Received: 3 January 2021 | Revised: 2 June 2021 | Accepted: 13 September 2021
DOI: 10.1002/mma.7891

RESEARCH ARTICLE

WILEY

General decay and blow up of solutions for a class of inverse problem with elasticity term and variable-exponent nonlinearities

Mohammad Shahrouzi

Department of Mathematics, Jahrom University, Jahrom, Iran

Correspondence
Mohammad Shahrouzi, Department of Mathematics, Jahrom University, Jahrom, Iran.
Email: mshahrouzi@jahromu.ac.ir

Communicated by: F. Colombo

In this paper, we study a class of Lamé inverse source problem with variable-exponent nonlinearities. Under some suitable conditions on the coefficients and initial data, we proved general decay of solutions when the integral overdetermination tends to zero as time goes to infinity in appropriate range of variable exponents. Furthermore, in the absence of damping term, we show that there are solutions under some conditions on initial data and variable exponents which blow up in finite time.

KEYWORDS
blowup, general decay, inverse problem, variable exponent

MSC CLASSIFICATION
35B35; 35B44; 35R30

1 | INTRODUCTION

This paper deals with the following Lamé inverse source problem of determining $\{u(x,t), f(t)\}$:

$$u_{tt} - \Delta_e u - div\left(|\nabla u|^{r(x)-2}\nabla u\right) + \beta u_t + h(x,t,u,\nabla u) + a|u_t|^{m(x)-2}u_t = b|u|^{p(x)-2}u + f(t)\omega(x),\ (x,t) \in \Omega \times (0,\infty) \tag{1}$$

$$u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,\infty) \tag{2}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega \tag{3}$$

$$\int_\Omega u(x,t)\omega(x)dx = \phi(t),\ t > 0, \tag{4}$$

where $\Omega \subset R^n (n \geq 1)$ is a bounded domain. Here, Δ_e denotes the elasticity operator, which is the $n \times n$ matrix-valued differential operator defined by

$$\Delta_e u = \mu\Delta u + (\alpha + \mu)\nabla(div\ \ u),\ u = (u_1, u_2, \dots, u_n)^T,$$

μ and α are the Lamé constants which satisfy the following conditions:

$$\mu > 0,\ \alpha + \mu \geq 0.$$

Also β, a, and b are positive constants, and $h(x,t,u,\nabla u)$ and $\omega(x)$ are real functions that satisfy specific conditions that will be enunciated later (c.f. A2–A5). In addition, we assume that

$(A1)$ $r(.)$, $m(.)$, and $p(.)$ are given continuous and measurable functions on $\overline{\Omega}$ such that:

$$2 < r_1 \leq r(x) \leq r_2 < \infty$$

$$2 < m_1 \leq m(x) \leq m_2 < \infty$$

$$2 < p_1 \leq p(x) \leq p_2 < \infty$$

with

$$r_1 := essinf_{x\in\overline{\Omega}} r(x),\ r_2 := esssup_{x\in\overline{\Omega}} r(x),$$

$$m_1 := essinf_{x\in\overline{\Omega}} m(x),\ m_2 := esssup_{x\in\overline{\Omega}} m(x),$$

$$p_1 := essinf_{x\in\overline{\Omega}} p(x),\ p_2 := esssup_{x\in\overline{\Omega}} p(x).$$

The theory of inverse problems yields a theoretical basis for remote sensing and nondestructive evaluation. Many inverse problems arise naturally and have important applications in identification of flying objects (airplanes missiles, etc.), objects immersed in water (submarines, paces of fish, etc.), and in many other situations. As a rule, these problems are rather difficult to solve for two reasons: They are nonlinear and they are improperly posed (see Isakov and Ramm[1,2]).

Before going any further, it is worth pointing out some important prior results in the inverse problems. Eden and Kalantarov[3] studied the following inverse problem:

$$u_t - \Delta u + b(x,t,u,\nabla u) - |u|^p u = F(t)\omega(x),\ x \in \Omega,\ t > 0$$

$$u(x,t) = 0,\ x \in \partial\Omega,\ t > 0$$

$$u(x,0) = u_0(x),\ x \in \Omega$$

$$\int_\Omega u(x,t)\omega(x)dx = \phi(t),\ t > 0.$$

They found conditions on data guaranteeing global nonexistence of solutions when $\phi(t) \equiv 1$, also established a stability result with the opposite sign on the power type nonlinearity and $b(x,t,u,\nabla u) \equiv 0$. Next Tahamtani and Shahrouzi[4] extend previous results to a Petrovsky inverse source problem (see also Tahamtani and Shahrouzi[5]). Shahrouzi[6] studied the following damped viscoelastic inverse problem:

$$u_{tt} - \nabla[(a_0 + a|\nabla u|^m)\nabla u] + \int_0^t e^{\lambda(t-\tau)} g(t-\tau)\Delta u(\tau)d\tau + bu_t = h(x,t,u,\nabla u) + |u|^p u + f(t)\omega(x),\ x \in \Omega,\ t > 0$$

$$u(x,t) = 0,\ x \in \Gamma,\ t > 0$$

$$u(x,0) = u_0(x),\ x \in \Omega$$

$$\int_\Omega u(x,t)\omega(x)dx = 1,\ t > 0,$$

and proved the blowup of solutions under sufficient conditions on initial functions by using the modified concavity argument. See, in this regard, Shahrouzi.[7,8]

On the other hand, it is known that modeling of some physical phenomena such as flows of electro-rheological fluids, nonlinear viscoelasticity, and image processing gives rise to equation with nonstandard growth conditions, that is, equations with variable exponents of nonlinearities. In direct problems, equations with nonlinearities of variable-exponent type have largely been discussed by several authors. For instant, Antontsev[9] considered the equation:

$$u_{tt} = div\left(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\right) + \alpha\Delta u_t + b(x,t)u|u|^{\sigma(x,t)-2} + f(x,t)$$

in $\Omega \subseteq R^n$, where $\alpha > 0$ is a constant and a, b, p, σ are given functions. For specific conditions on a, b, p, σ, the existence theorems for small and any finite time have been proved, and also blowup of solutions under some suitable conditions on data has been established. Messaoudi and Talahmeh[10] considered the following nonlinear equation with variable exponents:

$$u_{tt} - div\left(|\nabla u|^{r(.)-2}\nabla u\right) + a|u_t|^{m(.)-2}u_t = b|u|^{p(.)-2}u. \tag{5}$$

They proved a finite-time blowup result for the solutions with negative initial energy and also certain solutions with positive energy in appropriate range of $m(.), r(.)$ and $p(.)$. Next, Messaoudi[11] studied Equation (6) with $a = 1$, $b = 0$ in the presence of damping term $-\Delta u_t$. He proved several decay results depending on the range of variable exponents m and r. Recently, Antontsev and Ferreira[12] studied a nonlinear class viscoelastic plate equation with a lower order by perturbation of $\vec{p}(x,t)$-Laplace operator of the form

$$u_{tt} + \Delta^2 u - \Delta_{\vec{p}(x,t)}u + \int_0^t g(t-s)\Delta u(s)ds - \varepsilon\Delta u_t + f(u) = 0,$$

associated with initial and Dirichlet–Neumann boundary conditions. Here, $\Delta_{\vec{p}(x,t)}$ is the $\vec{p}(x,t)$-Laplace operator that defined as

$$\Delta_{\vec{p}(x,t)}u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\frac{\partial u}{\partial x_i}|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\right), \vec{p}(x,t) = (p_1, p_2, \dots, p_n).$$

They proved a blowup in finite time with negative initial energy under suitable conditions on g, f and the variable exponent of the $\vec{p}(x,t)$-Laplace operator. For more results regarding this matter, we refer the reader to previous studies[13–18] and the review paper.[19]

Motivated by the aforementioned works, in the present paper, we study a class of elastic inverse source problem with variable-exponent nonlinearities. As we mentioned before, existence of variable-exponent nonlinearities makes study of inverse problems difficult. However, we try to extend the previous results[7,8] to the inverse problems with variable-exponent nonlinearities. To the best of our knowledge, this is the first work dealing with inverse source problem subject to the variable-exponent nonlinearities.

