tau \right) \left(\int_0^t g_1(t-\tau) |\nabla u(t)| d\tau \right) dx \\
 & \leq \frac{1}{1-l} \int_{\Omega} \left(\int_0^t g_1(t-\tau) |\nabla u(t)| d\tau \right)^2 dx + \frac{1}{l} \int_{\Omega} \left(\int_0^t g_1(t-\tau) |\nabla u(\tau) - \nabla u(t)| d\tau \right)^2 dx \\
 & \leq \frac{1}{l} \int_{\Omega} \int_0^t g(s) ds \int_0^t g_1(t-\tau) |\nabla u(\tau) - \nabla u(t)|^2 d\tau dx + \frac{1}{1-l} \int_{\Omega} |\nabla u(t)|^2 \left(\int_0^t g_1(s) ds \right)^2 dx \\
 & \leq (1-l) \|\nabla u\|^2 + \frac{1-l}{l} (g_1 * \nabla u)(t). \tag{3.9}
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 |M\phi'(t) + N\phi(t)| \int_0^t g_1(t-\tau)(\nabla u(s), \nabla \omega) d\tau & \leq \frac{|M\phi'(t) + N\phi(t)|^2}{2} \|\nabla \omega\|^2 \\
 & \quad + \frac{1-l}{2} \|\nabla u\|^2 + \frac{1-l}{2l} (g_1 * \nabla u)(t). \tag{3.10}
 \end{aligned}$$

Applying estimations (3.5)–(3.8) and (3.10) in (3.4) yields the desired result. $\square$

Proof of Theorem 2.2 Inspired by the idea in [24], we define

$$F(t) = ME(t) + N(\psi_1(t) + \xi\psi_2(t)), \tag{3.11}$$

where

$$\psi_1(t) = \int_{\Omega} uu_t dx + \frac{1}{2} \|u\|^2, \quad \psi_2(t) = - \int_0^t \int_{\Omega} g_1(s-\tau) \|\nabla u(s) - \nabla u(\tau)\|^2 d\tau ds \tag{3.12}$$

for some $\xi > 0$ such that $\int_0^\infty g_1(s) ds \leq \frac{2\xi}{1+2\xi}$.

We differentiate (3.12) and use (3.1) to obtain

$$\begin{aligned}
 F'(t) & \leq (N - M) \|u_t\|^2 - N\xi (g_1 * \nabla u)(t) \\
 & \quad + N \int_{\Omega} uu_t dx + N(u_{tt}, u) + M(\widehat{h}(t, u), u_t) + M\phi'(t)f(t). \tag{3.13}
 \end{aligned}$$

It follows from (2.7) and (2.8) that

$$(u_{tt}, u) = -\|\nabla u\|^2 - \|\nabla u\|_{m+2}^{m+2} + \alpha \|u\|_{\Gamma_1}^2 - \|u\|_{p+2}^{p+2} - \int_{\Omega} u_t u dx$$

$$+ \int_0^t g_1(t-\tau)(\nabla u, \nabla u(\tau))d\tau + (\widehat{h}(t, u), u) + \phi(t)f(t). \quad (3.14)$$

Combining (3.13), (3.14) and (3.11) yields

$$\begin{aligned}
& F'(t) + \delta F(t) \\
& \leq \delta M E(t) + \delta N \int_{\Omega} u u_t dx + \frac{\delta N}{2} \|u\|^2 - \xi \delta N \int_0^t \int_0^s g_1(s - \tau) \|\nabla u(s) - \nabla u(\tau)\|^2 d\tau ds \\
& \quad - (M - N) \|u_t\|^2 - N \xi (g_1 * \nabla u)(t) + M(\widehat{h}(t, u), u_t) - N \|\nabla u\|^2 - N \|\nabla u\|_{m+2}^{m+2} \\
& \quad + N \alpha \|u\|_{\Gamma_1}^2 - N \|u\|_{p+2}^{p+2} + N \int_0^t g_1(t - \tau) (\nabla u, \nabla u(\tau)) d\tau \\
& \quad + N(\widehat{h}(t, u), u) + |M \phi'(t) + N \phi(t)| f(t).
\end{aligned} \tag{3.15}$$

