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ORIGINAL ARTICLE

Thermoelastic laminated beam with nonlocal delay

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Abstract
This manuscript deals with a thermoelastic laminated Timoshenko beam with a nonlocal integral condition on the transversal displacement and thermal dissipation in the equation that describes the dynamical of rotate angle. Using the Hille–Yosida Theorem, we prove the existence, uniqueness, and regularity of the solution. For the asymptotic behavior, we apply the energy method. Using suitable multipliers, we construct a Lyapunov functional, and then we obtain the exponential stability. To the best of our knowledge, thermoelastic laminated Timoshenko beams with nonlocal time delay conditions have not been considered previously.

Keywords Laminated beam · Thermoelasticity · Delay · Asymptotic behavior

Mathematics Subject Classification Primary: 35A01 · 35B40

1 Introduction

A single beam or more than one is present in structures like mechanical engineering, electrical engineering, civil engineering, and aerospace engineering. S.P. Timoshenko [46] came up with a pioneer theory for a beam that is better suited for engineering

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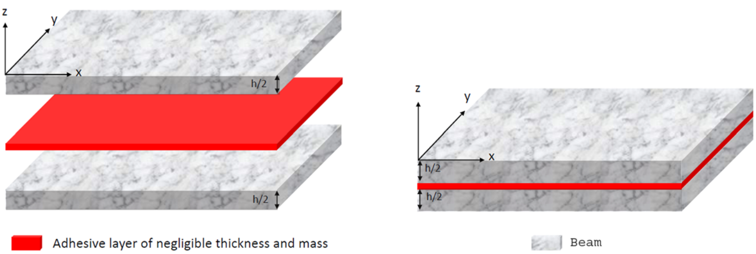

Fig. 1 Adhesive application (left) and beam in the press (right)

practice and is nowadays widely used for moderately thick beams. In Timoshenko's assumptions, the plane cross sections are perpendicular to the beam centerline and remain plane. Then, an additional kinematics variable representing the rotation angle of a filament of the beam ψ is added to the displacement assumptions u. Its mathematical formulation is given by a system of two partial differential equations

$$\rho A u_{tt} = \mathbb{S}_x \quad \text{and} \quad \rho I \psi_{tt} = \mathbb{M}_x - \mathbb{S}, \tag{1}$$

where M and S represent, respectively, the bending moment and the shear stress. The corresponding constitutive elastic laws are given by

$$\mathbb{S} = \kappa A(u_x + \psi) \quad \text{and} \quad \mathbb{M} = EI\psi_x, \tag{2}$$

where EI represents the flexural rigidity of the material, κ is a shear coefficient, ρ is the mass density, A and I denote the area and the inertial moment of a cross section.

The proposed linear model is a coupled partial differential equation obtained from (1) and (2) given by

$$\begin{aligned} \rho A u_{tt}(x,t) - \kappa A(u_x - \kappa A\psi)_x = 0, \quad & x \in (0,L),\ t > 0, \\ \rho I_\rho \varphi_{tt}(x,t) - EI\psi_{xx}(x,t) - \kappa A(u_x - \psi) = 0, \quad & x \in (0,L),\ t \geq 0, \end{aligned}$$

where $t > 0$ is the time variable, x is the space coordinate along the beam, of length L, in its equilibrium position. The coefficients I_ρ, E, and K are the polar moment of inertia of a cross section, Young's modulus of elasticity, and the shear modulus, respectively.

Hansen and Spies [14, 16] derived from Timoshenko's theory, a system of equations that describes the dynamics of a structure given by two identical Timoshenko beams with an adhesive layer (of negligible thickness and mass) bonding the two adjoining surfaces (Fig. 1).

The model derived is

$$\begin{aligned} \varrho u_{tt} + G(\psi - u_x)_x = 0,\ & x \in (0,L),\ t \geq 0, \\ I_\varrho(3S_{tt} - \psi_{tt}) - D(3S_{xx} - \psi_{xx}) - G(\psi - u_x) = 0,\ & x \in (0,L),\ t \geq 0, \\ 3I_\varrho S_{tt} - 3DS_{xx} + 3G(\psi - u_x) + 4\delta_0 S + 4\gamma_0 S_t = 0,\ & x \in (0,L),\ t \geq 0, \end{aligned} \tag{3}$$

where $u = u(x, t)$ is the transverse displacement, $\psi = \psi(x, t)$ is the rotation angle, $S = S(x, t)$ is proportional to the amount of slip along the interface. The positive parameters ϱ, I_ϱ, G, D, and $4\delta_0$, are the density, mass moment of inertia, shear stiffness, flexural rigidity, and adhesive stiffness, respectively. The non-negative parameter $4\gamma_0$ is called the adhesive damping, and S_t is the structural damping of the system. Laminated beams have gained popularity and importance in science and engineering fields.

When the Timoshenko system has not damped in all components, the system decays exponentially just under the so-called "equal wave speeds", see [27]. If the dampings are added in all equations, the energy of the system decay exponentially without none assumption over the coefficients relations, see [36]. In recent years, the control of partial differential equations with time delay effects has aroused the interest of several researchers. In [15], was studied the stability of Timoshenko's beam system with boundary time delay. In [45], the authors considered the interior damping and boundary delay. For exponential stability of the wave equation with boundary time-varying delay we cite [32]. In [20], the authors obtained the well-posedness and exponential stability for Timoshenko beam with delay on the frictional damping under the condition $\mu_1 > \mu_2 > 0$ and $\tau(t) = t$. In [19] was extended the result of [20] for $\tau(t)$ a time-varying function. For a transmission problem in the presence of memory and delay terms, under an appropriate hypothesis on the relaxation function and the relationship between the weight of the damping and the weight of the delay, in [22] was proved well-posedness by using the semigroup theory a decay result by introducing a suitable Lyapunov functional.

Time delays so often arise in many physical, chemical, biological, and economical phenomena, see [43] and references therein. Timoshenko's beam system, without delay, has been extensively studied by several authors. We can cite a few of them, [1, 2, 11, 13, 18, 25, 28, 29, 40, 41]. For Timoshenko system with delay, we cite [3, 12, 35].

From the laminated beam model introduced by Hansen in 1984, several works were produced in the last years, see for instance: [7, 10, 23, 24, 34, 38, 39, 44]. In [38], authors assumed that a container is fastened securely on the left; while on the right, it is free and has an attached container. Using the semigroup approach and a result of Borichev and Tomilov, they proved that the solution is polynomially stable. In another setting in [10], involving a nonlinear foundation, authors established the existence of smooth finite-dimensional global attractors for the corresponding solution semigroup.

Wang et al. [47] proved for the system (3) that the frictional damping S_t created by the interfacial slip alone is not enough to stabilize the system exponentially to its equilibrium state. Naturally, the question arises of studying the influence of additional stabilizing mechanisms on the model and including the situation without S_t.