The rest of paper is organized as follows. In Section 2, we recall some definitions and lemmas about the variable-exponent Lebesgue space, $L^{p(.)}(\Omega)$, the Sobolev space, $W^{1,p(.)}(\Omega)$, and additional conditions that be used for main results. In Section 3, we prove the general decay of solutions for appropriate initial data and variable exponents. Finally, the blowup result has been proved with positive initial energy and suitable conditions on data and variable exponents when $\phi(t) \equiv 1$ and $\alpha = \beta = 0$, in Section 4.

2 | PRELIMINARIES AND MAIN RESULT

In this section, we recall some notations and functionals. We denote by $\|.\|_q$ the L^q-norm over Ω , and in particular, the L^2-norm is denoted $\|.\|$ in Ω . We shall assume that the functions $h(x, t, u, \nabla u)$, $\omega(x)$ and the functions appearing in the data satisfy the following conditions:

$A(2)$

$$|h(x,t,u,\nabla u)| \leq M_1|u|^{\frac{p(x)}{2}} + M_2|\nabla u|^{\frac{r(x)}{2}},$$

with some positive M_1, M_2 and variable exponents satisfy (A1).

$(A3)$

$$u_0 \in H_0^2(\Omega) \cap L^{r(.)}(\Omega) \cap L^{p(.)}(\Omega),\ u_1 \in L^2(\Omega) \cap L^{m(.)}(\Omega),\ \int_\Omega u_0(x)\omega(x)dx = \phi(0).$$

$(A4)$

$$\omega \in H_0^2(\Omega) \cap L^{r(.)}(\Omega) \cap L^{m(.)}(\Omega) \cap L^{p(.)}(\Omega), \int_\Omega \omega^2(x)dx = 1.$$

In order to study problems (1)–(4), we need some hypotheses and theories about Lebesgue and Sobolev spaces with variable exponents (for detailed, see, other works[20–24]). Let $p(x)\geq 1$ and measurable, we assume that

$$C_+(\overline{\Omega}) = \{h | h \in C(\overline{\Omega}),\ \ h(x) > 1 \ \ for \ \ any \ \ x \in \overline{\Omega}\},$$

$$h^+ = \max_{\Omega} h(x),\ h^- = \min_{\Omega} h(x)\ for \ \ any \ \ h \in C(\overline{\Omega}),$$

$$L^{p(x)}(\Omega) = \left\{ u | \, u \ \ is \ \ a \ \ measurable \ \ real-valued \ \ function, \ \ \int_\Omega |u(x)|^{p(x)}dx < \infty \right\}.$$

We equip the Lebesgue space with a variable exponent, $L^{p(x)}(\Omega)$, with the following Luxembourg-type norm:

$$\|u\|_{p(x)} := \inf \left\{ \lambda > 0 \left| \int_\Omega \right| \frac{u(x)}{\lambda} |^{p(x)} dx \leq 1 \right\}$$

Lemma 2.1. *Diening et al. and Xianling and Zhang.*[20,24] *Let Ω be a bounded domain in R^n .*

(i) The space $(L^{p(x)}(\Omega), \|.\|_{p(x)})$ is a Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left| \int_\Omega uvdx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{p(x)} \|v\|_{q(x)}.$$

(ii) If $p, q \in C_+(\overline{\Omega})$, $q(x) \leq p(x)$ for any $x \in \overline{\Omega}$, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, and the imbedding is continuous.

The variable-exponent Lebesgue Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) | \nabla u\, exists\, and\, |\nabla u| \in L^{p(x)}(\Omega)\}.$$

This space is a Banach space with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$. Furthermore, let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The dual of $W_0^{1,p(x)}(\Omega)$ is defined as $W^{-1,p'(x)}(\Omega)$, by the same way as the usual Sobolev spaces, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

If we define

$$p^*(x) = \begin{cases} \frac{Np(x)}{esssup_{x\in\overline{\Omega}}(N-p(x))}, & p^+ < N \\ \infty, & p^+ \geq N, \end{cases}$$

then we have

Lemma 2.2. *Diening et al. and Xianling and Zhang.*[20,24] *Let Ω be a bounded domain in R^n then for any measurable bounded exponent $p(x)$ we have*

(i) $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable Banach spaces;
(ii) if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous;
(iii) if $p(x)$ is uniformly continuous in Ω then there exists a constant $C > 0$, such that

$$\|u\|_{p(x)} \leq C \|\nabla u\|_{p(x)} \ \ \forall u \in W_0^{1,p(x)}(\Omega).$$

By (iii) of Lemma 2.2, we know that the space $W_0^{1,p(x)}(\Omega)$ has an equivalent norm given by $\|u\|_{W^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)}$.

We recall the Young's inequality

$$ab \leq \theta a^{q(x)} + C(\theta, q(x)) b^{q'(x)}, \ a, b \geq 0, \ \beta > 0, \ \frac{1}{q(x)} + \frac{1}{q'(x)} = 1, \tag{6}$$

where $C(\theta, q(x)) = \frac{1}{q'(x)} (\theta q(x))^{-\frac{q'(x)}{q(x)}}$. In special case when $\theta = \frac{1}{q(x)}$, we have from (6)

$$ab \leq \frac{a^{q(x)}}{q(x)} + \frac{b^{q'(x)}}{q'(x)}. \tag{7}$$

Adapting the condition (A4) and multiplying Equation (1) in $\omega(x)$, the key observation is that the problems (1)–(4) are equivalent to the following direct problem:

$$u_{tt} - \Delta_e u - div\left(|\nabla u|^{r(x)-2} \nabla u\right) + \beta u_t + h(x, t, u, \nabla u) + a|u_t|^{m(x)-2} u_t = b|u|^{p(x)-2} u + f(t)\omega(x), \ (x, t) \in \Omega \times (0, \infty) \tag{8}$$

$$u(x, t) = \frac{\partial u}{\partial \nu}(x, t) = 0, \ (x, t) \in \partial\Omega \times (0, \infty) \tag{9}$$

$$u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \ x \in \Omega \tag{10}$$

in which the unknown function $f(t)$ is replaced by

$$\begin{aligned} f(t) = {} & \phi''(t) + \beta\phi'(t) + \mu \int_\Omega \nabla u \nabla \omega(x) dx + (\alpha + \mu) \int_\Omega (div\ u)(div\ \omega(x)) dx + \int_\Omega |\nabla u|^{r(x)-1} \nabla \omega(x) dx \\ & + \int_\Omega h(x, t, u, \nabla u) \omega(x) dx + a \int_\Omega |u_t|^{m(x)-1} \omega \, dx - b \int_\Omega |u|^{p(x)-1} \omega(x) dx. \end{aligned} \tag{11}$$

At this point, we state the local existence of solutions for the problems (1)–(3) that can be established employing the Galerkin method as in Antontsev.[9]

Theorem 2.3. *(Local existence) Let $u_0 \in W_0^{1,r(.)}(\Omega)$, $u_1 \in L^2(\Omega)$ and assume that (A1)–(A4) be satisfied, then problem (1)–(3) have a unique weak solution such that*

$$u \in L^\infty\left((0, T), W_0^{1,r(.)}(\Omega)\right) \cap L^{p(.)}((0, T), \Omega), \ u_t \in L^\infty\left((0, T), L^2(\Omega)\right) \cap L^{m(.)}((0, T), \Omega),$$

$$u_{tt} \in L^\infty\left((0, T), W_0^{-1,r'(.)}(\Omega)\right),$$

for any $T > 0$ and $\frac{1}{r(.)} + \frac{1}{r'(.)} = 1$.