Applying (2.11) into (3.15), we get

$$\begin{aligned}
F'(t) + \delta F(t) &\leq -\left(M - \frac{\delta M}{2} - N\right) \|u_t\|^2 - \left(N - \frac{\delta M}{2}\right) \|\nabla u\|^2 - \left(N\xi - \frac{\delta M}{2}\right) (g_1 * \nabla u)(t) \\
&\quad + M(\widehat{h}(t, u), u_t) - \left(N - \frac{\delta M}{m+2}\right) \|\nabla u\|_{m+2}^{m+2} + \left(\frac{\delta M}{p+2} - N\right) \|u\|_{p+2}^{p+2} \\
&\quad + \left(\alpha N - \frac{\alpha \delta M}{2}\right) \|u\|_{\Gamma_1}^2 + N(\widehat{h}(t, u), u) + N \int_0^t g_1(t-\tau)(\nabla u, \nabla u(\tau)) d\tau \\
&\quad + \delta N \int_{\Omega} u u_t dx + |M\phi'(t) + N\phi(t)|f(t),
\end{aligned} \tag{3.16}$$

where $1 - \int_0^t g_1(s)ds \leq 1$ has been used.

Now, by using the Young inequality (2.3) and (A3), the terms on the right-hand side of (3.16) can be estimated as follows:

$$\delta N \int_{\Omega} uu_t dx \leq \frac{\delta N}{2} \|u_t\|^2 + \frac{\delta N \theta^2}{2} \|\nabla u\|^2, \quad (3.17)$$

$$M|(\widehat{h}(t, u), u_t)| \leq \frac{M\gamma_0}{2}\|\nabla u\|_{m+2}^{m+2} + \frac{M\gamma_1}{2}\|u\|_{p+2}^{p+2} + \frac{L^2}{2M}\left(\frac{1}{\gamma_0} + \frac{1}{\gamma_1}\right)\|u_t\|^2, \quad (3.18)$$

$$N|\widehat{h}(t, u), u| \leq \frac{N\gamma_0}{2} \|\nabla u\|_{m+2}^{m+2} + \frac{N\gamma_1}{2} \|u\|_{p+2}^{p+2} + \frac{L^2\theta^2}{2N} \left(\frac{1}{\gamma_0} + \frac{1}{\gamma_1} \right) \|\nabla u\|^2, \quad (3.19)$$

$$N \int_0^t g_1(t-\tau) |(\nabla u, \nabla u(\tau))| d\tau \leq N \left(1 - \frac{l}{2}\right) \|\nabla u\|^2 + N \left(\frac{1-l}{2l}\right) (g_1 * \nabla u)(t). \quad (3.20)$$

Utilizing estimations (3.17)–(3.20) and Lemma 3.2 into (3.16), we get

$$\begin{aligned}
& F'(t) + \delta F(t) \\
& \leq - \left[N - \frac{\delta M}{2} - \frac{3\delta N\theta^2}{2} - N \left(1 - \frac{l}{2} \right) - \frac{1-l}{2} - \frac{L^2\theta^2}{2N} \left(\frac{1}{\gamma_0} + \frac{1}{\gamma_1} \right) \right] \|\nabla u\|^2 \\
& \quad + \alpha \left(N - \frac{\delta M}{2} \right) \|u\|_{\Gamma_1}^2 - \left[M - \frac{\delta(M+N)}{2} - N - \frac{L^2}{2M} \left(\frac{1}{\gamma_0} + \frac{1}{\gamma_1} \right) \right] \|u_t\|^2 \\
& \quad - \left(N\xi - \frac{\delta M}{2} - (N+1) \left(\frac{1-l}{2l} \right) \right) (g_1 * \nabla u)(t) - \left(N - \frac{\delta M}{m+2} - M\gamma_0 - \frac{N\gamma_0}{2} \right) \|\nabla u\|_{m+2}^{m+2} \\
& \quad - \left(N - \frac{\delta M}{p+2} - M\gamma_1 - \frac{N\gamma_1}{2} \right) \|u\|_{p+2}^{p+2} + H(t), \tag{3.21}
\end{aligned}$$

where $H(t)$ satisfies (3.3).