The following full damped system, derived of (3), with frictional dissipation in transversal displacement and in the rotation angle, was considered in [34]

$$\begin{aligned} \rho_1 u_{tt} + k(\psi - u_x)_x + \alpha u_t = 0,\\ \rho_2 (s-\psi)_{tt} - b(s-\psi)_{xx} - k(\psi - u_x) + \beta(s-\psi)_t = 0,\\ \rho_2 s_{tt} - b s_{xx} + 3k(\psi - u_x) + 4\delta s + 4\gamma s_t = 0, \end{aligned} \tag{4}$$

and established the exponential stability.

In [5] was done $\alpha = \gamma = 0$ in (4) and considered a single control in the form of a frictional damping ψ_t on the rotation angle,

$$\begin{aligned} \rho w_{tt} + G(\psi - w_x)_x &= 0, \\ I_\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - w_x) + \delta(3s - \psi)_t &= 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s &= 0, \end{aligned} \tag{5}$$

and proved that the unique dissipation through the frictional damping is strong enough to get exponential stability of the energy.

Whenever energy is transmitted from one place to another, there is a delay associated with the transmission, see [42]. A very delicate question to consider in the transmission of energy is that the delay can become a source of instability. A small delay in boundary control could turn the well-behaved hyperbolic system into a wild one, see [9].

In [4] was considered in the following thermoelastic laminated beam just one dissipation through heat conduction in the interfacial slip equation,

$$\begin{aligned} \rho w_{tt} + G(\psi - w_x)_x &= 0, \\ I_\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - w_x) &= 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + \delta\theta_x &= 0, \\ \rho_3\theta_t - \alpha\theta_{xx} + \delta s_{xt} &= 0. \end{aligned} \tag{6}$$

The author proved that this unique dissipation is strong enough to stabilize exponentially all the system provided the wave speeds of the system are equal. The result (6), in a way, extends previous works where additional internal or boundary controls were used together with frictional damping in the interfacial slip.

On the above scenario, the present paper is concerned with a nonlocal time delay, given by,

$$\begin{aligned} \rho u_{tt} + G(\psi - u_x)_x + \mu_1 u_t + \int_{\tau_1}^{\tau_2} \mu_2(\tau) u_t(x, t-\tau)\, d\tau &= 0, \quad (x,t) \in (0,L)\times(0,\infty), \\ I_\rho(3S - \psi)_{tt} - D(3S - \psi)_{xx} - G(\psi - u_x) + b\theta_x &= 0, \quad (x,t) \in (0,L)\times(0,\infty), \\ 3I_\rho S_{tt} - 3DS_{xx} + 3G(\psi - u_x) + 4\delta S &= 0, \quad (x,t) \in (0,L)\times(0,\infty), \\ k\theta_t - \alpha\theta_{xx} + b(3S - \psi)_{xt} &= 0, \quad (x,t) \in (0,L)\times(0,\infty), \end{aligned} \tag{7}$$

where τ_1, τ_2 are real numbers such that $0 \leq \tau_1 < \tau_2$, with Dirichlet–Neumann boundary conditions

$$u_x(0,t) = \psi(0,t) = S(0,t) = \theta_x(0,t) = u(L,t) = \psi_x(L,t) = S_x(L,t) = \theta(L,t) = 0, \tag{8}$$

and initial data

$$\begin{aligned} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \\ S(x,0) = S_0(x),\ S_t(x,0) = S_1(x),\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x), \\ u_t(x,-\tau) = f_0(x,\tau),\ \tau \in (\tau_1,\tau_2), \end{aligned} \tag{9}$$

with f_0 belong to a suitable Sobolev space. The function $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ are bounded satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(\tau)|\, d\tau < \mu_1. \tag{10}$$

We aim to study the well-posedness and take into account the stability number

$$\chi = \frac{G}{\rho} - \frac{D}{I_\rho}, \tag{11}$$

we will prove the exponential decay of the system. Note that $\chi = 0$ means that the wave speeds of the Eqs. of $(7)_1$, $(7)_2$ and $(7)_3$ are equal. Then, there is an effective transfer of the energy from the damped equation to the undamped one, and the delay can be a source of instability for all system.

By following [27], we assume that a thermal dissipation is applied on the bending moment. Note that $\mu_1 = 0$ and $\mu_2(\tau) = 0$ leads a problem equivalent to (6). The history of nonlocal problems with integral conditions for partial differential equations is recent and goes back to [8]. In [6], a review of the progress in the nonlocal models with integral type was given with many discussions related to physical justifications, advantages, and numerical applications. For a nonlocal problem for a hyperbolic equation with integral conditions of the 1st kind, we cite [33]. Well-posedness and exponential stability for a wave equation with nonlocal time-delayed were studied in [31, 37] by different techniques.

The paper is organized as follows. In the next section, we reformulate the system (7) introducing the new variable as in [30] and prove that the energy of the system is dissipative. In the Sect. 3, we present the semigroup configuration and using Hille–Yosida Theorem, we prove the existence, uniqueness, and regularity of the solution. In the Sect. 4, we prove the exponential stability. The tool is the energy method that consists of constructing a Lyapunov functional for the system.

2 Preliminaries

As in [30], we introduce the new variable

$$z(x,\eta,t,\tau) = u_t(x, t-\eta\tau) \text{ in } (0,L)\times(0,1)\times(0,\infty)\times(\tau_1,\tau_2).$$

It is easily verified that the new variable satisfies

$$\tau z_t(x,\eta,t,\tau)+z_\eta(x,\eta,t,\tau)=0 \text{ in } (0,L)\times(0,1)\times(0,\infty)\times(\tau_1,\tau_2). \quad (12)$$

Following the idea of [47], we denote the effective rotation angle by $\xi := 3S-\psi$. By (12), the differential equations (7) can be rewritten as follows:

$$\begin{aligned}
\rho u_{tt}+G(3S\psi-u_x)_x+\mu_1 u_t+\int_{\tau_1}^{\tau_2}\mu_2(\tau)z(x,1,t,\tau)\,d\tau &=0,\\
I_\rho\xi_{tt}-D\xi_{xx}-G(3S\psi-u_x)+b\theta_x &=0,\\
3I_\rho S_{tt}-3DS_{xx}+3G(3S\psi-u_x)+4\delta S &=0,\\
k\theta_t-\alpha\theta_{xx}+b(3S-\psi)_{xt} &=0,\\
\tau z_t+\xi_{xt} &=0,
\end{aligned} \quad (13)$$

subject to boundary conditions given in (8), that is,

$$u_x(0,t)=\xi(0,t)=S(0,t)=\theta_x(0,t)=u(L,t)=\xi_x(L,t)=S_x(L,t)=\theta(L,t)=0 \quad (14)$$

and initial conditions

$$\begin{aligned}
&u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x),\ \ \xi(x,0)=\xi_0(x),\ \ \xi_t(x,0)=\xi_1(x),\\
&S(x,0)=S_0(x),\ \ S_t(x,0)=S_1(x),\ \ \theta(x,0)=\theta_0(x),\ \ \theta_t(x,0)=\theta_1(x),\\
&z(x,\eta,0,\tau)=f_0(x,-\eta\tau)=u_2(x,\eta,\tau).
\end{aligned} \quad (15)$$