3 | GENERAL DECAY

In this section and in order to state and prove our result, we consider $b = 0$ and $h(x, t, u, \nabla u) \equiv 0$. The energy associated with problems (8)–(10) is given by

$$E(t) = \frac{1}{2}\left(\|u_t\|^2 + \mu\|\nabla u\|^2 + (\alpha + \mu) \int_\Omega |div\ u|^2 dx\right) + \int_\Omega \frac{1}{r(x)} |\nabla u|^{r(x)} dx. \tag{12}$$

To obtain the exponential decay result, we assume that
$(A5)$

$$a \geq \frac{m_2 - 1}{m_2 - \varepsilon(m_2 - 1)}, \ \frac{r_1(r_2 - 1)}{r_2(r_1 - 1)} \leq \varepsilon < \frac{m_2}{m_2 - 1},$$

where $\varepsilon \in (1, 2)$.

Our main result in this section reads in the following theorem:

Theorem 3.1. *Let the assumptions (A1) and (A3)–(A5) are satisfied and assume that μ and β are sufficiently large. Also, suppose that ϕ, ϕ', ϕ'' are continuous functions defined on $[0, \infty)$, such that ϕ'' is a bounded function and ϕ, ϕ' tend to zero as $t \to \infty$. Then the energy $E(t)$ of problems (1)–(4) generally decays to zero along the solution.*

To prove the above theorem, we need following Lemmas.

Lemma 3.2. *Under the conditions of Theorem 3.1, the energy functional $E(t)$, defined by (12), satisfies*

$$E'(t) = -\beta\|u_t\|^2 - a\int_\Omega |u_t|^{m(x)}dx + f(t)\phi'(t). \tag{13}$$

Proof. Multiplying Equation (8) by u_t and performing an integration by parts, we get

$$\frac{dE}{dt}(t) + \beta\|u_t\|^2 + a\int_\Omega |u_t|^{m(x)}dx = f(t)\int_\Omega \omega(x)u_t dx;$$

by using integral overdetermination (4), our conclusion follows. □

Lemma 3.3. *Under the conditions of Theorem 3.1, the function $f(t)$, defined by (11), for some $\varepsilon > 0$ satisfies*

$$|\phi'(t) + \varepsilon\phi(t)|f(t) \le \frac{\beta\varepsilon^2B_2^2}{2}\|\nabla u\|^2 + \frac{\varepsilon(\alpha+\mu)}{2}\int_\Omega |div\ u|^2dx + \frac{r_2-1}{r_2}\int_\Omega |\nabla u|^{r(x)}dx + \frac{m_2-1}{m_2}\int_\Omega |u_t|^{m(x)}dx + H(t), \tag{14}$$

where B_2 is the best constant in Poincaré's inequality, and

$$\begin{aligned} H(t) &= |\phi'(t) + \varepsilon\phi(t)||\phi''(t) + \beta\phi'(t)| + \frac{\mu^2|\phi'(t)+\varepsilon\phi(t)|^2}{2\beta\varepsilon^2B_2^2}\|\nabla\omega\|^2 + \frac{(\alpha+\mu)}{2\varepsilon}|\phi'(t)+\varepsilon\phi(t)|^2\int_\Omega |div\ \omega(x)|^2dx \\ &+ \int_\Omega \frac{|\phi'(t)+\varepsilon\phi(t)|^{r(x)}}{r(x)}|\nabla\omega(x)|^{r(x)}dx + \int_\Omega \frac{(a|\phi'(t)+\varepsilon\phi(t)|)^{m(x)}}{m(x)}|\omega(x)|^{m(x)}dx. \end{aligned} \tag{15}$$

Proof. We have from (11)

$$\begin{aligned} |\phi'(t) + \varepsilon\phi(t)|f(t) &= |\phi'(t)+\varepsilon\phi(t)|(\phi''(t) + \beta\phi'(t)) + \mu|\phi'(t)+\varepsilon\phi(t)|\int_\Omega \nabla u\nabla\omega(x)dx + |\phi'(t) \\ &+ \varepsilon\phi(t)|\int_\Omega |\nabla u|^{r(x)-1}\nabla\omega(x)dx + (\alpha+\mu)|\phi'(t)+\varepsilon\phi(t)|\int_\Omega (div\ \ u)(div\ \ \omega(x))dx \\ &+ a|\phi'(t)+\varepsilon\phi(t)|\int_\Omega |u_t|^{m(x)-1}\omega(x)dx. \end{aligned} \tag{16}$$

By using the Young's inequality and (A1), the last four terms in the right-hand side of (16) can be estimated as follows:

$$\mu|\phi'(t)+\varepsilon\phi(t)|.|\int_\Omega \nabla u\nabla\omega(x)dx| \le \frac{\beta\varepsilon^2B_2^2}{2}\|\nabla u\|^2 + \frac{\mu^2|\phi'(t)+\varepsilon\phi(t)|^2}{2\beta\varepsilon^2B_2^2}\|\nabla\omega\|^2, \tag{17}$$

$$\begin{aligned} |\phi'(t)+\varepsilon\phi(t)|.|\int_\Omega |\nabla u|^{r(x)-1}\nabla\omega(x)dx| &\le \int_\Omega \frac{r(x)-1}{r(x)}|\nabla u|^{r(x)}dx + \int_\Omega \frac{|\phi'(t)+\varepsilon\phi(t)|^{r(x)}}{r(x)}|\nabla\omega(x)|^{r(x)} \\ &\le \frac{r_2-1}{r_2}\int_\Omega |\nabla u|^{r(x)}dx + \int_\Omega \frac{|\phi'(t)+\varepsilon\phi(t)|^{r(x)}}{r(x)}|\nabla\omega(x)|^{r(x)}dx, \end{aligned} \tag{18}$$

$$(\alpha+\mu)|\phi'(t)+\varepsilon\phi(t)|.|\int_\Omega (div\ \ u)(div\ \ \omega(x))dx| \le \frac{\varepsilon(\alpha+\mu)}{2}\int_\Omega |div\ \ u|^2dx + \frac{(\alpha+\mu)}{2\varepsilon}|\phi'(t)+\varepsilon\phi(t)|^2\int_\Omega |div\ \ \omega(x)|^2dx, \tag{19}$$

$$a|\phi'(t)+\varepsilon\phi(t)|.|\int_\Omega |u_t|^{m(x)-1}\omega(x)dx| \le \int_\Omega \frac{m(x)-1}{m(x)}|u_t|^{m(x)}dx + \int_\Omega \frac{\left[a|\phi'(t)+\varepsilon\phi(t)|\right]^{m(x)}}{m(x)}|\omega(x)|^{m(x)}dx$$

$$\le \frac{m_2-1}{m_2}\int_\Omega |\nabla u|^{r(x)}dx + \int_\Omega \frac{\left[a|\phi'(t)+\varepsilon\phi(t)|\right]^{m(x)}}{m(x)}|\omega(x)|^{m(x)}dx. \tag{20}$$

Applying estimations (17)–(20) in (16) yields the desired result. □

Now define

$$F(t) = E(t) + \varepsilon\int_\Omega uu_t dx, \tag{21}$$

for some $\varepsilon > 0$.