By virtue of the trace embedding inequality (2.1) and for $\delta < \frac{2N}{M}$, we deduce

$$\begin{aligned}
 & F'(t) + \delta F(t) \\
 & \leq -\frac{1}{2M} \left[2 \left(1 - \frac{\delta}{2} \right) M^2 - 2 \left(N + \frac{\delta N}{2} \right) M - L^2 \left(\frac{1}{\gamma_0} + \frac{1}{\gamma_1} \right) \right] \|u_t\|^2 - \left(N\xi - \frac{\delta M}{2} \right. \\
 & \quad - (N+1) \left(\frac{1-l}{2l} \right) \left. \right) (g_1 * \nabla u)(t) - \frac{1}{2N} \left[(l - 2\alpha B_2^2 - 3\delta\theta^2) N^2 + (\alpha\delta M b_2^2 + l - 1 - \delta M) N \right. \\
 & \quad - L^2\theta^2 \left(\frac{1}{\gamma_0} + \frac{1}{\gamma_1} \right) \left. \right] \|\nabla u\|^2 - \left(N - \frac{\delta M}{m+2} - M\gamma_0 - \frac{N\gamma_0}{2} \right) \|\nabla u\|_{m+2}^{m+2} \\
 & \quad - \left(N - \frac{\delta M}{p+2} - M\gamma_1 - \frac{N\gamma_1}{2} \right) \|u\|_{p+2}^{p+2} + H(t).
 \end{aligned} \tag{3.22}$$

At this point if we choose $\gamma_0 = \gamma_1 = 1$ and

$$\begin{aligned}
 & \delta < \min \left\{ 2, \frac{l}{3\theta^2}, \frac{2N}{M} \right\}, \quad \alpha < \frac{l - 3\delta\theta^2}{2B_2^2}, \quad \int_0^\infty g_1(s)ds \leq \frac{2\xi}{1+2\xi}, \\
 & N > \max \left\{ \frac{2M(\delta + p + 2)}{p+2}, \frac{2M(\delta + m + 2)}{m+2}, \frac{\delta M l - l + 1}{2\xi l + l - 1}, N_0 \right\}, \quad M > M_0,
 \end{aligned}$$

such that N_0, M_0 are the maximum root of the following equations

$$\begin{aligned}
 & (l - 2\alpha B_2^2 - 3\delta\theta^2) N^2 + (\alpha\delta M b_2^2 + l - 1 - \delta M) N - 2L^2\theta^2 = 0, \\
 & 2 \left(1 - \frac{\delta}{2} \right) M^2 - 2 \left(N + \frac{\delta N}{2} \right) M - 2L^2 = 0,
 \end{aligned}$$

respectively, then we easily derive

$$F'(t) + \delta F(t) \leq H(t). \tag{3.23}$$

Thanks to the assumptions on $\phi(t)$, $\phi'(t)$ and $\phi''(t)$. Indeed, $\phi(t)$, $\phi'(t)$ tends to zero as t tends to infinity and $\phi''(t)$ is a bounded function, so we get

$$H(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Thus for some positive constant C , the result (2.12) follows from

$$E(t) \leq CF(t),$$

and the proof of Theorem 2.2 is complete.

4 Blow Up

In this section, we are going to prove that for sufficiently large initial data some of the solutions blow up in a finite time. To prove the blow-up result (Theorem 2.3) for certain solutions with positive initial energy, we need the following lemma for the problem (2.13)–(2.15).

Lemma 4.1 *Let the conditions (A1)–(A3) and (B1), (B2) be satisfied and*

$$\alpha \leq \frac{l(p+m+4)-4}{2B_2^2(p+m)}, \quad \int_0^\infty g(s)ds \leq \frac{p+m}{p+m+4}.$$

Then

$$E_\lambda(t) \geq E_\lambda(0) - \frac{2D_1}{\lambda(p+m+4)}$$

for $\lambda \geq \max\{\lambda_0, \frac{2(1-l)}{p+m}\}$ such that λ_0 is the maximum root of the following equation:

$$(p-m)(p+m+8)\lambda^2 + 4(p-m)\lambda - 4L^2(p+m+4) = 0.$$

Proof A multiplication of (2.13) by v_t and integrating over Ω gives

$$\begin{aligned} \frac{d}{dt}E_\lambda(t) &\geq (2\lambda+1)\|v_t\|^2 + \frac{\lambda p}{p+2}e^{\lambda p t}\|v\|_{p+2}^{p+2} - \frac{\lambda m}{m+2}e^{\lambda m t}\|v\|_{m+2}^{m+2} - e^{-\lambda t}(\widehat{h}(t, e^{\lambda t}v), v_t) \\ &\quad + k\lambda e^{-2\lambda t}f(t), \end{aligned} \quad (4.1)$$

where the condition (B1) has been used.

