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GENERAL DECAY AND BLOW-UP RESULTS FOR NONLINEAR FOURTH-ORDER INTEGRO-DIFFERENTIAL EQUATION

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This study aims at considering an initial-boundary value problem for nonlinear fourth-order viscoelastic equation in a bounded domain. Under suitable conditions of the initial data and of the relaxation function, it is proved that the solution energy is generally decayed. It is also shown that regarding arbitrary positive initial energy, certain solutions blow-up in a finite time.

Key words : General decay; blow up; viscoelastic; fourth-order.

1. Introduction

The current study investigates the solution behaviour of the nonlinear fourth-order equation

$$u_{tt} + \Delta[(a + b|\Delta u|^{m-2})\Delta u] - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau + cu_t = h(x,t,u,\Delta u)$$
$$+|u|^{p-2}u, \;\; x \in \Omega, \;\; t \geq 0, \tag{1.1}$$

supplemented by the initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega \tag{1.2}$$

and the following boundary conditions:

$$\begin{cases} u(x,t) = 0, & x \in \Gamma_0, t > 0 \\ a\Delta u(x,t) = \int_0^t g(t-\tau)\Delta u(\tau)d\tau - b|\Delta u|^{m-2}\Delta u, & x \in \Gamma_1, t > 0 \end{cases} \tag{1.3}$$

where Ω is a bounded domain of $R^n (n \geq 1)$ with smooth boundary $\Gamma_0 \cup \Gamma_1 = \partial\Omega$, $\bar{\Gamma}_0 \cup \bar{\Gamma}_1 = \emptyset$, and where Γ_0 and Γ_1 are closed with positive measures. Here a, b and c are nonnegative constants, moreover, p and m are assumed real numbers such that $p, m > 2$. In addition, $h(x, t, u, \Delta u)$ and $g(t)$ are functions satisfying specific conditions (c.f. (A2)-(A4)).

The one-dimension case of the fourth-order wave equation is written as

$$u_{tt} + u_{xxxx} - a(u_x^2)_x = f(x), \;\; x \in \Omega \subset R, \; t > 0, \tag{1.4}$$

which was first introduced in [1] to describe such important physical and biological phenomena as the analysis of elasto-plastic microstructure models for the longitudinal motion of an elasto-plastic bar. Zhao and Liu [21] have recently considered the following equation

$$u_{tt} + u_{xxxx} - \alpha u_{xxt} = f(u_x)_x, \tag{1.5}$$

subject to Dirichlet boundary condition and in a bounded domain $\Omega = (0, 1)$. By adopting and modifying the so-called concavity method, the authors obtained a blow-up result for positive initial energy. In another study, the following problem was examined by Messaoudi [12]:

$$u_{tt} + \Delta^2 u = b|u|^{p-2}u - a|u_t|^{m-2}u_t, \; x \in \Omega, \; t > 0$$

$$u(x, t) = \partial_\nu u(x, t) = 0, \; x \in \partial\Omega, t \geq 0,$$

$$u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x), \; x \in \Omega.$$

He proved the existence of a local weak solution and showed that this solution blows up in finite time with negative initial energy if $p > m$.

This study aims at considering an initial-boundary value problem for nonlinear fourth-order viscoelastic equation in a bounded domain. Under suitable conditions of the initial data and of the relaxation function, it is proved that the solution energy is generally decayed. It is also shown that regarding arbitrary positive initial energy, certain solutions blow-up in a finite time.

Shahrouzi [16] studied the blow up result for the following fourth-order equation:

$$u_{tt} + \Delta[(a_0 + a|\Delta u|^{m-2})\Delta u] - b\Delta u_t = g(x, t, u, \Delta u) + |u|^{p-2}u, \quad x \in \Omega, \; t > 0$$

with dissipative boundary condition

$$u(x, t) = 0, \quad \Delta u(x, t) = -c_0 \partial_\nu u(x, t), \quad x \in \partial\Omega, \; t > 0$$

He showed that when $p > m + 1 > 3$, there are solutions which blow up in finite time with positive initial energy. For the study of nonlinear Petrovsky type equation, we refer the reader to [4].