The following Lemma estimates an appropriate upper bound for $F'(t)$:

Lemma 3.4. *Under the assumptions of Theorem 3.1, $F(t)$ satisfies, along the solution, the estimate*

$$F'(t) \le -(\frac{\beta}{2}-\varepsilon)\|u_t\|^2 - a\left[1-\varepsilon\left(\frac{m_2-1}{m_2}\right)\right]\int_\Omega |u_t|^{m(x)}dx - \left(\varepsilon\mu - \frac{\varepsilon a\bar{C}}{m_1} - \frac{\beta\varepsilon^2 B_2^2}{2}\right)\|\nabla u\|^2$$

$$-\varepsilon(\alpha+\mu)\int_\Omega |div\ \ u|^2dx - \varepsilon\int_\Omega |\nabla u|^{r(x)}dx + |\phi'(t)+\varepsilon\phi(t)|f(t). \tag{22}$$

Proof. Differentiating (21) with respect to t, we have

$$F'(t) = -(\beta-\varepsilon)\|u_t\|^2 - a\int_\Omega |u_t|^{m(x)}dx - \varepsilon\mu\|\nabla u\|^2 - \varepsilon(\alpha+\mu)\int_\Omega |div\ \ u|^2dx - \varepsilon\int_\Omega |\nabla u|^{r(x)}dx - \varepsilon\beta\int_\Omega uu_t dx$$

$$-\varepsilon a\int_\Omega u|u_t|^{m(x)-1}dx + \left(\phi'(t)+\varepsilon\phi(t)\right)f(t), \tag{23}$$

where Lemma 3.2 has been used.

It is easy to see that

$$\varepsilon\beta|\int_\Omega uu_t dx| \le \frac{\beta}{2}\|u_t\|^2 + \frac{\beta\varepsilon^2 B_2^2}{2}\|\nabla u\|^2, \tag{24}$$

where the Young's inequality (6) has been used.

Again by using the Young's inequality (7) and Cauchy–Schwarz inequality, we get

$$|\int_\Omega u|u_t|^{m(x)-2}u_t dx| \le \int_\Omega \frac{1}{m(x)}|u|^{m(x)}dx + \int_\Omega \frac{m(x)-1}{m(x)}|u_t|^{m(x)}dx \le \frac{1}{m_1}\int_\Omega |u|^{m(x)}dx + \frac{m_2-1}{m_2}\int_\Omega |u_t|^{m(x)}dx. \tag{25}$$

Also, let c_* be the best constant of embedding $H_0^1 \hookrightarrow L^{m(.)}(\Omega)$, then we have

$$\begin{aligned}\int_\Omega |u|^{m(x)}dx &\le \max\{\left(\int_\Omega |u|^{m(x)}dx\right)^{\frac{m_1}{m(x)}}, \left(\int_\Omega |u|^{m(x)}dx\right)^{\frac{m_2}{m(x)}}\}\\ &\le \max\{c_*^{m_1}\|\nabla u\|^{m_1}, c_*^{m_2}\|\nabla u\|^{m_2}\}\\ &\le \max\{c_*^{m_1}\|\nabla u\|^{m_1-2}, c_*^{m_2}\|\nabla u\|^{m_2-2}\}\|\nabla u\|^2 \le \overline{C}\|\nabla u\|^2.\end{aligned} \tag{26}$$

Combining (25) with (26), we deduce

$$\left|\int_\Omega u|u_t|^{m(x)-2}u_t dx\right| \le \frac{\overline{C}}{m_1}\|\nabla u\|^2 + \frac{m_2-1}{m_2}\int_\Omega |u_t|^{m(x)}dx. \tag{27}$$

Replacing inequalities (24) and (27) in (23), then the proof of Lemma 3.4 is completed. □

Proof of Theorem 3.1. Inspired by the definition of $F(t)$ and Lemma 3.4, we have for any $\delta > 0$

$$F'(t)+\delta F(t) \leq -(\frac{\beta}{2}-\frac{\delta}{2}-\varepsilon)\|u_t\|^2 - a\left[1-\varepsilon(\frac{m_2-1}{m_2})\right]\int_\Omega |u_t|^{m(x)}dx - \left(\varepsilon\mu - \frac{\delta\mu}{2} - \frac{\varepsilon a\bar{C}}{m_1} - \frac{\beta\varepsilon^2 B_2^2}{2}\right)\|\nabla u\|^2$$

$$-(\alpha+\mu)\left[\varepsilon - \frac{\delta}{2}\right]\int_\Omega |div\ \ u|^2 dx - \left(\varepsilon - \frac{\delta}{r_1}\right)\int_\Omega |\nabla u|^{r(x)}dx + \delta\varepsilon\int_\Omega u u_t dx + |\phi'(t)+\varepsilon\phi(t)|f(t). \tag{28}$$

Utilizing Cauchy–Schwartz, Poincaré, and Young inequalities to get

$$\delta\varepsilon\left|\int_\Omega u u_t dx\right| \leq \frac{a\varepsilon\overline{C}}{m_1 B_2^2}\|u\|^2 + \frac{m_1\delta^2 B_2^2}{4a\overline{C}}\|u_t\|^2 \leq \frac{a\varepsilon\overline{C}}{m_1}\|\nabla u\|^2 + \frac{m_1\delta^2 B_2^2}{4a\overline{C}}\|u_t\|^2. \tag{29}$$

Combining Lemma 3.3 and (29) with (28) yields

$$F'(t)+\delta F(t) \leq -\left(\frac{\beta}{2}-\frac{\delta}{2}-\varepsilon-\frac{m_1\delta^2 B_2^2}{4a\overline{C}}\right)\|u_t\|^2 - a\left(\left[1-\varepsilon(\frac{m_2-1}{m_2})\right] - \frac{m_2-1}{m_2}\right)\int_\Omega |u_t|^{m(x)}dx$$

$$-\left(\varepsilon\mu - \frac{\delta\mu}{2} - \frac{2\varepsilon a\bar{C}}{m_1} - \beta\varepsilon^2 B_2^2\right)\|\nabla u\|^2 - \frac{(\alpha+\mu)}{2}(\varepsilon-\delta)\int_\Omega |div\ \ u|^2 dx$$

$$-\left(\varepsilon - \frac{\delta}{r_1} - \frac{r_2-1}{r_2}\right)\int_\Omega |\nabla u|^{r(x)}dx + H(t). \tag{30}$$

Consequently, let $\delta := \varepsilon$, then by (A5) and for sufficiently large μ and β, we deduce

$$F'(t)+\varepsilon F(t) \leq H(t), \tag{31}$$

and so by a simple integration over (t_0, t), we get

$$F(t) \leq e^{-\varepsilon(t-t_0)}\left(F(t_0) + \int_{t_0}^t e^{\varepsilon(t-t_0)}H(s)ds\right). \tag{32}$$

Since $H(t) \to 0$ as $t \to \infty$ by the condition imposed on $\phi(t)$, inequality (32) indicates that the functional $F(t)$ decays to zero in the sense of $\phi(t)$. Thus for some positive C_0, the result follows from $C_0 E(t) \leq F(t)$, and proof of Theorem 3.1 has been completed.

4 | BLOWUP

In this section, we are going to prove the blowup result for certain solutions with positive initial energy. Our approach in this section is based on concavity method.[6,14] In order to prove this result, we set $a = \beta = 0$ and $\phi(t) \equiv 1$ and define

$$v(x,t) = e^{-\lambda t}u(x,t). \tag{33}$$

A direct computation by substituting (33) into the problems (1)–(4) yields

$$v_{tt} + 2\lambda v_t + \lambda^2 v - \Delta_e v - div\left(e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)-2}\nabla v\right) + e^{-\lambda t}h(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v) = b e^{\lambda t(p(x)-2)}|v|^{p(x)-2}v$$

$$+ e^{-\lambda t}f(t)\omega(x),\ (x,t) \in \Omega \times (0,\infty) \tag{34}$$

$$v(x,t) = \frac{\partial v}{\partial \nu} = 0,\ (x,t) \in \partial\Omega \times (0,\infty) \tag{35}$$

$$v(x,0)=u_0(x),\ v_t(x,0)=u_1(x)-\lambda u_0(x),\qquad x\in\Omega, \tag{36}$$

$$\int_\Omega v(x,t)\omega(x)dx=e^{-\lambda t},\qquad t>0, \tag{37}$$

where the value of the parameter λ will be prescribed later.