Plugging definition of $f(t)$, (2.17), into (4.1), we obtain

$$\begin{aligned} \frac{d}{dt}E_\lambda(t) &\geq (2\lambda+1)\|v_t\|^2 + \frac{\lambda p}{p+2}e^{\lambda p t}\|v\|_{p+2}^{p+2} - \frac{\lambda m}{m+2}e^{\lambda m t}\|v\|_{m+2}^{m+2} - e^{-\lambda t}(\widehat{h}(t, e^{\lambda t}v), v_t) \\ &\quad + k\lambda e^{-\lambda t}(\nabla v, \nabla \omega) + k\lambda e^{\lambda(m-1)t}(|\nabla v|^m \nabla v, \nabla \omega) - k\lambda e^{\lambda(p-1)t}(|v|^p v, \omega) \\ &\quad - k\lambda e^{-2\lambda t}(\widehat{h}(t, e^{\lambda t}v), \omega) - k\lambda e^{-\lambda t} \int_{\Omega} \nabla \omega(x) \int_0^t g(t-\tau) \nabla v(\tau) d\tau dx. \end{aligned} \quad (4.2)$$

It is easy to verify that

$$e^{-\lambda t}|(\widehat{h}(t, v), v_t)| \leq \frac{\varepsilon_0}{2}e^{\lambda m t}\|\nabla v\|_{m+2}^{m+2} + \frac{\varepsilon_1}{2}e^{\lambda p t}\|v\|_{p+2}^{p+2} + \frac{L^2}{2}\left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_1}\right)\|v_t\|^2, \quad (4.3)$$

where (A3) and Young's inequality (2.1) have been used.

Combining (4.3) with (4.2), we obtain

$$\begin{aligned} \frac{d}{dt}E_\lambda(t) &\geq \left(2\lambda - \frac{L^2}{2}\left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_1}\right) + 1\right)\|v_t\|^2 + \left(\frac{\lambda p}{p+2} - \frac{\varepsilon_1}{2}\right)e^{\lambda p t}\|v\|_{p+2}^{p+2} - \left(\frac{\lambda m}{m+2} + \frac{\varepsilon_0}{2}\right)e^{\lambda m t} \\ &\quad \cdot \|\nabla v\|_{m+2}^{m+2} + k\lambda e^{-\lambda t}(\nabla v, \nabla \omega) + k\lambda e^{\lambda(m-1)t}(|\nabla v|^m \nabla v, \nabla \omega) - k\lambda e^{\lambda(p-1)t}(|v|^p v, \omega) \\ &\quad - k\lambda e^{-2\lambda t}(\widehat{h}(t, e^{\lambda t}v), \omega) - k\lambda e^{-\lambda t} \int_{\Omega} \nabla \omega(x) \int_0^t g(t-\tau) \nabla v(\tau) d\tau dx. \end{aligned} \quad (4.4)$$

Employing the last inequality, we obtain from (2.18) the following inequality:

$$\begin{aligned} &\frac{d}{dt}E_\lambda(t) - (\lambda p - \varepsilon_1(p+2))E_\lambda(t) \\ &\geq \left(\frac{\lambda p}{p+2} - \frac{\lambda p - \varepsilon_1(p+2)}{p+2} - \frac{\varepsilon_1}{2}\right)e^{\lambda p t}\|v\|_{p+2}^{p+2} - \frac{\alpha}{2}(\lambda p - \varepsilon_1(p+2))\|v\|_{\Gamma_1}^2 \\ &\quad + \left(\frac{\lambda p - \varepsilon_1(p+2)}{m+2} - \frac{\lambda m}{m+2} - \frac{\varepsilon_0}{2}\right)e^{\lambda m t}\|\nabla v\|_{m+2}^{m+2} + \left(2\lambda + \frac{\lambda p - \varepsilon_1(p+2)}{2}\right) \\ &\quad - \frac{L^2}{2}\left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_1}\right) + 1\|v_t\|^2 + \frac{\lambda p - \varepsilon_1(p+2)}{2}(g * \nabla v)(t) + \frac{\lambda(\lambda+1)(\lambda p - \varepsilon_1(p+2))}{2}\|v\|^2 \\ &\quad + \frac{\lambda p - \varepsilon_1(p+2)}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla v\|^2 + k\lambda e^{-\lambda t}(\nabla v, \nabla \omega) + k\lambda e^{\lambda(m-1)t}(|\nabla v|^m \nabla v, \nabla \omega) \\ &\quad - k\lambda e^{\lambda(p-1)t}(|v|^p v, \omega) - k\lambda e^{-2\lambda t}(\widehat{h}(t, e^{\lambda t}v), \omega) \\ &\quad - k\lambda e^{-\lambda t} \int_{\Omega} \nabla \omega(x) \int_0^t g(t-\tau) \nabla v(\tau) d\tau dx. \end{aligned} \quad (4.5)$$