It is known that viscoelastic materials show natural damping properties, which is due to the special property of these substances in keeping memory of their past history. From mathematical point of view, these damping effects are modeled by integro-differential operators. Such equations are inherently arduous to be analytically solved, and accordingly an efficient approximate solution would be required. These problems have largely been discussed by several authors in the last decades, and have made a lot of progress. Rivera *et al*. [14] studied the following Petrovsky viscoelastic equation:

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds = 0. \tag{1.6}$$

They showed that in a bounded domain $\Omega \subset R^N$, the solution energy decay exponentially provided that the kernel function does too.

Later, Tahamtani and Shahrouzi [17] investigated the equation (1.6) with source term $|u|^p u$, in a bounded domain and proved the existence of weak solutions. Furthermore, they showed that there are solutions under certain conditions of initial data which blow up in finite time with non-positive as well as positive initial energy.

Li and Gao [8] considered the nonlinear Petrovsky type equation

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds + |u_t|^{m-2}u_t = |u|^{p-2}u.$$

The authors obtained the blow-up results with upper bounded initial energy. Moreover, they proved that for the linear damping case and with non-positive initial energy the solution blows up in finite time.

Mustafa [15], investigated the following nonlinear viscoelastic problem

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds = u|u|^\gamma, \text{ in } \Omega \times (0,\infty)$$
$$u = 0, \text{ on } \partial\Omega \times (0,\infty)$$
$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega.$$

He established optimal explicit and general energy decay results with minimal conditions on the relaxation function g namely $g'(t) \le -\xi(t)H(g(t))$, where H is an increasing convex function near the origin and where ξ is a non-increasing function (see also [13]).

For more related results about the existence, finite time blow-up and asymptotic properties, we refer the reader to [2, 3, 9, 10, 18-20].

Motivated by the aforementioned works, our result here is twofold: First, it is assumed that $c = 1,\ h(x,t,u,\Delta u) \equiv 0$ and show that if the initial data and parameters are taken in the appropriate domain, then solutions of (1.1)-(1.3) uniformly decay to zero with same arbitrary rate as the memory kernel. Our approaches are based on the Lyapunov function and perturbed energy method. Second, if we take $a = b = 1$ and $c = 0$ then by using the modified concavity argument, the blow up of solutions for the problem (1.1)-(1.4) under the conditions of positive initial data and appropriate domain for parameters is verified.

2. Preliminaries and Main Results

This section discusses the material needed for the proof of our main results. We shall assume that the function $h(x,t,u,\Delta u)$ as well as the functions appearing in the data satisfy the following conditions

(A1)

$$u_0 \in H_0^1(\Omega) \cap L^p(\Omega) \cap L^m(\Omega), \qquad u_1 \in L^2(\Omega),$$

(A2)

$$|h(x,t,u,\Delta u)| \leq M(|\Delta u|^{\frac{m}{2}} + |u|^{\frac{p}{2}}),$$

with some positive $M > 0$.

Moreover, the following assumption is made for function $g(s)$:

(A3) $g : R^+ \to R^+$ is a non-increasing differentiable function which satisfies

$$g(0) > 0, \quad a - \int_0^{+\infty} g(s)ds = l > 0.$$

Throughout this paper all the functions considered are real-valued. We denote by $\|.\|_q$ the L^q-norm over Ω and $\|.\|_{q,\partial\Omega}$ the L^q-norm over $\partial\Omega$. In particular, the L^2-norm is denoted $\|.\|$ in Ω and $\|.\|_{\partial\Omega}$ in $\partial\Omega$ also $(.,.)$ denotes the usual L^2-inner product. We use familiar function spaces H_0^1.

We use the Young's inequality

$$ab \leq \beta a^q + C(\beta,q)b^{q'}, \ \ a,b \geq 0, \ \ \beta > 0, \ \ \frac{1}{q} + \frac{1}{q'} = 1, \tag{2.1}$$

where $C(\beta,q) = \frac{1}{q'}(\beta q)^{-\frac{q'}{q}}$.

For the sake of simplicity, to prove the asymptotic stability result, we assume that $c = 1$ and $h(x,t,u,\Delta u) \equiv 0$.