Similarly, adopting to the condition (A4) and integral overdetermination, the inverse problems (34)–(37) are equivalent to the direct problems (34)–(36) when the unknown function $f(t)$ is replaced with

$$f(t)=\mu e^{\lambda t}\int_\Omega \nabla v\nabla\omega(x)dx+(\alpha+\mu)e^{\lambda t}\int_\Omega (div\ \ v)(div\ \ \omega(x))dx+\int_\Omega e^{\lambda t(r(x)-1)}|\nabla v|^{r(x)-1}\nabla\omega(x)dx$$

$$+\int_\Omega \omega(x)h(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)dx-b\int_\Omega \omega(x)e^{\lambda t(p(x)-1)}|u|^{p(x)-1}dx. \tag{38}$$

The energy function related with problems (34)–(36) is given by

$$E_\lambda(t)=\int_\Omega \frac{b}{p(x)}e^{\lambda t(p(x)-2)}|v|^{p(x)}dx-\int_\Omega \frac{1}{r(x)}e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx-\frac{1}{2}I(t), \tag{39}$$

where

$$I(t)=\|v_t\|^2+\lambda^2\|v\|^2+\mu\|\nabla v\|^2+(\lambda+\mu)\int_\Omega |div\ \ v|^2dx.$$

Now we are in a position to state our blowup result as follows:

Theorem 4.1. *Let the conditions (A1)–(A4) are satisfied. Assume that* $p_1>2r_2$ *and* $E_\lambda(0)>\frac{2D_1}{\lambda(p_1+2)}+\frac{D_2}{2r_2}$, *where*

$$D_1=\frac{\mu\lambda}{p_1-2}\|\nabla\omega\|^2+\frac{\lambda(\lambda+\mu)}{p_1-2}\int_\Omega |div\ \ \omega(x)|^2dx+\int_\Omega \frac{1}{r(x)}(\lambda|\nabla\omega(x)|)^{r(x)}dx+\frac{\lambda}{2}(\sqrt{p_2M_1^3}+\sqrt{r_2M_2^3})\|\omega\|^2$$

$$+\int_\Omega \frac{1}{p(x)}(b\lambda|\omega(x)|)^{p(x)}dx,$$

$$D_2=\frac{1}{2r_2\mu}\|\nabla\omega\|^2+\frac{(\alpha+\mu)}{2r_2}\int_\Omega |div\ \ \omega(x)|^2dx+\int_\Omega \frac{1}{r(x)}(|\nabla\omega(x)|)^{r(x)}dx+\frac{1}{2\sqrt{2}}(\sqrt{p_2M_1^3}+\sqrt{p_1M_2^3})\|\omega\|^2$$

$$+\int_\Omega \frac{1}{p(x)}(b|\omega(x)|)^{p(x)}dx.$$

Then for sufficiently large λ, b *there exists a finite time* t^* *such that the solution of the problems (1)–(3) blows up in a finite time, that is,*

$$\|u(t)\|\to+\infty \ as\ t\to t^*. \tag{40}$$

To prove the blowup result, we need the following Lemma.

Lemma 4.2. *Under the conditions of Theorem 4.1, the energy functional* $E_\lambda(t)$ *, defined by (39), satisfies*

$$E_\lambda(t)\ge E_\lambda(0)-\frac{2D_1}{\lambda(p_1+2)}\qquad \forall t\in R^+, \tag{41}$$

Proof. A multiplication of Equation (34) by v_t and integrating over Ω gives

$$E'_\lambda(t)=2\lambda\|v_t\|^2+e^{-\lambda t}\int_\Omega v_th(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)dx-\lambda\int_\Omega \frac{(r(x)-2)}{r(x)}e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx$$

$$+b\lambda\int_\Omega \frac{(p(x)-2)}{p(x)}e^{\lambda t(p(x)-2)}|v|^{p(x)}dx+\lambda e^{-2\lambda t}f(t). \tag{42}$$

Employing the last equation, we obtain from (39) the following inequality for some $\varepsilon > 0$:

$$E'_\lambda(t) - \varepsilon E_\lambda(t) = \frac{\varepsilon\lambda^2}{2}\|v\|^2 + \frac{\varepsilon\mu}{2}\|\nabla v\|^2 + \frac{\varepsilon(\alpha+\mu)}{2}\int_\Omega |div\ \ v|^2 dx + (2\lambda + \frac{\varepsilon}{2})\|v_t\|^2 + e^{-\lambda t}\int_\Omega v_t h(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v)dx$$

$$-\int_\Omega \frac{[\lambda(r(x)-2)-\varepsilon]}{r(x)} e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx + b\int_\Omega \frac{[\lambda(p(x)-2)-\varepsilon]}{p(x)} e^{\lambda t(p(x)-2)}|v|^{p(x)}dx + \lambda e^{-2\lambda t}f(t). \tag{43}$$

Taking into account (A1) and (38), we deduce

$$E'_\lambda(t) - \varepsilon E_\lambda(t) \geq \frac{\varepsilon\lambda^2}{2}\|v\|^2 + \frac{\varepsilon\mu}{2}\|\nabla v\|^2 + \frac{\varepsilon(\alpha+\mu)}{2}\int_\Omega |div\ \ v|^2 dx + (2\lambda + \frac{\varepsilon}{2})\|v_t\|^2 + e^{-\lambda t}\int_\Omega v_t h(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v)dx$$

$$-\frac{[\lambda(r_2-2)-\varepsilon]}{r_2}\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx + \frac{b[\lambda(p_1-2)-\varepsilon]}{p_1}\int_\Omega e^{\lambda t(p(x)-2)}|v|^{p(x)}dx + \lambda\mu e^{-\lambda t}\int_\Omega \nabla v\nabla\omega(x)dx$$

$$+\lambda(\alpha+\mu)e^{-\lambda t}\int_\Omega (div\ \ v)(div\ \ \omega(x))dx + \lambda e^{-\lambda t}\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)-1}\nabla\omega(x)dx$$

$$+\lambda e^{-2\lambda t}\int_\Omega \omega(x)h(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v)dx - b\lambda e^{-\lambda t}\int_\Omega e^{\lambda t(p(x)-2)}|u|^{p(x)-1}\omega(x)dx. \tag{44}$$

Now, the terms on the right-hand side of (44) can be estimated as follows. First recalling (A2), Cauchy–Schwarz inequality, and Young's inequality (6), we get

$$|e^{-\lambda t}\int_\Omega v_t h(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v)dx| \leq M_1\int_\Omega |v_t|.e^{-\lambda t}|e^{\lambda t}v|^{\frac{p(x)}{2}}dx + M_2\int_\Omega |v_t|.e^{-\lambda t}|e^{\lambda t}\nabla v|^{\frac{r(x)}{2}}dx$$

$$\leq M_1\left(\frac{p_2^2M_1^2}{2}\|v_t\|^2 + \frac{e^{-2\lambda t}}{2p_2M_1}\int_\Omega |e^{\lambda t}v|^{p(x)}dx\right) + M_2\left(\frac{r_2^2M_2^2}{2}\|v_t\|^2 + \frac{e^{-2\lambda t}}{2r_2M_2}\int_\Omega |e^{\lambda t}\nabla v|^{r(x)}dx\right)$$

$$\leq \frac{(p_2^2M_1^3 + r_2^2M_2^3)}{2}\|v_t\|^2 + \frac{1}{2p_2}\int_\Omega e^{\lambda t(p(x)-2)}|v|^{p(x)}dx + \frac{1}{2r_2}\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx, \tag{45}$$

$$\lambda e^{-2\lambda t}|\int_\Omega \omega(x)h(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v)dx| \leq e^{-2\lambda t}\left(M_1\int_\Omega |e^{\lambda t}v|^{\frac{p(x)}{2}}.\lambda|\omega(x)|dx + M_2\int_\Omega |e^{\lambda t}\nabla v|^{\frac{r(x)}{2}}.\lambda|\omega(x)|dx\right)$$

$$\leq M_1e^{-2\lambda t}\left(\frac{1}{2p_2M_1}\int_\Omega |e^{\lambda t}v|^{p(x)}dx + \frac{\lambda^2\sqrt{p_2M_1}}{2}\|\omega(x)\|^2\right) + M_2e^{-2\lambda t}\left(\frac{1}{2r_2M_2}\int_\Omega |e^{\lambda t}\nabla v|^{r(x)}dx + \frac{\lambda^2\sqrt{r_2M_2}}{2}\|\omega(x)\|^2\right)$$

$$\leq \frac{1}{2p_2}\int_\Omega e^{\lambda t(p(x)-2)}|v|^{p(x)}dx + \frac{1}{2r_2}\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx + \frac{\lambda^2}{2}\left(\sqrt{p_2M_1^3} + \sqrt{r_2M_2^3}\right)e^{-2\lambda t}\|\omega\|^2. \tag{46}$$

