# Optimal Partitioned State Kalman Estimator

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***Abstract*-- A new general two-stage algorithm was originally proposed to reduce the computational effort of the augmented state Kalman estimator. The conventional input estimation techniques assume constant input level and there are not covered a generalized input modeling. In this paper an innovative scheme is developed to overcome these drawbacks by using a new partitioned input dynamic modeling. In addition, authors propose a modified two-stage Kalman estimator with a new structure, which is an extension of the conventional input estimation techniques and is optimal for general, linear discrete-time systems.**

**Keywords- Optimal two-stage Kalman estimators; input estimation; augmented state Kalman estimators; maneuvering target tracking.**

## 1. Introduction

The general state estimation problem in a stochastic linear system with unknown input variable, is solved by the well-known augmented state Kalman estimator (AUSKE) or "full state" method. The AUSKE solves the problems by including the input parameters as a part of an augmented state to be estimated [1], [2]. However "reduced state" methods do not augment the state, and usually yield a better performance [3]. The AUSKE suffers from complexity of computational effort and numerical problems when state dimensions are large. The input detection and estimation (IDE) algorithm was first developed by Chan et al., in [4] using a simplified batch least square data. Although IDE approach is attractive since it intends to relax restrictive assumptions about input dynamics modeling, it suffers from a major deficiency, being that little prior knowledge is available for dynamics estimation [5]. For example, we can cite Wang et. al., [6] used the IDE approach in the maneuvering target tracking problem. In [6], the predicted states for the maneuvering target are related to the corresponding states without maneuvering assuming constant input or constant acceleration (CA). Therefore, the performance of the estimation is reduced when target moves with non-constant acceleration. In [7] the unknown input defined as a sum of elementary time functions. Although this input modeling is more general than the constant-input model of the original IDE algorithm, the performance is reduced if there is any input dynamics.

Friedland [8] introduced a method of separating estimation of the unknown input from the dynamic variables and Blair used this method in the MTT problem [9]. The basic idea was to decouple the augmented Kalman filter (AKF) into two-stage filters in order to reduce computation and memory requirements [10]-[13]. Recently, Hsieh and Chen [10], [11] derived an optimal two-stage Kalman estimator (OTSKE) for a general case to reduce the computational complexity of AUSKE. The two-stage filtering method, suggested for MTT problem in [9] suffers from two major drawbacks. These drawbacks stem from assuming constant acceleration and assuming the input term is observable from the measurement equation (also in [10] and [13]).

The objective of this paper is to propose a new partitioned two-stage Kalman estimator, which is optimal in the minimum mean square error (MMSE) sense. The Optimal Partitioned state Kalman Estimator (OPSKE) may serve as an alternative solution of the OTSKE proposed in [10]. It is shown that the maneuver tracking algorithm proposed in [6] and [9] are special case of our proposed method. The motivation of our proposed method is the generation of a two-stage structure to obtain the optimal performance when the input term is not observable through the measurements. The computation cost will compare with the AUSKE and OPSKE at another paper.

## 2. Statement of the Problem

The problem of interest is described by the discretized equation set

$$X_{k+1} = A_k X_k + B_k U_k + W_k^x \tag{1}$$

$$U_{k+1} = C_k U_k + W_k^u \tag{2}$$

$$Z_k = H_k X_k + V_k \tag{3}$$

Where $X_k \in R^n$ is the system state, $U_k \in R^m$ and $Z_k \in R^p$ are the input and the measurement vectors, respectively. Matrices $A_k$, $B_k$, $C_k$ and $H_k$ are assumed to be known functions of the time interval $k$ and are of appropriate dimensions. Matrix $C_k$ is assumed nonsingular. The process noises $W_k^x$, $W_k^u$ and the measurement noise $V_k$ are zero-mean white Gaussian sequences with the following covariances: $E[W_k^x (W_l^x)'] = Q_k^x \delta_{kl}$ , $E[W_k^u (W_l^u)'] = Q_k^u \delta_{kl}$ , $E[W_k^x (W_l^u)'] = Q_k^{xu} \delta_{kl}$ , $E[V_k V_l'] = R_k \delta_{kl}$ , $E[W_k^x V_l'] = 0$ and $E[W_k^u V_l'] = 0$ , where $'$ denotes transpose and $\delta_{kl}$ denotes the Kronecker delta function. The initial states $X_0$ and $U_0$ are assumed to be uncorrelated with the sequences $W_k^x$, $W_k^u$ and

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$V_k$ . The initial conditions are assumed to be Gaussian random variables with $E[X_0]=\hat{X}_0$ , $E[X_0 X_0^{'}]=P_0^x$ , $E[U_0]=\hat{U}_0$ , $E[U_0 U_0^{'}]=P_0^u$ , $E[X_0 U_0^{'}]=P_0^{xu}$ . As we can see in (3) the input vector $U_k$ is not observable through the measurement process, in spite of the assumptions in [9], [10] and [13].