In order to formulate our result, the energy of the problem (1.1)-(1.3) is introduced first:

$$E(t) = I(t) - \frac{1}{p}\|u\|_p^p, \tag{2.2}$$

where

$$I(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}(a - \int_0^t g(s)ds)\|\Delta u\|^2 + \frac{1}{2}(g * \Delta u)(t) + \frac{b}{m}\|\Delta u\|_m^m. \tag{2.3}$$

where $(g * v)(t) = \int_0^t g(t-\tau)\|v(t) - v(\tau)\|^2 d\tau$.

Also, to show that the solutions decay uniformly to zero with the same arbitrary rate as the memory kernel, the following conditions are considered:

(A4) There exists a non-increasing differentiable function $\xi : R^+ \to R^+$ such that

$$\xi(0) > 0, \;\; g'(t) \leq -\xi(t)g(t), \;\; \int_0^{+\infty} \xi(s)ds = +\infty.$$

(A5) For certain sufficiently small constants $\varepsilon_1, \varepsilon_2$ such that $\varepsilon_2 \in (0, \frac{1}{3})$ and $\varepsilon_1 \in (\varepsilon_2, 2\varepsilon_2)$, it is assume that

$$p < \frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_2}.$$

Now, we are in a position to state our general decay result:

***Theorem* 2.1** — *Suppose that conditions $(A1)$ and $(A3) - (A5)$ are satisfied. Then the energy $E(t)$ of problem (1.1)-(1.3) satisfies the following general estimate for the two constants k and K:*

$$E(t) \leq KE(0)e^{-k\int_0^t \xi(s)ds}, \;\; t \geq 0. \tag{2.4}$$

Finally, to prove the blow-up result, the following problem is obtained from (1.1)-(1.3) when $a = b = 1$ and $c = 0$, where $e^{\lambda t}v(x,t)$ is substituted for $u(x,t)$:

$$v_{tt} + 2\lambda v_t + \lambda^2 v + \Delta[(1 + e^{\lambda(m-2)t}|\Delta v|^{m-2})\Delta v] - \int_0^t g_1(t-\tau)\Delta^2 v(\tau)d\tau$$
$$= e^{-\lambda t}\hat{h}(t,v) + e^{\lambda(p-2)t}|v|^{p-2}v, \qquad x \in \Omega, t \geq 0, \tag{2.5}$$

$$\begin{cases} v(x,t) = 0, & x \in \Gamma_0, t > 0 \\ \Delta v(x,t) = \int_0^t g_1(t-\tau)\Delta v(\tau)d\tau - e^{\lambda(m-2)t}|\Delta v|^{m-2}\Delta v, & x \in \Gamma_1, t > 0 \end{cases} \tag{2.6}$$

$$v(x,0) = u_0(x), \;\; v_t(x,0) = u_1(x) - \lambda u_0(x), \qquad x \in \Omega, \tag{2.7}$$

where

$$g_1(s) = e^{-\lambda s}g(s), \qquad \hat{h}(t,v) = h(x,t,e^{\lambda t}v, e^{\lambda t}\Delta v), \tag{2.8}$$

the value of the parameter λ will be prescribed later.

The energy function related to problem (2.4)-(2.7) is given by

$$E_\lambda(t) = \frac{1}{p}e^{\lambda(p-2)t}\|v\|_p^p - \frac{1}{2}J(t), \tag{2.9}$$

$$J(t) = \|v_t\|^2 + \lambda^2\|v\|^2 + (1 - \int_0^t g_1(s)ds)\|\Delta v\|^2 + (g_1 * \Delta v)(t)$$
$$+ \frac{2}{m} e^{\lambda(m-2)t}\|\Delta v\|_m^m$$

In order to prove the blow-up result, the following lemma is stated in the sequel and will be used later in section 3. This lemma was introduce in [5].

Lemma 2.2 — Let $\mu > 0, c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. Assume that $\psi(t)$ is a twice differentiable positive function such that

$$\psi^{''}\psi - (1+\mu)[\psi^{'}]^2 \geq -2c_1\psi\psi' - c_2[\psi]^2, \tag{2.10}$$

for all $t \geq 0$. If

$$\psi(0) > 0 \qquad \text{and} \qquad \psi'(0) + \gamma_2\mu^{-1}\psi(0) > 0, \tag{2.11}$$

then

$$\psi(t) \to +\infty \qquad \text{as} \qquad t \to t_1 \leq t_2 = \frac{1}{2\sqrt{c_1^2 + \mu c_2}} \log \frac{\gamma_1\psi(0) + \mu\psi'(0)}{\gamma_2\psi(0) + \mu\psi'(0)}. \tag{2.12}$$

Here

$$\gamma_1 = -c_1 + \sqrt{c_1^2 + \mu c_2} \qquad \text{and} \qquad \gamma_2 = -c_1 - \sqrt{c_1^2 + \mu c_2}.$$

Now we are in a position to state our blow-up result as follows.