Also by using Cauchy–Schwarz inequality and Young's inequality (6), we have

$$\lambda\mu e^{-\lambda t}|\int_\Omega \nabla v\nabla\omega(x)|dx \leq \frac{\varepsilon\mu}{2}\|\nabla v\|^2+\frac{\lambda^2\mu}{2\varepsilon}e^{-2\lambda t}\|\nabla\omega\|^2, \tag{47}$$

$$\lambda(\alpha+\mu)e^{-\lambda t}|\int_\Omega (div\ \ v)(div\ \ \omega(x))dx| \leq \frac{\varepsilon(\alpha+\mu)}{2}\int_\Omega |div\ \ v|^2dx+\frac{\lambda^2(\alpha+\mu)e^{-2\lambda t}}{2\varepsilon}\int_\Omega |div\ \ \omega(x)|^2dx. \tag{48}$$

Using (A1) and Young's inequality (7) yields

$$\begin{aligned}
\lambda e^{-\lambda t}|\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)-1}\nabla\omega(x)dx| &= \lambda e^{-2\lambda t}|\int_\Omega |e^{\lambda t}\nabla v|^{r(x)-1}\nabla\omega(x)dx|\\
&\leq e^{-2\lambda t}\left(\int_\Omega \frac{r(x)-1}{r(x)}|e^{\lambda t}\nabla v|^{r(x)}dx+\int_\Omega \frac{1}{r(x)}|\lambda\nabla\omega(x)|^{r(x)}dx\right)\\
&\leq \left(\frac{r_2-1}{r_2}\right)\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx+e^{-2\lambda t}\int_\Omega \frac{1}{r(x)}|\lambda\nabla\omega(x)|^{r(x)}dx,
\end{aligned} \tag{49}$$

$$\begin{aligned}
b\lambda e^{-\lambda t}|\int_\Omega e^{\lambda t(p(x)-2)}|u|^{p(x)-1}\omega(x)dx| &= e^{-2\lambda t}\int_\Omega |e^{\lambda t}v|^{p(x)-1}.b\lambda\omega(x)dx|\\
&\leq e^{-2\lambda t}\left(\int_\Omega \frac{p(x)-1}{p(x)}|e^{\lambda t}v|^{p(x)}dx+\int_\Omega \frac{1}{p(x)}|b\lambda\omega(x)|^{p(x)}dx\right)\\
&\leq \left(\frac{p_2-1}{p_2}\right)\int_\Omega e^{\lambda t(p(x)-2)}|v|^{p(x)}dx+e^{-2\lambda t}\int_\Omega \frac{1}{p(x)}|b\lambda\omega(x)|^{p(x)}dx.
\end{aligned} \tag{50}$$

Combining (45)–(50) with (44) to get

$$\begin{aligned}
E'_\lambda(t)-\varepsilon E_\lambda(t) &\geq \frac{\varepsilon\lambda^2}{2}\|v\|^2+\left(2\lambda+\frac{\varepsilon}{2}-\frac{1}{2}(p_2^2M_1^3+r_2M_2^3)\right)\|v_t\|^2+\left(\frac{\varepsilon-\lambda(r_2-2)}{r_2}-1\right)\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx\\
&+\left(\frac{b[\lambda(p_1-2)-\varepsilon]}{p_1}-1\right)\int_\Omega e^{\lambda t(p(x)-2)}|v|^{p(x)}dx-e^{-2\lambda t}\left(\frac{\lambda^2\mu}{2\varepsilon}\|\nabla\omega\|^2+\frac{\lambda^2(\alpha+\mu)}{2\varepsilon}\int_\Omega |div\ \ \omega(x)|^2dx\right.\\
&\left.+\int_\Omega \frac{1}{r(x)}(\lambda|\nabla\omega(x)|)^{r(x)}dx+\frac{\lambda}{2}(\sqrt{p_2M_1^3}+\sqrt{r_2M_2^3})\|\omega\|^2+\int_\Omega \frac{1}{p(x)}(b\lambda|\nabla\omega(x)|)^{p(x)}dx\right).
\end{aligned} \tag{51}$$

At this point, we set $\varepsilon := \frac{\lambda(p_1-2)}{2}$; therefore, we obtain

$$\begin{aligned}
E'_\lambda(t)-\frac{\lambda(p_1-2)}{2}E_\lambda(t) &\geq \frac{\varepsilon\lambda^2}{2}\|v\|^2+\frac{1}{4}\left(\lambda(p_1+6)-2(p_2^2M_1^3+r_2M_2^3)\right)\|v_t\|^2\\
&+\left(\frac{\lambda}{2r_2}(p_1-2r_2+2)-1\right)\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx+\left(\frac{b\lambda(p_1-2)}{2p_2}-1\right)\int_\Omega e^{\lambda t(p(x)-2)}|v|^{p(x)}dx-e^{-2\lambda t}D_1,
\end{aligned} \tag{52}$$

where D_1 satisfied hypotheses of Theorem 4.1.

Finally, since $p_1 > 2r_2$ if we choose λ sufficiently large as

$$\lambda \geq \lambda^* = \max\left\{ \frac{2(p_2^2 M_1^3 + r_2 M_2^3)}{p_1 + 6}, \frac{2r_2}{p_1 - 2r_2 + 2}, \frac{2p_2}{b(p_1 - 2)} \right\},$$

then we have

$$E'_\lambda(t) - \frac{\lambda(p_1 - 2)}{2} E_\lambda(t) \geq -e^{-2\lambda t} D_1. \tag{53}$$

Integrating (53) between 0 and t, we observe that

$$E_\lambda(t) \geq E_\lambda(0) - \frac{2D_1}{\lambda(p_1 + 2)}, \ \forall t \geq 0,$$

and proof of Lemma 4.1 is complete. □

Proof of Theorem 4.1. To obtain the blowup result, the choice of the following functional is standard

$$\psi(t) = \|v(t)\|^2, \tag{54}$$

then

$$\psi'(t) = 2\int_\Omega v v_t dx, \tag{55}$$

$$\psi''(t) = 2\int_\Omega v v_{tt} dx + 2\|v_t\|^2. \tag{56}$$

A multiplication of Equation (34) by v and integrating over Ω gives

$$\int_\Omega v v_{tt} dx = -2\lambda \int_\Omega v v_t dx - \lambda^2 \|v\|^2 - \mu \|\nabla v\|^2 - (\alpha + \mu)\int_\Omega |div \ \ v|^2 dx - \int_\Omega e^{\lambda t(r(x)-2)} |\nabla v|^{r(x)} dx$$

$$-e^{-\lambda t}\int_\Omega v h(x, t, e^{\lambda t} v, e^{\lambda t}\nabla v) dx + b\int_\Omega e^{\lambda t(p(x)-2)} |v|^{p(x)} dx + e^{-2\lambda t} f(t). \tag{57}$$