Now, the five terms on the right-hand side of (4.5) can be estimated as follows:

$$k\lambda e^{-\lambda t}|(\nabla v, \nabla \omega)|$$

$$\leq \frac{l(\lambda p - \varepsilon_1(p+2))}{4} \|\nabla v\|^2 + \frac{\lambda^2(M_1 + M_2)}{4} \|v\|^2 + \frac{k^2 \lambda^2 e^{-2\lambda t}}{l(\lambda p - \varepsilon_1(p+2))} \|\nabla \omega\|, \quad (4.6)$$

$$k\lambda e^{\lambda(m-1)t} |(|\nabla v|^m \nabla v, \nabla \omega)| \leq \frac{\varepsilon_0}{4} e^{\lambda m t} \|\nabla v\|_{m+2}^{m+2} + \frac{k^{m+2} \lambda^{m+2} e^{-2\lambda t}}{(m+2) \left[\frac{\varepsilon_0(m+2)}{4m+4} \right]^{m+1}} \|\nabla \omega\|_{m+2}^{m+2}, \quad (4.7)$$

$$k\lambda e^{-2\lambda t} |(\widehat{h}(t, e^{\lambda t} v), \omega)| \leq \frac{\varepsilon_0}{4} e^{\lambda m t} \|\nabla v\|_{m+2}^{m+2} + \frac{\varepsilon_1}{4} e^{\lambda p t} \|v\|_{p+2}^{p+2} + L^2 \left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_1} \right) e^{-2\lambda t} \|\omega\|^2, \quad (4.8)$$

$$\begin{aligned} & k\lambda e^{-\lambda t} \int_{\Omega} \omega(x) \int_0^t g(t-\tau) \Delta v(\tau) d\tau dx \\ & \leq \frac{K^2 \lambda^2 e^{-2\lambda t}}{2} \|\nabla \omega\|^2 + \frac{1-l}{2} \|\nabla v\|^2 + \frac{1-l}{2l} (g * \nabla v)(t), \end{aligned} \quad (4.9)$$

$$k\lambda e^{\lambda(p-1)t} |(|v|^p v, \omega)| \leq \frac{\varepsilon_1}{4} e^{\lambda p t} \|v\|_{p+2}^{p+2} + \frac{k^{p+2} \lambda^{p+2} e^{-2\lambda t}}{(p+2) \left[\frac{\varepsilon_1(p+2)}{4p+4} \right]^{p+1}} \|\omega\|_{p+2}^{p+2}. \quad (4.10)$$

Taking into account estimates (4.6)–(4.10) and trace embedding inequality (2.1) in relation with (4.5), we get

$$\begin{aligned} & \frac{d}{dt} E_{\lambda}(t) - (\lambda p - \varepsilon_1(p+2)) E_{\lambda}(t) \\ & \geq + \left(2\lambda + \frac{\lambda p - \varepsilon_1(p+2)}{2} - \frac{L^2}{2} \left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_1} \right) + 1 \right) \|v_t\|^2 \\ & \quad + \left(\frac{\lambda(p-m) - \varepsilon_1(p+2)}{m+2} - \varepsilon_0 \right) e^{\lambda m t} \|\nabla v\|_{m+2}^{m+2} \\ & \quad + \left(\frac{l}{4} (\lambda p - \varepsilon_1(p+2)) - \frac{\alpha B_2^2}{2} (\lambda p - \varepsilon_1(p+2)) - \frac{1-l}{2} \right) \|\nabla v\|^2 \\ & \quad + \frac{\lambda(\lambda+1)(\lambda p - \varepsilon_1(p+2))}{2} \|v\|^2 \\ & \quad + \left(\frac{\lambda p - \varepsilon_1(p+2)}{2} - \frac{1-l}{2l} \right) (g * \nabla v)(t) - e^{-2\lambda t} D_1, \end{aligned} \quad (4.11)$$

where D_1 satisfies (2.21).