***Theorem* 2.3** — *Let the conditions (A1)-(A3), be satisfied. For $p > m > 3$, it is assumed that $E_\lambda(0) > 0$ and*

$$\lambda \geq M\sqrt{\frac{2(p+m)}{(p-m)(p+m-4)}}, \quad \int_0^\infty g(s)ds \leq \frac{p+m-6}{p+m+2}, \tag{2.13}$$

then there exists a finite time t_1 such that the solution of the problem (1.1)-(1.3) blows up in a finite time, that is

$$\|u(t)\| \to +\infty \;\; as \;\; t \to t_1. \tag{2.14}$$

3. General Decay

As we mentioned earlier, in this section we shall prove a general decay for the solution energy. In order to carry the proof of Theorem 2.1, the following Lemmas are required. First, we define

$$F(t) = E(t) + \varepsilon_1\psi_1(t) + \varepsilon_2\psi_2(t), \tag{3.1}$$

where

$$\psi_1(t) = \int_\Omega u u_t dx + \frac{1}{2}\|u\|^2, \tag{3.2}$$

$$\psi_2(t) = -\int_0^t \int_\Omega |u(x,s)|^{p-1} u(x,s) dx ds. \tag{3.3}$$

It can clearly be seen that for sufficiently small ε_2, the functional $F(t)$ is equivalent to $E(t)$.

Lemma 3.1 — Under the conditions of Theorem 2.1, the energy functional $E(t)$, defined by (2.1), satisfies

$$\frac{d}{dt}E(t) = \frac{1}{2}(g' * \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|^2 - \|u_t\|^2 \leq 0. \tag{3.4}$$

PROOF : Multiplying equation (1.1) by u_t, performing an integration by parts and using (A3) and (2.1) our conclusion follows. □

Lemma 3.2 — Under the assumptions of Theorem 2.1, there exists a constant $\alpha \in (0, \frac{l}{2})$ such that satisfies, along the solution, the estimate

$$\psi_1'(t) \leq -E(t) + \frac{3}{2}\|u_t\|^2 + \frac{1}{2}\big(1 + \frac{1}{2\alpha}\int_0^\infty g(s)ds\big)(g * \Delta u)(t) + (1 - \frac{1}{p})\|u\|_p^p. \tag{3.5}$$

PROOF : Differentiating with respect to t we have

$$\psi_1'(t) = \|u_t\|^2 + \|u\|_p^p - a\|\Delta u\|^2 - b\|\Delta u\|_m^m + \int_\Omega \Delta u \int_0^t g(t-\tau)\Delta u(\tau) d\tau dx. \tag{3.6}$$

Using Young's inequality, we clearly see that for any $\alpha > 0$

$$\Big|\int_\Omega \Delta u(t) \int_0^t g(t-\tau)\Delta u(\tau) d\tau dx\Big| \leq \Big(\alpha + \int_0^t g(s)ds\Big)\|\Delta u\|^2 + \frac{1}{4\alpha}\big(\int_0^\infty g(s)ds\big)(g * \Delta u)(t). \tag{3.7}$$

Hence, by combining (3.4) and (3.5), the following estimation is obtained

$$\psi_1'(t) \leq \|u_t\|^2 + \|u\|_p^p - (a - \alpha - \int_0^t g(s)ds)\|\Delta u\|^2 - b\|\Delta u\|_m^m + \frac{1}{4a}\big(\int_0^\infty g(s)ds\big)(g * \Delta u)(t). \tag{3.8}$$

Applying the definition of energy functional (2.1) into (3.6), we get

$$\psi_1'(t) \leq -E(t) + \frac{3}{2}\|u_t\|^2 - \big[\frac{1}{2}(a - \int_0^t g(s)ds) - \alpha\big]\|\Delta u\|^2 + (1 - \frac{1}{p})\|u\|_p^p$$

$$-b(1-\frac{1}{m})\|\Delta u\|_m^m+\frac{1}{2}\big(1+\frac{1}{2\alpha}\int_0^\infty g(s)ds\big)(g*\Delta u)(t). \tag{3.9}$$