We define the energy of the solution of problem (13)-(15) by

$$\begin{aligned}
E(t)=\frac{1}{2}\int_0^L\Big[&\rho u_t^2+G(3S-\xi-u_x)^2+I_\rho\xi_t^2+D\xi_x^2+3I_\rho S_t^2\\
&+3DS_x^2+4\delta S^2+k\theta^2+\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|z^2\,d\tau\,d\eta\Big]dx.
\end{aligned} \quad (16)$$

Our first result states that the energy is a nonincreasing function and uniformly bounded above by $E(0)$.

Lemma 2.1 *Let $(u(t),u_t(t),\xi(t),\xi_t(t),S(t),S_t(t),\theta(t),z(t))$ be a solution to the system (13)-(15). Then, the energy functional defined by (16) satisfies*

$$\frac{d}{dt}E(t)\le -m_0\int_0^L u_t^2dx-\alpha\int_0^L\theta_x^2\,dx\le 0, \quad (17)$$

where $m_0=\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\,d\tau>0.$

Proof Multiplying (13)$_{1,2,3,4}$ by u_t, ξ_t, S_t and θ, respectively, integrating by parts each over $(0, L)$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^L \rho u_t^2\,dx - G\int_0^L (3S-\xi-u_x)u_{xt}\,dx\\&+\mu_1\int_0^L u_t^2\,dx + \int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|z(x,1,t,\tau)u_t\,dx = 0,\end{aligned} \tag{18}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left(I_\rho\xi_t^2 + D\xi_x^2\right)dx - G\int_0^L (3S-\xi-u_x)\xi_t\,dx + b\int_0^L \theta_x\xi_t\,dx = 0, \tag{19}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left(3I_\rho S_t^2 + 3DS_x^2 + 4\delta S^2\right)dx + 3G\int_0^L (3S-\xi-u_x)S_t\,dx = 0, \tag{20}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^L k\theta^2\,dx + \alpha\int_0^L \theta_x^2\,dx - b\int_0^L \xi_t\theta_x\,dx = 0. \tag{21}$$

Now, multiplying (13)$_5$ by $|\mu_2(\tau)|z$ and integrating by parts over $(0, L)\times(0, 1)\times(\tau_1, \tau_2)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|z^2(x,\eta,t,\tau)\,d\tau\,d\eta\,dx\\&\quad= -\frac{1}{2}\frac{d}{dt}\int_0^L\int_{\tau_1}^{\tau_2}\int_0^1|\mu_2(\tau)|\frac{\partial}{\partial\eta}z^2(x,\eta,t,\tau)\,d\eta\,d\tau\,dx\\&\quad= -\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|z^2(x,1,t,\tau)\,d\tau\,dx + \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|u_t^2\,d\tau\,dx.\end{aligned} \tag{22}$$

Combining (18)–(22), we arrive at

$$\begin{aligned}\frac{d}{dt}E(t) =& -\left(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\,d\tau\right)\int_0^L u_t^2 dx - \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|z^2(x,1,t,\tau)\,d\tau\,dx\\&-\int_0^L\int_{\tau_1}^{\tau_2}\mu_2(\tau)z(x,1,t,\tau)u_t\,d\tau\,dx.\end{aligned} \tag{23}$$

Using Young's inequality, we see that last term in (23) satisfies

$$\begin{aligned}-\int_0^L\int_{\tau_1}^{\tau_2}\mu_2(\tau)z(x,1,t,\tau)u_t\,d\tau\,dx \le& \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\,d\tau\int_0^L u_t^2\,dx\\&+\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|z^2(x,1,t,\tau)\,d\tau\,dx.\end{aligned} \tag{24}$$

We complete the proof of (17) by substituting (24) in (23), and using (10). □

3 Well-posedness

In this section, we give an existence and uniqueness result for problem (13)-(15) using the semigroup theory. Introducing the vector function $U = (u, w, \xi, v, S, y, \theta, z)^T$, where $w = u_t,\ v = \xi_t$ and $y = S_t$, the system (13)-(15) can be written as

$$
\begin{aligned}
&U_t - \mathcal{A}U = 0,\\
&U(x,0) = U_0(x) = (u_0, u_1, \xi_0, \xi_1, S_0, S_1, \theta_0, u_2)^T,
\end{aligned}
\tag{25}
$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is a linear operator defined by

$$
\mathcal{A}U = \begin{pmatrix}
w\\
-\rho^{-1}\left[G(3S-\xi-u_x)_x + \mu_1 w + \int_{\tau_1}^{\tau_2}\mu_2(\tau)z(x,1,t,\tau)\,d\tau\right]\\
v\\
I_\varrho^{-1}\left[D\xi_{xx} + G(3S-\xi-u_x) - b\theta_x\right]\\
y\\
I_\varrho^{-1}\left[DS_{xx} - G(3S-\xi-u_x) - \frac{4\delta}{3}S\right]\\
k^{-1}\left(\alpha\theta_{xx} - \xi_{xt}\right)\\
-\tau^{-1}z_\eta
\end{pmatrix}.
$$

We consider the following spaces

$$
\begin{aligned}
H_a^1 &:= \{f \in H^1(0,L);\ f(L) = 0\},\\
H_b^1 &:= \{f \in H^1(0,L);\ f(0) = 0\},\\
L_*^2 &:= \{f \in L^2(0,L);\ f(L) = 0\}.
\end{aligned}
$$

Let

$$
\mathcal{H} = H_a^1 \times L^2(0,L) \times \left[H_b^1 \times L^2(0,L)\right]^2 \times H_a^1 \times L_*^2 \times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2))
$$

be the Hilbert space equipped with the following inner product

$$
\begin{aligned}
\langle U, \tilde{U}\rangle_{\mathcal{H}} =& \rho\int_0^L w\tilde{w}\,dx + D\int_0^L \xi_x\tilde{\xi}_x\,dx + I_\varrho\int_0^L v\tilde{v}\,dx + 3D\int_0^L S_x\tilde{S}_x\,dx\\
&+ 4\delta\int_0^L S\tilde{S}\,dx + 3I_\varrho\int_0^L y\tilde{y}\,dx + G\int_0^L (3S-\xi-u_x)(3\tilde{S}-\tilde{\xi}-\tilde{u}_x)\,dx\\
&+ k\int_0^L \theta\tilde{\theta}\,dx + \int_0^L\int_0^1\int_{\tau_1}^{\tau_2}\tau\mu_2(\tau)z\tilde{z}\,d\tau\,d\eta\,dx,
\end{aligned}
\tag{26}
$$

for $U = (u, w, \xi, v, S, y, \theta, z)^T$ and $\tilde{U} = (\tilde{u}, \tilde{w}, \tilde{\xi}, \tilde{v}, \tilde{S}, \tilde{y}, \tilde{\theta}, \tilde{z})^T$.