At this point if we choose $\varepsilon_0 = \frac{\lambda(p-m)}{2m+4}$ and $\varepsilon_1 = \frac{\lambda(p-m)}{2p+4}$, then we obtain

$$\begin{aligned} \frac{d}{dt} E_{\lambda}(t) - \frac{\lambda(p+m)}{2} E_{\lambda}(t) & \geq ((p-m)(p+m+8)\lambda^2 + 4(p-m)\lambda - 4L^2(p+m+4)) \|v_t\|^2 \\ & \quad + \left(\frac{p+m}{8} (l - 2\alpha B^2) - \frac{1-l}{2} \right) \|\nabla v\|^2 \\ & \quad + \left(\frac{\lambda}{4} (p+m) - \frac{1-l}{2l} \right) (g * \nabla v)(t) - e^{-2\lambda t} D_1. \end{aligned} \quad (4.12)$$

At this point if we choose

$$\lambda \geq \max \left\{ \lambda_0, \frac{2(1-l)}{p+m} \right\}, \quad \alpha \leq \frac{l(p+m+4) - 4}{2B_2^2(p+m)}, \quad \int_0^{\infty} g(s) ds \leq \frac{p+m}{p+m+4},$$

where λ_0 is the maximum root of the following equation:

$$(p-m)(p+m+8)\lambda^2 + 4(p-m)\lambda - 4L^2(p+m+4) = 0.$$

Then we deduce

$$\frac{d}{dt} E_{\lambda}(t) - \frac{\lambda(p+m)}{2} E_{\lambda}(t) \geq -e^{-2\lambda t} D_1. \quad (4.13)$$

Finally, we end up by integrating (4.13) between 0 and t :

$$E_\lambda(t) \geq E_\lambda(0) - \frac{2D_1}{\lambda(p+m+4)}, \quad \forall t \geq 0, \quad (4.14)$$

and the proof of Lemma 4.1 is complete. $\square$

Proof of Theorem 2.3 To obtain the blow-up result, the choice of the following functional is standard

$$\chi(t) = \|v(t)\|^2. \quad (4.15)$$

Then

$$\chi'(t) = 2(v, v_t), \quad (4.16)$$

$$\chi''(t) = 2(v, v_{tt}) + 2\|v_t\|^2. \quad (4.17)$$

A multiplication of (2.13) by v and integrating over Ω gives

$$\begin{aligned} (v_{tt}, v) &= -2(\lambda+1)(v_t, v) - \lambda(\lambda+1)\|v\|^2 - \|\nabla v\|^2 + \alpha\|v\|_{\Gamma_1}^2 - e^{\lambda mt}\|\nabla v\|_{m+2}^{m+2} \\ &\quad + \int_0^t g(t-\tau)(\nabla v, \nabla v(\tau))d\tau + e^{\lambda pt}\|v\|_{p+2}^{p+2} + e^{-\lambda t}(\widehat{h}(t, e^{\lambda t}v), v) + ke^{-2\lambda t}f(t). \end{aligned} \quad (4.18)$$

Taking into account definition of unknown function (2.17), we obtain

$$\begin{aligned} (v_{tt}, v) &= -2(\lambda+1)(v_t, v) - \lambda(\lambda+1)\|v\|^2 - \|\nabla v\|^2 + \alpha\|v\|_{\Gamma_1}^2 - e^{\lambda mt}\|\nabla v\|_{m+2}^{m+2} \\ &\quad + \int_0^t g(t-\tau)(\nabla v, \nabla v(\tau))d\tau + e^{\lambda pt}\|v\|_{p+2}^{p+2} + e^{-\lambda t}(\widehat{h}(t, e^{\lambda t}v), v) + ke^{-2\lambda t} \\ &\quad + ke^{-\lambda t}(\nabla v, \nabla \omega) - ke^{-2\lambda t}(\widehat{h}(t, e^{\lambda t}v), \omega) + ke^{\lambda(m-1)t}(\nabla v|^m \nabla v, \nabla \omega) \\ &\quad - ke^{-\lambda t} \int_0^t g(t-\tau)(\nabla v(\tau), \nabla \omega)d\tau - ke^{\lambda(p-1)t}(|v|^p v, \omega). \end{aligned} \quad (4.19)$$