Finally, if we choose $\alpha \in (0, \frac{l}{2})$, the proof of Lemma 3.2 is completed. □

PROOF OF THEOREM 2.1. : Differentiating of $F(t)$ with respect to t, and taking (3.3) and (3.4) into account, we obtain

$$\xi(t)F'(t)=\xi(t)E'(t)+\varepsilon_1\xi(t)\psi_1'(t)+\varepsilon_2\xi(t)\psi_2'(t)$$

$$\leq -\varepsilon_1\xi(t)E(t)-\xi(t)\big(1-\frac{3\varepsilon_1}{2}\big)\|u_t\|^2-\xi(t)\big[\varepsilon_2-\varepsilon_1(1-\frac{1}{p})\big]\|u\|_p^p$$

$$+\frac{\varepsilon_1}{2}\xi(t)\big(1+\frac{1}{2\alpha}\int_0^\infty g(s)ds\big)(g*\Delta u)(t). \tag{3.10}$$

Thanks to the assumption (A5), we get

$$\xi(t)F'(t)\leq -\varepsilon_1\xi(t)E(t)+C\varepsilon_1\xi(t)(g*\Delta u)(t), \tag{3.11}$$

where $2C=1+\frac{1}{2\alpha}\int_0^\infty g(s)ds$.

Utilizing (A4) and (3.3) we have

$$\xi(t)F'(t)\leq -\varepsilon_1\xi(t)E(t)-C\varepsilon_1(g'*\Delta u)(t)$$

$$\leq -\varepsilon_1\xi(t)E(t)-2C\varepsilon_1E'(t). \tag{3.12}$$

Let is define

$$L(t)=\xi(t)F(t)+2C\varepsilon_1E(t), \tag{3.13}$$

it can clearly be observed that there exists a constant β_0 such that $L(t)\leq\beta_0E(t)$. Differentiating of $L(t)$, and using (A4) and (3.11), we get

$$L'(t)\leq -\varepsilon_1\xi(t)E(t)\leq -\frac{\varepsilon_1}{\beta_0}\xi(t)L(t). \tag{3.14}$$

A simple integration over $(0,t)$ we then get

$$L(t)\leq L(0)e^{-\frac{\varepsilon_1}{\beta_0}\int_0^t\xi(s)ds}. \tag{3.15}$$

Finally, since $L(t)\leq\beta_0E(t)$ and $F\sim E$, (3.14) accomplishes the proof.

4. Blow-up

In this section we prove that for sufficiently large initial data some of the solutions blow up in a finite time. To prove the blow-up result for certain solutions with positive initial energy, the following Lemma is required.

Lemma 4.1 — Under the conditions of Theorem 2.3, the energy functional $E_\lambda(t)$, defined by (2.9), satisfies

$$E_\lambda(t) \geq E_\lambda(0) \qquad \forall t \in R^+. \tag{4.1}$$

PROOF : A multiplication of equation (2.5) by v_t and integrating over Ω gives

$$\frac{d}{dt}E_\lambda(t) = 2\lambda\|v_t\|^2 + \frac{\lambda(p-2)}{p}e^{\lambda(p-2)t}\|v\|_p^p - \frac{\lambda(m-2)}{m}e^{\lambda(m-2)t}\|\Delta v\|_m^m$$
$$-\frac{1}{2}(g_1' * \Delta v)(t) - e^{-\lambda t}(\hat{h}(t,v), v_t), \tag{4.2}$$

It is easy to verify that $(\varepsilon_3, \varepsilon_4 > 0)$

$$e^{-\lambda t}|(\hat{h}(t,v), v_t)| \leq \varepsilon_3 e^{\lambda(m-2)t}\|\Delta v\|_m^m + \varepsilon_4 e^{\lambda(p-2)t}\|v\|_p^p + \frac{M^2}{4}(\frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4})\|v_t\|^2, \tag{4.3}$$

where (A2) and Young's inequality (2.1) have been used.
Taking into account (A3) and estimate (4.3) in relation with (4.2), we get

$$\frac{d}{dt}E_\lambda(t) \geq [\frac{\lambda(p-2)}{p} - \varepsilon_4]e^{\lambda(p-2)t}\|v\|_p^p + [2\lambda - \frac{M^2}{4}(\frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4})]\|v_t\|^2$$
$$-[\varepsilon_3 + \frac{\lambda(m-2)}{m}]e^{\lambda(m-2)t}\|\Delta v\|_m^m. \tag{4.4}$$