The domain of $\mathcal{A}$ is defined by

$$D(\mathcal{A})=\left\{(u,w,\xi,v,S,y,\theta,z)\in\mathcal{H}\left|\begin{array}{l}u,\xi,S\in H^2(0,L),\ \theta,w\in H_a^1,\ \xi,S\in H_b^1\\ z\in L^2\Big((\tau_1,\tau_2);\,H^1((0,L)\times(0,1))\Big)\\ z(\cdot,0,\cdot)=w\text{ in }(0,L)\\ u_x(0)=\xi(L)=S_x(L)=\theta_x(0)=0\end{array}\right.\right\}. \tag{27}$$

Note that $D(\mathcal{A})$ is independent of $t>0$. Furthermore, clearly $D(\mathcal{A})$ dense in $\mathcal{H}$.

To prove the existence and uniqueness of solutions, observe that, as $E(t)=\frac{1}{2}\|U\|_{\mathcal{H}}^2$, for all $U\in D(\mathcal{A})$, a simple differentiation gives

$$\langle U_t,U\rangle_{\mathcal{H}}=\frac{d}{dt}E(t)\Leftrightarrow\langle\mathcal{A}U,U\rangle_{\mathcal{H}}=\frac{d}{dt}E(t). \tag{28}$$

Then, from (17), we obtain

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}}\leq-m_0\int_0^L u_t^2dx-\alpha\int_0^L\theta_x^2\,dx\leq 0, \tag{29}$$

where m_0 is defined in (2.1). Hence, the operator $\mathcal{A}$ is dissipative.

The Hille–Yosida Theorem give us the conditions for a linear (unbounded) operator $\mathcal{A}$ to be generator of a C_0-semigroup of contractions $\mathcal{S}(t)$ in a Banach space.

Theorem 3.1 *(Hille-Yosida) A linear (unbounded) operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $\mathcal{S}(t)$, $t\geq 0$, if and only if,*

(i)$\mathcal{A}$is closed and$\overline{D(\mathcal{A})}=\mathcal{H}$,

(ii) the resolvent set$\rho(\mathcal{A})$of$\mathcal{A}$contains$\mathbb{R}^+$and for every$\lambda>0$, $\|(\lambda I-\mathcal{A})^{-1}\|\leq\frac{1}{\lambda}$.

Proof See [26, Theorem 3.1, page 8] □

However, for Hilbert space the Hille–Yosida Theorem leads to the following result, see [21, Theorem 1.2.2, page 3].

Theorem 3.2 *Let $\mathcal{A}$ be a densely defined linear operator on a Hilbert space $\mathcal{H}$. Then, $\mathcal{A}$ generates a C_0-semigroup of contractions $\mathcal{S}(t)$ on $\mathcal{H}$ if and only if $\mathcal{A}$ is dissipative and $R(I-\mathcal{A})=\mathcal{H}$.*

Using the result above, we prove the following:

Lemma 3.3 *$\mathcal{A}$ generates a C_0-semigroup of contractions $\mathcal{S}(t)$ on $\mathcal{H}$.*

Proof Since $\mathcal{A}$ is dissipative and $D(\mathcal{A})$ is dense in $\mathcal{H}$, to prove that $\mathcal{A}$ generates a C_0-semigroup of contractions $\mathcal{S}(t)$ on $\mathcal{H}$ it is sufficient to show that $R(I-\mathcal{A})=\mathcal{H}$.

Given

$$F = (f_1, \ldots, f_8)^T \in \mathcal{H},$$

we must show that there exists

$$U = (u, w, \xi, v, S, y, \theta, z)^T \in D(\mathcal{A})$$

satisfying

$$(I - \mathcal{A})U = F$$

which is equivalent to

$$\begin{aligned} u - w &= f_1, \\ \rho w + G(3S - \xi - u_x)_x + \mu_1 w + \int_{\mu_1}^{\mu_2} \mu_2(\tau) z(x, 1, t, \tau)\, d\tau &= \rho f_2, \\ \xi - v &= f_3, \\ I_\rho v - D\xi_{xx} - G(3S - \xi - u_x) + b\theta_x &= I_\rho f_4, \\ S - y &= f_5, \\ 3I_\rho y - 3DS_{xx} + 3G(3S - \xi - u_x) + 4\delta S &= 3I_\rho f_6, \\ k\theta - \alpha\theta_{xx} + bv_x &= kf_7, \\ \tau z + z_\eta &= \tau f_8. \end{aligned} \tag{30}$$

Suppose that we have found u, ξ and S with the appropriated regularity. Therefore, (30)$_{1,3,5}$ give

$$\begin{aligned} w &= u - f_1, \\ v &= \xi - f_3, \\ y &= S - f_5. \end{aligned} \tag{31}$$

It is clear that $w \in H_a^1$ and $v,\ y \in H_b^1$. Furthermore, following the same approach as in [31], we obtain that

$$z(x, \eta, \tau) = w(x)e^{-\eta\tau} + \tau e^{-\eta\tau} \int_0^\eta e^{\sigma\tau} f_8(x, \sigma, \tau)\, d\sigma$$

is solution of the (30)$_8$ satisfying

$$z(x, 0, \tau) = w(x), \quad \text{for } x \in (0, L),\ \tau \in (\tau_1, \tau_2). \tag{32}$$

So, from (31)$_1$,

$$z(x, \eta, \tau) = u(x)e^{-\eta\tau} - f_1(x)e^{-\eta\tau} + \tau e^{-\eta\tau} \int_{\tau_1}^{\tau_2} e^{\sigma\tau} f_8(x, \sigma, \tau)\, d\sigma,$$

and, in particular,

$$z(x,1,\tau)=u(x)e^{-\tau}+z_0(x,\tau), \tag{33}$$

where $z_0(x,\tau)\in L^2((0,L)\times(\tau_1,\tau_2))$ defined by

$$z_0(x,\tau)=-f_1(x)e^{-\tau}+\tau e^{-\tau}\int_0^1 e^{\sigma\tau}f_8(x,\sigma,\tau)\,d\sigma.$$