Again by exploiting the condition (A3), Young's inequality (2.1), we can obtain the following estimations for the terms on the right-hand side of (4.19):

$$\int_0^t g(t-\tau)|(\nabla v(\tau), \nabla v)|d\tau \leq \left(1 - \frac{l}{2}\right)\|\nabla v\|^2 + \frac{1-l}{2l}(g * \nabla v)(t), \quad (4.20)$$

$$e^{-\lambda t}|(\widehat{h}(t, v), v)| \leq \frac{\mu_0}{4}e^{\lambda mt}\|\nabla v\|_{m+2}^{m+2} + \frac{\mu_1}{4}e^{\lambda pt}\|v\|_{p+2}^{p+2} + L^2\left(\frac{1}{\mu_0} + \frac{1}{\mu_1}\right)\|v\|^2, \quad (4.21)$$

$$ke^{-\lambda t}|(\nabla v, \nabla \omega)| \leq \frac{(p+m)l}{8}\|\nabla v\|^2 + \frac{k^2e^{-2\lambda t}}{4l(p+m)}\|\nabla \omega\|^2, \quad (4.22)$$

$$ke^{\lambda(m-1)t}|(|\nabla v|^m \nabla v, \nabla \omega)| \leq \frac{\mu_0}{2}e^{\lambda mt}\|\nabla v\|_{m+2}^{m+2} + \frac{k^{m+2}e^{-2\lambda t}}{(m+2)[\frac{\mu_0(m+2)}{2m+2}]^{m+1}}\|\nabla \omega\|_{m+2}^{m+2}, \quad (4.23)$$

$$ke^{-2\lambda t}|(\widehat{h}(t, e^{\lambda t}v), \omega)| \leq \frac{\mu_0}{4}e^{\lambda mt}\|\nabla v\|_{m+2}^{m+2} + \frac{\mu_1}{4}e^{\lambda pt}\|v\|_{p+2}^{p+2} + L^2\left(\frac{1}{\mu_0} + \frac{1}{\mu_1}\right)\|\omega\|^2, \quad (4.24)$$

$$\begin{aligned} &ke^{-\lambda t} \int_0^t g(t-\tau)|(\nabla v(\tau), \nabla \omega)|d\tau \\ &\leq \left(\frac{1-l}{2}\right)\|\nabla v\|^2 + \frac{1-l}{2l}(g * \nabla v)(t) + \frac{k^2e^{-2\lambda t}}{2}\|\nabla \omega\|^2, \end{aligned} \quad (4.25)$$

$$ke^{\lambda(p-1)t}(|v|^p v, \omega) \leq \frac{\mu_1}{2}e^{\lambda pt}\|v\|_{p+2}^{p+2} + \frac{k^{p+2}e^{-2\lambda t}}{(p+2)[\frac{\mu_1(p+2)}{2p+2}]^{p+1}}\|\omega\|_{p+2}^{p+2}. \quad (4.26)$$

Combining estimations (4.20)–(4.26) with (4.19), we obtain

$$\begin{aligned} (v_{tt}, v) \geq & -(2\lambda + 1)(v_t, v) - \left(\lambda^2 + \lambda + L^2 \left(\frac{1}{\mu_0} + \frac{1}{\mu_1} \right) \right) \|v\|^2 - \left(\frac{(p+m)l}{8} + \frac{5}{2} - l \right) \|\nabla v\|^2 \\ & + \alpha \|v\|_{\Gamma_1}^2 - (\mu_0 + 1)e^{\lambda mt} \|\nabla v\|_{m+2}^{m+2} - \frac{1-l}{l} (g * \nabla v)(t) \\ & - (\mu_1 - 1)e^{\lambda pt} \|v\|_{p+2}^{p+2} - D_2, \end{aligned} \quad (4.27)$$

where D_2 satisfies (2.22).