Employing the last inequality, we obtain from (2.9) the following inequality

$$\frac{d}{dt}E_\lambda(t) - [\lambda(p-2) - \varepsilon_4 p]E_\lambda(t) \geq [\frac{\lambda(p+2) - \varepsilon_4 p}{2} - \frac{M^2}{4}(\frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4})]\|v_t\|^2$$
$$+[\frac{\lambda(p-m) - \varepsilon_4 p}{m} - \varepsilon_3]e^{\lambda(m-2)t}\|\Delta v\|_m^m + \frac{1}{2}[\lambda(p-2) - \varepsilon_4 p](g_1 * \Delta v)(t)$$
$$+\frac{\lambda^2}{2}[\lambda(p-2) - \varepsilon_4 p]\|v\|^2 + \frac{l}{2}[\lambda(p-2) - \varepsilon_4 p]\|\Delta v\|^2, \tag{4.5}$$

where we used $1 - \int_0^t g_1(s)ds > l, (1 - \int_0^t g_1(s)ds > 1 - \int_0^t g(s)ds > 1 - \int_0^\infty g(s)ds = l)$.
At this point, if we choose $\varepsilon_3 = \frac{\lambda(p-m)}{2m}$ and $\varepsilon_4 = \frac{\lambda(p-m)}{2p}$ then the following inequality is obtained from (4.5)

$$\frac{d}{dt}E_\lambda(t) - \frac{\lambda}{2}(p+m-4)E(t) \geq [2\lambda + \frac{\lambda}{4}(p+m-4) - \frac{M^2}{2}(\frac{m+p}{\lambda(p-m)})]\|v_t\|^2$$

$$+\frac{\lambda^3}{4}(p+m-4)\|v\|^2+\frac{l\lambda}{4}(p+m-4)\|\Delta v\|^2$$
$$+\frac{\lambda}{4}(p+m-4)(g_1*\Delta v)(t). \tag{4.6}$$

Thanks to the hypothesises of Theorem 2.3, since we have $p>m>2$ and $\lambda \geq M\sqrt{\frac{2(p+m)}{(p-m)(p+m-4)}} > M\sqrt{\frac{2(p+m)}{(p-m)(p+m+4)}}$, we get

$$\frac{d}{dt}E_\lambda(t)-\frac{\lambda}{2}(p+m-4)E(t)\geq 0. \tag{4.7}$$

$E(t)\geq e^{\frac{\lambda}{2}(p+m-4)}E(0)\geq E(0)$ is obtained from (4.7) and the proof of this Lemma is competed. □

PROOF OF THEOREM 2.3. : For obtaining the blow-up result, the choice of the following functional is standard (see [6, 7])

$$\psi(t)=\|v(t)\|^2, \tag{4.8}$$

then

$$\psi'(t)=2(v,v_t), \tag{4.9}$$
$$\psi''(t)=2(v,v_{tt})+2\|v_t\|^2. \tag{4.10}$$

A multiplication of equation (2.5) by v and integrating over Ω gives

$$(v_{tt},v)=-2\lambda(v_t,v)-\lambda^2\|v\|^2-\|\Delta v\|^2-e^{\lambda(m-2)t}\|\Delta v\|_m^m+e^{\lambda(p-2)t}\|v\|_p^p$$
$$+e^{-\lambda t}(\hat{h}(t,v),v)+\int_0^t g_1(t-\tau)(\Delta v(\tau),\Delta v)d\tau. \tag{4.11}$$

Due to the condition (A2) and Young's inequality (2.1) we have $(\alpha_0,\alpha_1>0)$

$$e^{-\lambda t}|(\hat{h}(t,v),v)|\leq \alpha_0 e^{\lambda(m-2)t}\|\Delta v\|_m^m+\alpha_1 e^{\lambda(p-2)t}\|v\|_p^p+\frac{M^2}{4}\left(\frac{1}{\alpha_0}+\frac{1}{\alpha_1}\right)\|v\|^2, \tag{4.12}$$