Utilizing (38) and (39) in (57), for any $\delta > 0$, we obtain

$$\int_\Omega v v_{tt} dx = \delta E_\lambda(t) - \delta b \int_\Omega \frac{1}{p(x)} e^{\lambda t(p(x)-2)} |v|^{p(x)} dx + \delta \int_\Omega \frac{1}{r(x)} e^{\lambda t(r(x)-2)} |\nabla v|^{r(x)} dx + \lambda^2 (\frac{\delta}{2} - 1)\|v\|^2 + \frac{\delta}{2}\|v_t\|^2$$

$$+\mu(\frac{\delta}{2} - 1)\|\nabla v\|^2 + (\alpha + \mu)(\frac{\delta}{2} - 1)\int_\Omega |div \ \ v|^2 dx - 2\lambda\int_\Omega v v_t dx - \int_\Omega e^{\lambda t(r(x)-2)} |\nabla v|^{r(x)} dx$$

$$-e^{-\lambda t}\int_\Omega v h(x, t, e^{\lambda t} v, e^{\lambda t}\nabla v) dx + b\int_\Omega e^{\lambda t(p(x)-2)} |v|^{p(x)} dx + \mu e^{-\lambda t}\int_\Omega \nabla v \nabla \omega(x) dx$$

$$+(\alpha + \mu) e^{-\lambda t}\int_\Omega (div \ \ v)(div \ \ \omega(x)) dx + e^{-\lambda t}\int_\Omega (e^{\lambda t(r(x)-2)} |\nabla v|^{r(x)-1})\nabla \omega(x) dx$$

$$+e^{-2\lambda t}\int_\Omega \omega(x) h(x, t, e^{\lambda t} v, e^{\lambda t}\nabla v) dx - b e^{-\lambda t}\int_\Omega (e^{\lambda t(p(x)-2)} |v|^{p(x)-1})\omega(x) dx. \tag{58}$$

Again, the terms on the right-hand side of (58) can be estimated as follows. Recalling (A2), Cauchy–Schwartz inequality, and Young's inequality (6), we deduce

$$|e^{-\lambda t}\int_\Omega vh(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)dx| \le M_1\int_\Omega |v|.e^{-\lambda t}|e^{\lambda t}v|^{\frac{p(x)}{2}}dx + M_2\int_\Omega |v|.e^{-\lambda t}|e^{\lambda t}\nabla v|^{\frac{r(x)}{2}}dx$$

$$\le M_1\left(\frac{p_2^2M_1^2}{8}\|v\|^2+\frac{1}{p_2M_1}\int_\Omega e^{\lambda t(p(x)-2)}|v|^{p(x)}dx\right)+M_2\left(\frac{p_1^2M_2^2}{8}\|v\|^2+\frac{1}{p_1M_2}\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx\right)$$

$$\le \frac{(p_2^2M_1^3+p_1^2M_2^3)}{8}\|v\|^2+\frac{1}{p_2}\int_\Omega e^{\lambda t(p(x)-2)}|v|^{p(x)}dx+\frac{1}{p_1}\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx, \tag{59}$$

$$e^{-2\lambda t}|\int_\Omega \omega(x)h(x,t,e^{\lambda t}v,e^{\lambda t}\nabla v)dx| \le e^{-2\lambda t}\left(M_1\int_\Omega |e^{\lambda t}v|^{\frac{p(x)}{2}}.|\omega(x)|dx + M_2\int_\Omega |e^{\lambda t}\nabla v|^{\frac{r(x)}{2}}.|\omega(x)|dx\right)$$

$$= M_1\int_\Omega e^{-\lambda t}|e^{\lambda t}v|^{\frac{p(x)}{2}}.e^{-\lambda t}\omega(x)dx + M_2\int_\Omega e^{-\lambda t}|e^{\lambda t}\nabla v|^{\frac{r(x)}{2}}.e^{-\lambda t}\omega(x)dx$$

$$\le M_1\left(\frac{1}{p_2M_1}\int_\Omega e^{\lambda t(p(x)-2)}|v|^{p(x)}dx+\frac{\sqrt{p_2M_1}e^{-2\lambda t}}{2\sqrt{2}}\|\omega\|^2\right)+M_2\left(\frac{1}{p_1M_2}\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx+\frac{\sqrt{p_1M_2}e^{-2\lambda t}}{2\sqrt{2}}\|\omega\|^2\right)$$

$$\le \frac{1}{p_2}\int_\Omega e^{\lambda t(p(x)-2)}|v|^{p(x)}dx+\frac{1}{p_1}\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx+\frac{1}{2\sqrt{2}}\left(\sqrt{p_2M_1^3}+\sqrt{p_1M_2^3}\right)e^{-2\lambda t}\|\omega\|^2. \tag{60}$$

Also by using Cauchy–Schwartz inequality and Young's inequality (6), we have

$$\mu e^{-\lambda t}|\int_\Omega \nabla v\nabla\omega(x)|dx \le \frac{\delta\mu}{4}\|\nabla v\|^2+\frac{e^{-2\lambda t}}{\delta\mu}\|\nabla\omega\|^2, \tag{61}$$

$$(\alpha+\mu)e^{-\lambda t}|\int_\Omega (div\ \ v)(div\ \ \omega(x))dx| \le \frac{\delta(\alpha+\mu)}{4}\int_\Omega |div\ \ v|^2dx+\frac{(\alpha+\mu)}{\delta}e^{-2\lambda t}\int_\Omega |div\ \ \omega(x)|^2dx. \tag{62}$$

Using (A1) and Young's inequality (7) yields

$$e^{-\lambda t}|\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)-1}\nabla\omega(x)dx| = e^{-2\lambda t}|\int_\Omega |e^{\lambda t}\nabla v|^{r(x)-1}\nabla\omega(x)dx|$$

$$\le e^{-2\lambda t}\left(\int_\Omega \frac{r(x)-1}{r(x)}|e^{\lambda t}\nabla v|^{r(x)}dx+\int_\Omega \frac{1}{r(x)}|\nabla\omega|^{r(x)}dx\right)$$

$$\le \left(\frac{r_2-1}{r_2}\right)\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx+e^{-2\lambda t}\int_\Omega \frac{1}{r(x)}|\nabla\omega|^{r(x)}dx, \tag{63}$$

$$be^{-\lambda t}|\int_\Omega e^{\lambda t(p(x)-2)}|u|^{p(x)-1}\omega(x)dx| = e^{-2\lambda t}\int_\Omega |e^{\lambda t}v|^{p(x)-1}.b\omega(x)dx|$$

$$\le e^{-2\lambda t}\left(\int_\Omega \frac{p(x)-1}{p(x)}|e^{\lambda t}v|^{p(x)}dx+\int_\Omega \frac{1}{p(x)}|b\omega|^{p(x)}dx\right)$$

$$\le \left(\frac{p_2-1}{p_2}\right)\int_\Omega e^{\lambda t(p(x)-2)}|v|^{p(x)}dx+e^{-2\lambda t}\int_\Omega \frac{1}{p(x)}|b\omega(x)|^{p(x)}dx. \tag{64}$$

Let $\delta := 2r_2$, then by combining estimations (59)–(64) with (58), we obtain

$$\int_\Omega vv_{tt}dx \geq 2r_2E_\lambda(t) + \left[\lambda^2(r_2-1) - \frac{1}{8}(p_2^2M_1^3 + p_1^2M_2^3)\right]\|v\|^2 + r_2\|v_t\|^2 + \mu(\frac{r_2}{2}-1)\|\nabla v\|^2$$

$$+(\alpha+\mu)(\frac{r_2}{2}-1)\int_\Omega |div\ \ v|^2dx - 2\lambda\int_\Omega vv_tdx + \left(\frac{p_1-2r_2}{p_1r_2}\right)\int_\Omega e^{\lambda t(r(x)-2)}|\nabla v|^{r(x)}dx$$

$$+\left[\frac{b(p_1-2r_2)}{p_1} - \frac{p_2+1}{p_2}\right]\int_\Omega e^{\lambda t(p(x)-2)}|v|^{p(x)}dx - e^{-2\lambda t}D_2, \tag{65}$$

where D_2 satisfied Theorem 4.1.