By (30) and (31), we see that the functions u, ξ, S and θ satisfy the following system

$$\begin{aligned} \varsigma u+G(3S-\xi-u_x)_x&=h_1,\\ I_\rho\xi-D\xi_{xx}-G(3S-\xi-u_x)+b\theta_x&=h_2,\\ \gamma S-3DS_{xx}+3G(3S-\xi-u_x)&=h_3,\\ k\theta_x-\alpha\theta_{xx}+b\xi_x&=h_4, \end{aligned} \tag{34}$$

where

$$\begin{aligned} \varsigma=\rho+\mu_1+\int_{\tau_1}^{\tau_2}\mu_2(\tau)e^{-\tau}\,d\tau,\quad \gamma=3I_\rho+4\delta,\\ h_1=(\rho-\mu_1)f_1+\rho f_2-\int_{\tau_1}^{\tau_2}\mu_2(\tau)z_0(x,\tau)\,d\tau,\quad h_2=I_\rho f_3+I_\rho f_4,\\ h_3=3I_\rho f_5+3I_\rho f_6\quad\text{and}\quad h_4=bf_{3,x}+kf_7. \end{aligned}$$

Solving the system (34) is equivalent to finding

$$(u,\xi,S,\theta)\in H^2(0,L)\cap H_a^1\times\left(H^2(0,L)\cap H_b^1\right)^2\times H_a^1,$$

such that

$$\begin{aligned} \varsigma\int_0^L u\tilde{u}\,dx-G\int_0^L(3S-\xi-u_x)\tilde{u_x}\,dx&=\int_0^L h_1\tilde{u}\,dx,\\ I_\rho\int_0^L\xi\tilde{\xi}\,dx+D\int_0^L\xi_x\tilde{\xi}_x\,dx-G\int_0^L(3S-\xi-u_x)\tilde{\xi}\,dx+b\int_0^L\theta_x\tilde{\xi}\,dx&=\int_0^L h_2\tilde{\xi}\,dx,\\ \gamma\int_0^L S\tilde{S}\,dx+3D\int_0^L S_x\tilde{S}_x\,dx+3G\int_0^L(3S-\xi-u_x)\tilde{S}\,dx&=\int_0^L h_3\tilde{S}\,dx,\\ k\int_0^L\theta\tilde{\theta}\,dx+\alpha\int_0^L\theta_x\tilde{\theta}_x\,dx+b\int_0^L\xi_x\tilde{\theta}\,dx&=\int_0^L h_4\tilde{\theta}\,dx, \end{aligned} \tag{35}$$

for all $\left(\tilde{u},\tilde{\xi},\tilde{S},\tilde{\theta}\right)\in H_a^1\times H_b^1\times H_b^1\times L_*^2$.

Now, we observe that solving the system (35) is equivalent to solve the problem

$$\Upsilon\left((u,\xi,S,\theta),(\tilde{u},\tilde{\xi},\tilde{S},\tilde{\theta})\right)=L(\tilde{u},\tilde{\xi},\tilde{S},\tilde{\theta}), \tag{36}$$

where the bilinear form

$$\Upsilon : [H_a^1 \times H_b^1 \times H_b^1 \times L_*^2]^2 \to \mathbb{R}$$

and the linear form

$$L : H_a^1 \times H_b^1 \times H_b^1 \times L_*^2 \to \mathbb{R}$$

are defined by

$$\begin{aligned}\Upsilon\left((u,\xi,S,\theta),(\tilde{u},\tilde{\xi},\tilde{S},\tilde{\theta})\right) = &\, \varsigma\int_0^L u\tilde{u}\,dx + G\int_0^L (3S-\xi-u_x)(3\tilde{S}-\tilde{\xi}-\tilde{u}_x)\,dx \\ &+ I_\rho\int_0^L \xi\tilde{\xi}\,dx + D\int_0^L \xi_x\tilde{\xi}_x\,dx + b\int_0^L \theta_x\tilde{\xi}\,dx \\ &+ \gamma\int_0^L S\tilde{S}\,dx + 3D\int_0^L S_x\tilde{S}_x\,dx + k\int_0^L \theta\tilde{\theta}\,dx \\ &+ \alpha\int_0^L \theta_x\tilde{\theta}_x\,dx + b\int_0^L \xi_x\tilde{\theta}\,dx\end{aligned}$$

and

$$\begin{aligned}L\left(\tilde{u},\tilde{\xi},\tilde{S},\tilde{\theta}\right) &= \int_0^L h_1\tilde{u}\,dx + \int_0^L h_2\tilde{\xi}\,dx + \int_0^L h_3\tilde{S}\,dx + \int_0^L h_4\tilde{\theta}\,dx \\ &= \left((h_1,h_2,h_3,h_4),\ (\tilde{u},\tilde{\xi},\tilde{S},\tilde{\theta})\right).\end{aligned}$$

Now, we introduce the Hilbert space $V = H_a^1 \times H_b^1 \times H_b^1 \times L_*^2$ equipped with the norm

$$\begin{aligned}\|(u,\xi,S,\theta)\|_V^2 = &\|u\|_{L^2(0,L)}^2 + \|3S-\xi-u_x\|_{L^2(0,L)}^2 + \|\xi_x\|_{L^2(0,L)}^2 \\ &+ \|S_x\|_{L^2(0,L)}^2 + \|\theta_x\|_{L^2(0,L)}^2.\end{aligned}$$

It is clear that Υ and L are bounded. Furthermore, using integration by parts, we can obtain that there exists a positive constant m such that

$$\begin{aligned}\Upsilon\left((u,\xi,S,\theta),(u,\xi,S,\theta)\right) = &\, \varsigma\int_0^L u^2\,dx + G\int_0^L (3S-\xi-u_x)^2\,dx + I_\rho\int_0^L \xi^2\,dx \\ &+ D\int_0^L \xi_x^2\,dx + \gamma\int_0^L S^2\,dx + 3D\int_0^L S_x^2\,dx \\ &+ k\int_0^L \theta^2\,dx + \alpha\int_0^L \theta_x^2\,dx \\ \geq &\, m\,\|(u,\xi,S,\theta)\|_V^2,\end{aligned}$$

which implies that Υ is V-elliptic.

Hence, we assert that Υ is continuous and V-elliptic bilinear form on $V \times V$, and L is continuous form on V. We are in conditions of the Lax-Milgram Theorem, [17, Theorem 3.1.4, page 115].