We now estimate the last term in the right-hand side of (4.11) as follows: [11]

$$\int_0^t g_1(t-\tau)(\Delta v(\tau),\Delta v)d\tau \leq \frac{1}{2}\|\Delta v\|^2+\frac{1}{2}\int_\Omega\Big(\int_0^t g_1(t-\tau)|\Delta v(\tau)|d\tau\Big)^2dx$$
$$\leq \frac{1}{2}\|\Delta v\|^2+\frac{1}{2}\int_\Omega\Big(\int_0^t g_1(t-\tau)(|\Delta v(\tau)-\Delta v|+|\Delta v|)d\tau\Big)^2dx. \tag{4.13}$$

We then use the Cauchy-Schwarz, Poincare and Young's inequalities, and the fact that $\int_0^t g_1(s)ds \leq \int_0^\infty g_1(s)ds<1-l$, to obtain, for any $\beta_1>0$,

$$\int_\Omega\Big(\int_0^t g_1(t-\tau)(|\Delta v(\tau)-\Delta v|+|\Delta v|)d\tau\Big)^2dx$$

$$\leq \int_\Omega \Big(\int_0^t g_1(t-\tau)(|\Delta v(\tau) - \Delta v|)d\tau \Big)^2 dx + \int_\Omega \Big(\int_0^t g_1(t-\tau)|\Delta v|)d\tau \Big)^2 dx$$
$$+2\int_\Omega \Big(\int_0^t g_1(t-\tau)(|\Delta v(\tau) - \Delta v|)d\tau \Big)\Big(\int_0^t g_1(t-\tau)|\Delta v|)d\tau \Big) dx$$
$$\leq (1+\beta_1) \int_\Omega \Big(\int_0^t g_1(t-\tau)|\Delta v|)d\tau \Big)^2 dx$$
$$+\big(1+\frac{1}{\beta_1}\big) \int_\Omega \Big(\int_0^t g_1(t-\tau)(|\Delta v(\tau) - \Delta v|)d\tau \Big)^2 dx$$
$$\leq \big(1+\frac{1}{\beta_1}\big)(1-l)(g_1 * \Delta v)(t) + (1+\beta_1)(1-l)^2\|\Delta v\|^2. \tag{4.14}$$

By applying (4.12) and (4.14) into (4.11), we see that

$$(v_{tt}, v) \geq -2\lambda(v_t, v) - \lambda^2\|v\|^2 - \frac{3}{2}\|\Delta v\|^2 - e^{\lambda(m-2)t}\|\Delta v\|_m^m + e^{\lambda(p-2)t}\|v\|_p^p$$
$$-\alpha_0 e^{\lambda(m-2)t}\|\Delta v\|_m^m - \alpha_1 e^{\lambda(p-2)t}\|v\|_p^p - \frac{M^2}{4}\big(\frac{1}{\alpha_0} + \frac{1}{\alpha_1}\big)\|v\|^2$$
$$-\big(1+\frac{1}{\beta_1}\big)\frac{(1-l)}{2}(g_1 * \Delta v)(t) - \frac{1}{2}(1+\beta_1)(1-l)^2\|\Delta v\|^2. \tag{4.15}$$

By using definition of $E_\lambda(t)$ in (2.9), we have

$$(v_{tt}, v) \geq \frac{p+m}{2}E_\lambda(t) - 2\lambda(v_t, v) + \big(1 - \alpha_1 - \frac{p+m}{2p}\big)e^{\lambda(p-2)t}\|v\|_p^p$$
$$+\frac{p+m}{4}\|v_t\|^2 + \Big(\frac{\lambda^2}{4}(p+m) - \lambda^2 - \frac{M^2}{4}\big(\frac{1}{\alpha_0} + \frac{1}{\alpha_1}\big)\Big)\|v\|^2$$
$$+\Big(\frac{p+m}{4} - \big(1+\frac{1}{\beta_1}\big)\frac{(1-l)}{2}\Big)(g_1 * \Delta v)(t) + \big(\frac{p+m}{2m} - \alpha_0 - 1\big)e^{\lambda(m-2)t}\|\Delta v\|_m^m$$
$$+\Big(\frac{p+m}{4}\big(1 - \int_0^t g_1(s)ds\big) - \frac{1}{2}(1+\beta_1)(1-l)^2 - \frac{3}{2}\Big)\|\Delta v\|^2. \tag{4.16}$$