Treating $X_k$ and $U_k$ as the augmented system state [10], the AUSKE is described by

$$X_{k+1|k+1}^{Aug} = X_{k+1|k}^{Aug} + K_{k+1}^{Aug}(Z_{k+1} - H_{k+1}^{Aug} X_{k+1|k}^{Aug}) \quad (4)$$

$$X_{k+1|k}^{Aug} = A_k^{Aug} X_{k|k}^{Aug} \quad (5)$$

$$K_{k+1}^{Aug} = P_{k+1|k}(H_{k+1}^{Aug})^{'}[H_{k+1}^{Aug} P_{k+1|k}(H_{k+1}^{Aug})^{'} + R_k]^{-1} \quad (6)$$

$$P_{k+1|k} = A_k^{Aug} P_{k|k}(A_k^{Aug})^{'} + Q_k \quad (7)$$

$$P_{k+1|k+1} = (I - K_{k+1}^{Aug} H_{k+1}^{Aug})P_{k+1|k} \quad (8)$$

where

$$X_k^{Aug} = \begin{bmatrix} X_k \\ U_k \end{bmatrix}, \quad K_k^{Aug} = \begin{bmatrix} K_k^x \\ K_k^u \end{bmatrix}, \quad P_k = \begin{bmatrix} P_k^x & P_k^{xu} \\ (P_k^{xu})^{'} & P_k^u \end{bmatrix}$$

$$A_k^{Aug} = \begin{bmatrix} A_k & B_k \\ 0_{m\times n} & C_k \end{bmatrix}, H_k^{Aug} = \begin{bmatrix} H_k \\ 0_{p\times m} \end{bmatrix}^{'} Q_k = \begin{bmatrix} Q_k^x & Q_k^{xu} \\ (Q_k^{xu})^{'} & Q_k^u \end{bmatrix}$$

Where the superscript 'Aug' denotes the augmented system state, $I$ denotes the identity matrix of any dimension and $0_{m\times n}$ is a $m\times n$ zero matrix. It is clear from (4)-(8) that the computational cost of the AUSKE increases with the augmented state dimension [14]. The reason for this computational complexity is the extra computation of $P_k^{xu}$ terms in each sample time $k$ [10]. Therefore, if this term can be eliminated, one can reduce the complexity of computational effort. In this paper, we propose a new optimal two-stage Kalman estimator without calculating the term of $P_k^{xu}$ explicitly. Therefore, the proposed scheme is developed to reduced the computational cost of an AUSKE by partitioning the equations (4)-(8) into two subsystems.

## 3. DERIVATION OF THE OPTIMAL PARTITIONED STATE KALMAN ESTIMATOR

The design of a new two-stage estimator is described as follows. First, define a modified input-free model and design a modified input-free filter by ignoring the input term. Second, derive an input filter to compensate the modified input-free filter in order to minimize mean square error. These two filters are used to build a new scheme, which is equivalent to the AUSKE. The major derivation is the relation between the measurement residues of the two different filters. One is the measurement residue of the input-free filter, which does not consider unknown input vector, and the other is the measurement residue of the input filter. Based on the measurement residues of the two filters, an input estimation algorithm is derived using the minimum mean square estimation technique.

The input-free model can be obtained by ignoring the input term ( $U_k = 0$ ) in (1) as below:

$$\overline{X}_{k+1} = A_k \overline{X}_k + W_k^x \quad (9)$$

where the state vector of the input-free model is denoted by $\overline{X}_k$ . The input-free filter is just a Kalman filter based on the model (9) and (3) as below:

$$\hat{\overline{X}}_{k+1|k+1} = \hat{\overline{X}}_{k+1|k} + K_{k+1}(Z_{k+1} - H_{k+1}\hat{\overline{X}}_{k+1|k}) \quad (10)$$

$$\hat{\overline{X}}_{k+1|k} = A_k \hat{\overline{X}}_{k|k} \quad (11)$$

$$K_{k+1} = P_{k+1|k}^x H_{k+1}^{'}[H_{k+1}P_{k+1|k}^x (H_{k+1})^{'} + R_k]^{-1} \quad (12)$$