Theorem 3.4 *(Lax-Milgram) Let V be a Hilbert space and $\Upsilon(\cdot\,,\,\cdot)$ a continuous and V-elliptic bilinear form on $V \times V$. Then, given $f \in V$, there exists a unique $u \in V$ such that $\Upsilon(u\,,\,v) = (f\,,\,v)\,,\ \forall v \in V$.*

So, applying the Lax–Milgram Theorem, we deduce that for all

$$\left(\tilde{u}, \tilde{\xi}, \tilde{S}, \tilde{\theta}\right) \in H_a^1 \times H_b^1 \times H_b^1 \times L_*^2,$$

the problem (36) admits a unique solution

$$(u, \xi, S, \theta) \in H_a^1 \times H_b^1 \times H_b^1 \times L_*^2.$$

Applying the classical elliptic regularity, it follows from (35) that

$$(u, \xi, S, \theta) \in H^2(0, L)^3 \times H_a^1.$$

On the other hand, $(35)_1$ also holds true for any $\varphi \in C^1(0, L)$ with $\varphi(L) = 0$, then

$$\varsigma \int_0^L u\varphi\, dx + 3G \int_0^L S_x\varphi\, dx - G \int_0^L u_{xx}\varphi\, dx = \int_0^L h_1\varphi\, dx,$$

which, using integration by parts, implies

$$G u_x(0)\varphi(0) = 0.$$

Hence,

$$u_x(0) = 0.$$

Similarly, we can get

$$\xi_x(L) = S_x(L) = \theta_x(0) = 0.$$

Therefore, the operator $I - \mathcal{A}$ is surjective, that is, $R(I - \mathcal{A}) = \mathcal{H}$ and then $\mathcal{A}$ generates a C_0-semigroup of contractions $\mathcal{S}(t) = e^{t\mathcal{A}}$ on $\mathcal{H}$. □

Thus, we have the following result of existence and uniqueness:

Theorem 3.5 *Let $U_0 \in \mathcal{H}$, then there exists a unique weak solution U of (25) satisfying*

$$U \in C([0, \infty); \mathcal{H}). \tag{37}$$

Moreover, if $U_0 \in D(\mathcal{A})$, *then*

$$U \in C([0,\infty); D(\mathcal{A})) \cap C^1([0,\infty); \mathcal{H}). \tag{38}$$

In this case, it is called a strong solution.

Proof From semigroup theory, $U(t) = e^{t\mathcal{A}} U_0$ is the unique solution of (25) satisfying (37) and (38). The proof is complete. □

4 Asymptotic behavior

In this section, our objective is to show the exponential stability of the system (13)-(15). For our goal, we need to use the stability number (11).

4.1 Technical lemmas

Lemma 4.1 *Consider*

$$I_1(t) = I_\rho \int_0^L \xi \xi_t \, dx. \tag{39}$$

Let

$$U(t) = (u(t), u_t(t), \xi(t), \xi_t(t), S(t), S_t(t), \theta(t), z(t))$$

be a solution of (13)–(15). Then the functional I_1, *satisfies the estimative*

$$\frac{d}{dt} I_1(t) \leq -\frac{D}{2} \int_0^L \xi_x^2 \, dx + I_\rho \int_0^L \xi_t^2 \, dx + c_1 \int_0^L (3S - \xi - u_x)^2 \, dx \\ + c_1 \int_0^L \theta_x^2 \, dx, \tag{40}$$

for any constant $c_1 > 0$.

Proof Differentiating $I_1(t)$, using (13) and integration by parts, we arrive at

$$\frac{d}{dt} I_1(t) = I_\rho \int_0^L \xi_t^2 \, dx - D \int_0^L \xi_x^2 \, dx \\ + G \int_0^L \xi(3S - \xi - u_x) \, dx - b \int_0^L \xi \theta_x \, dx$$

Estimate (40) follows thanks to Young's and Poincaré's inequalities. □

Now, let us introduce the functional

$$I_2(t) = 3I_\rho \int_0^L S S_t \, dx. \tag{41}$$

Lemma 4.2 *Let*

$$U(t) = (u(t), u_t(t), \xi(t), \xi_t(t), S(t), S_t(t), \theta(t), z(t))$$

be a solution of (13)-(15). Then the functional I_2, satisfies the following estimative

$$\begin{aligned} \frac{d}{dt} I_2(t) \leq -\delta \int_0^L S^2 \, dx - 3D \int_0^L S_x^2 \, dx \\ + I_\rho \int_0^L S_t^2 \, dx + c_2 \int_0^L (3S - \xi - u_x)^2 \, dx, \end{aligned} \tag{42}$$

for any constant $c_2 > 0$.

Proof By differentiating $I_2(t)$, using (13) together with integration by parts, we obtain

$$\frac{d}{dt} I_2(t) = I_\rho \int_0^L S_t^2 \, dx - 3D \int_0^L S_x^2 \, dx - 3G \int_0^L (3S - \xi - u_x) S \, dx - 4\delta \int_0^L S^2 \, dx.$$

We then use Young's inequality to obtain (42). □

Now, we introduce another functional

$$I_3(t) = \frac{k I_\rho}{b} \int_0^L \xi_t \int_0^x \theta(r) \, dr \, dx. \tag{43}$$

Lemma 4.3 *Let*

$$U(t) = (u(t), u_t(t), \xi(t), \xi_t(t), S(t), S_t(t), \theta(t), z(t))$$

be a solution of (13)-(15). Then the functional I_3, satisfies

$$\begin{aligned} \frac{d}{dt} I_3(t) \leq & -\frac{I_\rho}{2} \int_0^L \xi_t^2 \, dx + \varepsilon_3 \int_0^L (3S - \xi - u_x)^2 \, dx \\ & + \varepsilon_3 \int_0^L \xi_x^2 \, dx + c_3 \left(1 + \frac{1}{\varepsilon_3}\right) \int_0^L \theta_x^2 \, dx \end{aligned} \tag{44}$$

for any constants $\varepsilon_3 > 0$ and $c_3 > 0$.