At this point by choosing $\alpha_0 = \frac{p-m}{2m}$ and $\alpha_1 = \frac{p-m}{2p}$, we deduce

$$(v_{tt}, v) \geq \frac{p+m}{2}E(t) - 2\lambda(v_t, v) + \frac{1}{4}\Big(\lambda^2(p+m-4) - 2M^2\big(\frac{p+m}{p-m}\big)\Big)\|v\|^2$$
$$+\frac{p+m}{4}\|v_t\|^2 + \Big(\frac{p+m}{4} - \big(1+\frac{1}{\beta_1}\big)\frac{(1-l)}{2}\Big)(g_1 * \Delta v)(t)$$
$$+\Big(\frac{(p+m)l}{4} - \frac{1}{2}(1+\beta_1)(1-l)^2 - \frac{3}{2}\Big)\|\Delta v\|^2, \tag{4.17}$$

where $1 - \int_0^t g_1(s)ds > l$ has been used.

Now if we choose $\beta_1 = \frac{l}{1-l}$ and apply the following assumptions of Theorem 2.3

$$\int_0^\infty g(s)ds \leq \frac{p+m-6}{p+m+2},$$

then we arrive at

$$(v_{tt}, v) \geq \frac{p+m}{2} E_\lambda(t) - 2\lambda(v_t, v) + \frac{1}{4}\Big(\lambda^2(p+m-4) - 2M^2(\frac{p+m}{p-m})\Big)\|v\|^2$$
$$+\frac{p+m}{4}\|v_t\|^2. \tag{4.18}$$

Now, by using Lemma 4.1 and since $E_\lambda(0) > 0$ we get from (4.18)

$$(v_{tt}, v) \geq -2\lambda(v_t, v) + \frac{1}{4}\Big(\lambda^2(p+m-4) - 2M^2(\frac{p+m}{p-m})\Big)\|v\|^2 + \frac{p+m}{4}\|v_t\|^2. \tag{4.19}$$

By substituting (4.8)-(4.10) in (4.19) we get

$$\psi''(t) \geq (\frac{p+m+4}{2})\|v_t\|^2 + \frac{1}{4}\Big(\lambda^2(p+m-4) - 2M^2(\frac{p+m}{p-m})\Big)\psi(t) - 2\lambda\psi'(t),$$

thus

$$\psi''(t)\psi(t) \geq (\frac{p+m+4}{8})[\psi'(t)]^2 + \frac{1}{4}\Big(\lambda^2(p+m-4) - 2M^2(\frac{p+m}{p-m})\Big)\psi^2(t)$$
$$-2\lambda\psi'(t)\psi(t), \tag{4.20}$$

where

$$[\psi'(t)]^2 \leq 4\|v_t\|^2\|v\|^2$$

has been used.

To this end, if we choose $\lambda \geq M\sqrt{\frac{2(p+m)}{(p-m)(p+m-4)}}$ then we derive

$$\psi''(t)\psi(t) \geq (\frac{p+m+4}{8})[\psi'(t)]^2 - 2\lambda\psi'(t)\psi(t). \tag{4.21}$$

Hence, we see that the hypotheses of Lemma 2.2 are fulfilled with

$$\mu = \frac{p+m}{8}, \qquad c_1 = \lambda, \qquad c_2 = 0,$$

and the conclusion of Lemma 2.2 demonstrates that some solutions of problem (2.5)-(2.7) blow up in a finite time t_1. Since this system is equivalent to (1.1)-(1.3), the proof is complete.

Remark 4.2 : Upper bound of the blow-up time t_1, for the problem (1.1)-(1.3) can be estimated as follows:

$$t_1 \leq \frac{1}{2\lambda} ln \frac{(p+m)\big(\int_\Omega u_0 u_1 dx - \lambda\|u_0\|^2\big)}{(p+m)\int_\Omega u_0 u_1 dx - \lambda(p+m+8)\|u_0\|^2},$$

where λ and p satisfy conditions of Theorem 2.3.

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