Now if we choose

$$\lambda \geq \max\{\lambda^*, \sqrt{\frac{M_1^3p_2^2 + M_2^3p_1^2}{8(r_2-1)}}\},\ b \geq \frac{p_1(p_2+1)}{p_2(p_1-2r_2)},$$

then since $p_1 > 2r_2$, we deduce

$$\int_\Omega vv_{tt}dx \geq 2r_2E_\lambda(t) + r_2\|v_t\|^2 - 2\lambda\int_\Omega vv_tdx - e^{-2\lambda t}D_2. \tag{66}$$

Now, by using Lemma 4.2 and since $E_\lambda(0) > \frac{2D_1}{\lambda(p_1+2)} + \frac{D_2}{2r_2}$, we get from (66)

$$\int_\Omega vv_{tt}dx \geq -2\lambda\int_\Omega vv_tdx + r_2\|v_t\|^2. \tag{67}$$

By substituting (56) in (67), we get

$$\psi''(t) \geq -4\lambda\int_\Omega vv_tdx + 2(r_2+1)\|v_t\|^2,$$

thus

$$\psi''(t)\psi(t) \geq -2\lambda\psi'(t)\psi(t) + \frac{(r_2+1)}{2}[\psi'(t)]^2, \tag{68}$$

where

$$[\psi'(t)]^2 \leq 4\|v_t\|^2\|v\|^2$$

has been used.

Hence, the concavity argument gives us

$$\lim_{t\to t^*}\psi(t) = \infty;$$

this yields solutions of problems (34)–(36) blowup in a finite time t^*. Since this system is equivalent to (1)–(4), the proof of Theorem 4.1 is complete. □

ACKNOWLEDGEMENT

There are no funders to report for this submission. Moreover, I would like to express my sincere thanks to the anonymous referee for his(her) valuable comments and suggestions which helped improving the original manuscript.

CONFLICT OF INTEREST

There are no conflicts of interest to this work.

ORCID

Mohammad Shahrouzi https://orcid.org/0000-0001-9308-0115

REFERENCES

1. Isakov V. *Inverse Source Problems*, Math Surveys and Monographs Series, vol. 34. Providence R.I.: AMS; 1990.
2. Ramm AG. Inverse Problems. Mathematical and Analytical Techniques with Applications to Engineering, Springer; 2005.
3. Eden A, Kalantarov VK. On global behavior of solutions to an inverse problem for nonlinear parabolic equations. *J Math Anal Appl*. 2005;307(1):120-133. https://doi.org/10.1016/j.jmaa.2005.01.007
4. Tahamtani F, Shahrouzi M. Global nonexistence and stability of solutions of inverse problems for a class of Petrovsky systems. *Georgian Math J*. 2012;19:575-586. https://doi.org/10.1515/gmj-2012-0028
5. Tahamtani F, Shahrouzi M. Asymptotic stability and blow up of solutions for a Petrovsky inverse source problem with dissipative boundary condition. *Math Meth Appl Sci*. 2013;36:829-839. https://doi.org/10.1002/mma.2629
6. Shahrouzi M. Blow up of solutions to a class of damped viscoelastic inverse source problem. *Differ Equ Dyn Syst*. 2020;28:889-899. https://doi.org/10.1007/s12591-017-0356-8
7. Shahrouzi M. Asymptotic stability and blowup of solutions for a class of viscoelastic inverse problem with boundary feedback. *Math Meth Appl Sci*. 2016;39:2368-2379. https://doi.org/10.1002/mma.3646
8. Shahrouzi M. On behavior of solutions to a class of nonlinear hyperbolic inverse source problem. *Acta Math Sinica, English Series*. 2016;32(6):683-698. https://doi.org/10.1007/s10114-016-5081-7
9. Antontsev S. Wave equation with $p(x,t)-$Laplacian and damping term: existence and blow-up. *Differ Equ Appl*. 2011;3(4):503-525. https://doi.org/10.7153/dea-03-32
10. Messaoudi SA, Talahmeh AA. Blowup in solutions of a quasilinear wave equation with variable-exponent nonlinearities. *Math Meth Appl Sci*. 2017;40(18):6976-6986. https://doi.org/10.1002/mma.4505
11. Messaoudi SA. On the decay of solutions of a damped quasilinear wave equation with variable-exponent nonlinearities. *Math Meth Appl Sci*. 2020;43(8):5114-5126. https://doi.org/10.1002/mma.6254
12. Antontsev S, Ferreira J. A nonlinear viscoelastic plate equation with $\rightarrow p$ Laplace operator: Blow up of solutions with negative initial energy. *Nonlinear Anal Real World Appl*. 2021;59:103240. https://doi.org/10.1016/j.nonrwa.2020.103240
13. Shahrouzi M. On behaviour of solutions for a nonlinear viscoelastic equation with variable-exponent nonlinearities. *Comp Math Appl*. 2018;75:3946-3956. https://doi.org/10.1016/j.camwa.2018.03.005
14. Shahrouzi M, Kargarfard F. Blow-up of solutions for a Kirchhoff type equation with variable-exponent nonlinearities. *J Appl Anal*. 2021;27(1):97-105. https://doi.org/10.1515/jaa-2020-2036
15. Shahrouzi M. Global nonexistence of solutions for a class of viscoelastic Lamé system. *Indian J Pure Appl Math*. 2020;51(4):1383-1397. https://doi.org/10.1007/s13226-020-0472-y
16. Mahdi FZ, Hakem A. Existence and blow up for a nonlinear viscoelastic hyperbolic problem with variable exponents. *Ser Math Inform*. 2020;35(3):647-672. https://doi.org/10.22190/FUMI2003647M
17. Ghegal S, Hamchi I, Messaoudi SA. Global existence and stability of a nonlinear wave equation with variable-exponent nonlinearities. *Appl Anal*. 2020;99(8):1333-1343. https://doi.org/10.1080/00036811.2018.1530760
18. Park SH, Kang JR. Blow-up of solutions for a viscoelastic wave equation with variable exponents. *Math Meth Appl Sci*. 2019;42:2083-2097. https://doi.org/10.1002/mma.5501
19. Messaoudi SA, Talahmeh AA. On wave equation: review and recent results. *Arab J Math*. 2018;7:113-145. https://doi.org/10.1007/s40065-017-0190-4
20. Diening L, Petteri H, Hasto P, et al. Lebesgue and Sobolev spaces with variable exponents. In Lecture Note Mathematics Vol 2017.
21. Edmunds D, Rakosnik J. Sobolev embeddings with variable exponent. *Stud Math*. 2000;143:267-293.
22. Edmunds D, Rakosnik J. Sobolev embeddings with variable exponent II. *Math Nachr*. 2002;246:53-67.
23. Xianling F, Zhao D. On the spaces $L^{p(x)}$ and $W^{m,p(x)}(\Omega)$. *J Math Anal Appl*. 2001;263:424-446. https://doi.org/10.1006/jmaa.2000.7617
24. Xianling F, Zhang QH. Existence of solutions for $p(x)-$Laplacian Dirichlet problem. *Nonlinear Anal*. 2003;52:1843-1852. https://doi.org/10.1016/S0362-546X(02)00150-5

How to cite this article: Shahrouzi M. General decay and blow up of solutions for a class of inverse problem with elasticity term and variable-exponent nonlinearities. *Math Meth Appl Sci*. 2022;45:1864-1878. https://doi.org/10.1002/mma.7891