Applying (2.18) into (4.27), yields

$$\begin{aligned} (v_{tt}, v) \geq & \frac{p+m+4}{2} E_\lambda(t) - (2\lambda + 1)(v_t, v) - \left(\lambda^2 + \lambda + L^2 \left(\frac{1}{\mu_0} + \frac{1}{\mu_1} \right) \right. \\ & \left. - \frac{\lambda(\lambda+1)(p+m+4)}{4} \right) \|v\|^2 - \left(\frac{(p+m)l}{8} + \frac{5}{2} - l \right) \|\nabla v\|^2 - \frac{\alpha(p+m)}{4} \|v\|_{\Gamma_1}^2 \\ & - \left(\mu_0 + 1 - \frac{p+m+4}{2m+4} \right) e^{\lambda mt} \|\nabla v\|_{m+2}^{m+2} + \left(\frac{p+m+4}{2} - \frac{1-l}{l} \right) (g * \nabla v)(t) \\ & - \left(\mu_1 + \frac{p+m+4}{2p+4} - 1 \right) e^{\lambda pt} \|v\|_{p+2}^{p+2} + \frac{p+m+4}{2} \|v_t\|^2 - D_2, \end{aligned} \quad (4.28)$$

and by using trace embedding inequality (2.1), we obtain

$$\begin{aligned} (v_{tt}, v) \geq & \frac{p+m+4}{2} E_\lambda(t) - (2\lambda + 1)(v_t, v) - \left(\lambda^2 + \lambda + L^2 \left(\frac{1}{\mu_0} + \frac{1}{\mu_1} \right) \right. \\ & \left. - \frac{\lambda(\lambda+1)(p+m+4)}{4} \right) \|v\|^2 - \left(\frac{\alpha B_2^2(p+m)}{4} - \frac{(p+m)l}{8} + \frac{5}{2} - 2l \right) \|\nabla v\|^2 \\ & - \left(\mu_0 + 1 - \frac{p+m+4}{2m+4} \right) e^{\lambda mt} \|\nabla v\|_{m+2}^{m+2} + \left(\frac{p+m+4}{2} - \frac{1-l}{l} \right) (g * \nabla v)(t) \\ & - \left(\mu_1 + \frac{p+m+4}{2p+4} - 1 \right) e^{\lambda pt} \|v\|_{p+2}^{p+2} + \frac{p+m+4}{2} \|v_t\|^2 - D_2, \end{aligned} \quad (4.29)$$

where the inequality $\frac{p+m+4}{4}(1 - \int_0^t g(s)ds) > l$ has been used.

At this point, we choose $\mu_0 = \frac{p-m}{2m+4}$ and $\mu_1 = \frac{p-m}{2p+4}$ and $p > m \geq 2$. Also suppose that

$$\lambda \geq \lambda_1, \quad \alpha \leq \frac{p+m-4}{2B_2(p+m)}, \quad \int_0^{+\infty} g(s)ds \leq \min \left\{ \frac{p+m+4}{p+m+6}, \frac{p+m-4-2\alpha B_2^2(p+m)}{p+m+16} \right\},$$

where λ_1 is the maximum root of the following equation

$$\left(\frac{p+m-4}{4} \right) \lambda^2 + \frac{p+m-4}{4} \lambda - \frac{2L^2}{p-m} (p+m+4) = 0.$$

Therefore, we deduce the following inequality:

$$(v_{tt}, v) \geq \frac{p+m+4}{2} E_\lambda(t) + \frac{p+m+4}{2} \|v_t\|^2 - (2\lambda + 1)(v_t, v) - D_2. \quad (4.30)$$

Then by using (2.20) and Lemma 4.1, we obtain, from (4.30),

$$(v_{tt}, v) \geq \frac{p+m+4}{2} \|v_t\|^2 - (2\lambda + 1)(v_t, v). \quad (4.31)$$

To this end, by substituting (4.15)–(4.17) in (4.31), we arrive at

$$\chi''(t) \geq 4 \left(1 + \frac{p+m}{8} \right) \|v_t\|^2 - (2\lambda + 1)\chi'(t),$$

thus

$$\chi''(t)\chi(t) \geq (1 + \frac{p+m}{8})[\chi'(t)]^2 - (2\lambda + 1)\chi'(t)\chi(t), \quad (4.32)$$

where

$$[\chi'(t)]^2 \leq 4\|v_t\|^2\|v\|^2$$

has been used.

Hence we see that the hypotheses of Lemma 2.1 are fulfilled with

$$\kappa = \frac{p+m}{8}, \quad c_1 = \frac{1+2\lambda}{2}, \quad c_2 = 0,$$

and the conclusion of Lemma 2.1 gives us that some solutions of problem (2.13)–(2.16) blow up in a finite time and since this system is equivalent to (1.1)–(1.4), the proof is complete.

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