Proof We differentiate $I_3(t)$, use (13) and integrating by parts, yield

$$\frac{d}{dt}I_3(t) = -\frac{kD}{b}\int_0^L \xi_x\theta\,dx + \frac{kG}{b}\int_0^L (3S-\xi-u_x)\int_0^x \theta(r)\,dr\,dx$$
$$+k\int_0^L \theta^2\,dx + \frac{\alpha I_\rho}{b}\int_0^L \xi_t\theta_x\,dx - I_\rho\int_0^L \xi_t^2\,dx.$$

Exploiting Young's, Poincaré's, and Cauchy-Schwarz inequalities, we have the estimates (44) and conclude the prove. □

Now, let us consider the following functional

$$I_4(t) = -I_\rho\int_0^L \xi_t(3S-\xi-u_x)\,dx + \frac{\rho D}{G}\int_0^L u_t\xi_x\,dx. \tag{45}$$

Lemma 4.4 *Assume that $\chi = 0$ holds. Let*

$$U(t) = (u(t), u_t(t), \xi(t), \xi_t(t), S(t), S_t(t), \theta(t), z(t))$$

be a solution of (13)-(15). Then, the functional I_4, satisfies the estimative

$$\begin{aligned}\frac{d}{dt}I_4(t) \le &-\frac{G}{2}\int_0^L (3S-\xi-u_x)^2\,dx + \varepsilon_4\int_0^L \xi_x^2\,dx + \varepsilon_4\int_0^L S_t^2\,dx\\ &+\frac{c_4}{\varepsilon_4}\int_0^L u_t^2\,dx + \frac{c_4}{\varepsilon_4}\int_0^L\int_{\tau_1}^{\tau_2} |\mu_2(\tau)|z^2(x,1,t,\tau)\,d\tau\,dx\\ &+c_4\int_0^L \theta_x^2\,dx + c_4\left(1+\frac{1}{\varepsilon_4}\right)\int_0^L \xi_t^2\,dx.\end{aligned} \tag{46}$$

for any constants $\varepsilon_4 > 0$ and $c_4 > 0$.

Proof Derivative of $I_4(t)$, using (13) and integrating by parts, yields

$$\begin{aligned}\frac{d}{dt}I_4(t) = &-G\int_0^L (3S-\xi-u_x)^2\,dx + b\int_0^L \theta_x(3S-\xi-u_x)\,dx - I_\rho\int_0^L \xi_t\psi_t\,dx\\ &-\frac{\mu_1 D}{G}\int_0^L u_t\xi_x\,dx - \frac{D}{G}\int_0^L\int_{\tau_1}^{\tau_2} \mu_2(\tau)z(x,1,t,\tau)\xi_x\,d\tau\,dx\\ &+\left(\frac{\rho D}{G} - I_\rho\right)\int_0^L \xi_{xt}u_t\,dx.\end{aligned}$$

Since $\chi = 0$, coupled with the fact that $\psi_t = -\xi_t + 3S_t$, we have that

$$
\begin{aligned}
\frac{d}{dt} I_4(t) =& - G \int_0^L (3S - \xi - u_x)^2 \, dx + b \int_0^L \theta_x (3S - \xi - u_x) \, dx + I_\rho \int_0^L \xi_t^2 \, dx \\
& - 3I_\rho \int_0^L \xi_t S_t \, dx - \frac{\mu_1 D}{G} \int_0^L u_t \xi_x \, dx \\
& - \frac{D}{G} \int_0^L \int_{\tau_1}^{\tau_2} \mu_2(\tau) z(x, 1, t, \tau) \xi_x \, dx.
\end{aligned}
$$

Estimate (46) follows thanks to Young's inequality. □

Now, we introduce another functional

$$
I_5(t) = -3I_\rho \int_0^L S_t (3S - \xi - u_x) \, dx + 3I_\rho \int_0^L u_t S_x \, dx. \tag{47}
$$

Lemma 4.5 *Assume that* $\chi = 0$ *holds. Let*

$$
U(t) = (u(t), u_t(t), \xi(t), \xi_t(t), S(t), S_t(t), \theta(t), z(t))
$$

be a solution of (13)-(15). Then, the functional I_5 *satisfies the estimative*

$$
\begin{aligned}
\frac{d}{dt} I_5(t) \leq & - \frac{9I_\rho}{2} \int_0^L S_t^2 \, dx + \varepsilon_5 \int_0^L S_x^2 \, dx + c_5 \int_0^L \xi_t^2 \, dx \\
& + \frac{c_5}{\varepsilon_5} \int_0^L u_t^2 \, dx + \frac{c_5}{\varepsilon_5} \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(\tau)| z^2(x, 1, t, \tau) \, d\tau \, dx \\
& + c_5 \left(1 + \frac{1}{\varepsilon_5}\right) \int_0^L (3S - \xi - u_x)^2 \, dx,
\end{aligned} \tag{48}
$$

for any constants $\varepsilon_5 > 0$ *and* $c_5 > 0$.

Proof Taking derivative of I_5, use (13), integrating by parts, coupled with fact that $\psi_t = -\xi_t + 3S_t$, we obtain

$$
\begin{aligned}
\frac{d}{dt} I_5(t) =& 3 \left(D - \frac{G I_\rho}{\rho}\right) \int_0^L S_x (3S - \xi - u_x) \, dx + 3G \int_0^L (3S - \xi - u_x)^2 \, dx \\
& + 4\delta \int_0^L S (3S - \xi - u_x) \, dx + 3I_\rho \int_0^L S_t \xi_t \, dx - 9I_\rho \int_0^L S_t^2 \, dx \\
& - \frac{3\mu_1 I_\rho}{\rho} \int_0^L u_t S_x \, dx - \frac{3I_\rho}{\rho} \int_0^L \int_{\tau_1}^{\tau_2} \mu_2(\tau) z(x, 1, t, \tau) S_x \, d\tau \, dx.
\end{aligned}
$$

Since $\chi = 0$ and using Young's, Poincaré's inequalities to obtain (48). □

Finally, we define the functional

$$I_6(t) = \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} \tau e^{-\tau\eta} |\mu_2(\tau)| z^2(x, \eta, \tau)\, d\tau\, d\eta\, dx. \tag{49}$$

Lemma 4.6 *Let*

$$U(t) = (u(t), u_t(t), \xi(t), \xi_t(t), S(t), S_t(t), \theta(t), z(t))$$

be a solution of (13)-(15). Then, the functional I_6, satisfies

$$\begin{aligned} \frac{d}{dt} I_6(t) \leq &- c_6 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} \tau |\mu_2(\tau)| z^2(x, \eta, \tau)\, d\tau\, d\eta\, dx \\ &- c_6 \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(\tau)| z^2(x, 1, \tau)\, d\tau\, d\eta\, dx + \mu_1 \int_0^L u_t^2 dx, \end{aligned} \tag{50}$$

for some constant $c_6 > 0$.

Proof Taking derivative of $I_6(t)$, using (13) and the fact that $z(x, 0, \tau, t) = u_t(x, t)$ as follows

$$\begin{aligned} \frac{d}{dt} I_6(t) = &- \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(\tau)| e^{-\tau} z^2(x, 1, \tau)\, d\tau\, dx + \int_{\tau_1}^{\tau_2} |\mu_2(\tau)|\, d\tau \int_0^L u_t^2\, dx \\ &- \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} \tau e^{-\tau\eta} |\mu_2(\tau)| z^2(x, \eta, \tau)\, d\tau\, d\eta\, dx. \end{aligned} \tag{51}$$

Since $e^{-\tau} \leq e^{-\tau\eta} \leq 1$ for all $\eta \in (0, 1)$ and from (10), we obtain

$$\begin{aligned} \frac{d}{dt} I_6(t) \leq &- \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(\tau)| e^{-\tau} z^2(x, 1, \tau)\, d\tau\, dx + \mu_1 \int_0^L u_t^2 dx \\ &- \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} \tau e^{-\tau} |\mu_2(\tau)| z^2(x, \eta, \tau)\, d\tau\, d\eta\, dx. \end{aligned} \tag{52}$$

Since $-e^{-\tau}$ is an increasing function, $-e^{-\tau} \leq -e^{\tau_2}$ for all $\tau \in [\tau_1, \tau_2]$, we can choose $c_6 > 0$ such that $c_6 = e^{\tau_2}$ and, hence, we arrive at (50). □

4.2 Exponential stability

We define the Lyapunov functional $\mathcal{L}(t)$ as follows:

$$\mathcal{L}(t) := N E(t) + \sum_{i=1}^{6} N_i I_i(t), \quad \forall t \geq 0, \tag{53}$$

where N_i $(i = 1, \ldots, 6)$ are positive constants to be fixed later. The functionals $I_1, \ldots, I_6$ satisfy the Lemmas 4.1–4.6, respectively. First of all, we prove that $\mathcal{L}(t)$ and $E(t)$ are equivalent.

Proposition 4.1 *There exist positive constants γ_1 and γ_2 such that*

$$\gamma_1 E(t) \leq \mathcal{L}(t) \leq \gamma_2 E(t), \quad \forall t \geq 0. \tag{54}$$

Proof By definition of $\mathcal{L}(t)$, we have

$$|\mathcal{L}(t) - NE(t)| \leq \sum_{i=1}^{6} N_i |I_i(t)|. \tag{55}$$

It follows from (16), Young's, Poincaré's, and Hölder's inequalities, and from the fact that $e^{-\tau\eta} \leq 1$ for all $\eta \in (0, 1)$, for some constant $\gamma_3 > 0$, we deduce that

$$|\mathcal{L}(t) - NE(t)| \leq \gamma_3 E(t). \tag{56}$$

So, we can choose N large enough that $\gamma_1 := N - \gamma_3$ and $\gamma_1 := N + \gamma_3$, then (54) holds. □

Now, we are in a position to prove our main result.

Theorem 4.7 *Let $U(t) = (u(t), u_t(t), \xi(t), \xi_t(t), S(t), S_t(t), \theta(t), z(t))$ be a solution of (13)-(15) with initial data $U_0 \in D(\mathcal{A})$. Then, there exists positive constants M and γ such that*

$$E(t) \leq ME(0)e^{-\gamma t}, \quad \forall t \geq 0. \tag{57}$$

Proof Taking derivative $\mathcal{L}(t)$, substituting the estimates (17), (40), (42), (44), (46), (48), (50), and setting

$$N_1 = N_2 = 1, \quad \varepsilon_3 = \frac{D}{8N_3}, \quad \varepsilon_4 = \frac{D}{8N_4} \quad \text{and} \quad \varepsilon_5 = \frac{D}{2N_5},$$

we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq & -\left(m_0 N - \frac{8c_4}{D}N_4^2 - \frac{2c_5}{D}N_5^2 - \mu_1 N_6\right)\int_0^L u_t^2\,dx \\
& -\left(\alpha N - c_1 - c_3\left(1+\frac{8}{D}N_3\right)N_3 - c_4 N_4\right)\int_0^L \theta_x^2\,dx \\
& -\frac{D}{4}\int_0^L \xi_x^2\,dx - \delta\int_0^L S^2\,dx - \frac{5D}{2}\int_0^L S_x^2\,dx \\
& -\left(\frac{I_\rho}{2}N_3 - I_\rho - c_4\left(1+\frac{8}{D}N_4\right)N_4 - c_5 N_5\right)\int_0^L \xi_t^2\,dx \\
& -\left(\frac{G}{2}N_4 - c_1 - c_2 - \frac{D}{8} - c_5\left(1+\frac{2}{D}N_5\right)N_5\right)\int_0^L (3S-\xi-u_x)^2\,dx \\
& -\left(\frac{9I_\rho}{2}N_5 - I_\rho - \frac{D}{8}\right)\int_0^L S_t^2\,dx \\
& -\left(c_6 N_6 - \frac{8c_4}{D}N_4^2 - \frac{2c_5}{D}N_5^2\right)\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|z^2(x,1,\tau)\,d\tau\,dx \\
& - c_6 N_6\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|z^2(x,\eta,\tau)\,d\tau\,d\eta\,dx.
\end{aligned}
$$

First, let us choose N_5 large enough such that

$$\frac{9I_\rho}{2}N_5 - I_\rho - \frac{D}{8} > 0.$$

Once N_5 is fixed, we proceed to choose N_4 large enough such that

$$\frac{G}{2}N_4 - c_1 - c_2 - \frac{D}{8} - c_5\left(1+\frac{2}{D}N_5\right)N_5 > 0.$$

Now, once N_4 and N_5 are fixed, we select N_3 and N_6 large enough so that

$$\frac{I_\rho}{2}N_3 - I_\rho - c_4\left(1+\frac{8}{D}N_4\right)N_4 - c_5 N_5 > 0 \quad \text{and} \quad c_6 N_6 - \frac{8c_4}{D}N_4^2 - \frac{2c_5}{D}N_5^2 > 0.$$

Lastly, choosing N sufficiently large enough and applying Poincaré's inequality, we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &\leq -\gamma_0\int_0^L\Big[u_t^2 + (3S-\xi-u_x)^2 + \xi_t^2 + \xi_x^2 + S_t^2 + S_x^2 + S^2 + \theta^2 \\
&\qquad + \int_{\tau_1}^{\tau_2}|\mu_2(\tau)|z^2(x,1,\tau)\,d\tau + \int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|z^2(x,\eta,\tau)\,d\tau\,d\eta\Big]\,dx \\
&\leq -\gamma_0 E(t),
\end{aligned}
$$

for some positive constant γ_0. Moreover, from the equivalence between $\mathcal{L}(t)$ and $E(t)$ according to inequality (54), we obtain

$$E(t) \leq ME(0)e^{-\gamma t}, \quad \forall t \geq 0,$$

where $M \geq 1$ and $\gamma := \gamma_0/\gamma_1$. □

Final comment

In this manuscript, we use a nonlocal delay condition $\int_{\tau_1}^{\tau_2} \mu_2(\tau) u_t(x, t-\tau)\, d\tau$ and additionally, we are considering the non-constant delay coefficient $\mu_2(\tau)$, which makes the result more comprehensive and realistic. Combining the semigroup technique with the energy method, we obtain the existence and uniqueness of a strong solution and the exponential decay of the solution.

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Declarations

Conflict of interest The author declares no conflict of interest.

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