$$P_{k+1|k}^x = A_k P_{k|k}^x (A_k)^{'} + Q_k^x \quad (13)$$

$$P_{k+1|k+1}^x = (I - K_{k+1}H_{k+1})P_{k+1|k}^x \quad (14)$$

In the following, we propose an expression which relates the state vector of the input model $X_k$ to the state vector of the input-free model $\overline{X}_k$ . The state vector of the input model in (1) using the state of the input-free model in (9) can be calculated for each sample time:

$$\begin{aligned} X_1 &= A_0 X_0 + B_0 U_0 + W_0^x \\ &= A_0 X_0 + B_0[C_0^{-1}U_1 - C_0^{-1}W_0^u] + W_0^x \\ &= \overline{X}_1 + B_0 C_0^{-1} U_1 - B_0 C_0^{-1} W_0^u \end{aligned} \quad (15)$$

Since $C_k$ is a nonsingular matrix, $U_0$ in (15) replaced by (3). It is assumed in (15) that $X_0 = \overline{X}_0$ , so $\overline{X}_1 = A_0 \overline{X}_0 + W_0^x = A_0 X_0 + W_0^x$ . Following the same procedure, we can define and derive the expression for $X_2$ by using (15):

$$\begin{aligned} X_2 &= A_1 X_1 + B_1 U_1 + W_1^x \\ &= A_1[\overline{X}_1 + B_0 C_0^{-1} U_1 - B_0 C_0^{-1} W_0^u] + B_1 U_1 + W_1^x \\ &= \overline{X}_2 + [A_1 B_0 C_0^{-1} + B_1]U_1 - A_1 B_0 C_0^{-1} W_0^u \\ &= \overline{X}_2 + [A_1 B_0 C_0^{-1} + B_1][C_1^{-1}U_2 - C_1^{-1}W_1^u] - A_1 B_0 C_0^{-1} W_0^u \\ &= \overline{X}_2 + [A_1 B_0 C_0^{-1} + B_1]C_1^{-1}U_2 - [A_1 B_0 C_0^{-1} + B_1]C_1^{-1}W_1^u \\ &\quad - A_1 B_0 C_0^{-1} W_0^u \end{aligned} \quad (16)$$

For an arbitrary sample time $k$ , we will have

$$X_{k+1} = \overline{X}_{k+1} + M_{k+1}U_{k+1} - \sum_{i=0}^{k} \omega_i W_i^u \quad (17)$$

where

$$\begin{aligned} M_{k+1} &= [A_k M_k + B_k]C_k^{-1}, \qquad k = 2,3,.... \\ M_1 &= B_0 C_0^{-1} \end{aligned} \quad (18)$$

$$\begin{aligned} \omega_i &= [\omega_{i-1} + (\prod_{j=0}^{k-i-1} A_{k-j})B_i]C_i^{-1}, \qquad i = 1,2,..,k \\ \omega_0 &= (\prod_{j=0}^{k-1} A_{k-j})B_0 C_0^{-1} \end{aligned} \quad (19)$$

By using zero-mean property of $W_k^u$ and one-step prediction logic from dynamic equation (17) the predicted state is obtained as follow:

$$\hat{X}_{k+1|k} = \hat{\hat{X}}_{k+1|k} + M_{k+1}U_{k+1} \tag{20}$$

The updated state for the input model can be considered using the state of the input-free model as below:

$$\hat{X}_{k+1|k+1} = \hat{\hat{X}}_{k+1|k+1} + N_{k+1}U_{k+1} \tag{21}$$

where $N_{k+1}$ in (21) must be calculated. Hence, the relation between two innovation matrices $M_k$ and $N_k$ needs to be determined. Suppose that the updated state $\hat{X}_{k+1|k+1}$ obtains from the Kalman filter framework:

$$\hat{X}_{k+1|k+1} = \hat{X}_{k+1|k} + K_{k+1}(Z_{k+1} - H_{k+1}\hat{X}_{k+1|k}) \tag{22}$$

By some manipulation and using (10) and (22), an expression to relate $N_{k+1}$ and $M_{k+1}$ can be obtained as below:

$$\begin{aligned} N_{k+1}U_{k+1} &= \hat{X}_{k+1|k+1} - \hat{\hat{X}}_{k+1|k+1} \\ &= \hat{X}_{k+1|k} + K_{k+1}[Z_{k+1} - H_{k+1}\hat{X}_{k+1|k}] \\ &\quad - \hat{\hat{X}}_{k+1|k} - K_{k+1}[Z_{k+1} - H_{k+1}\hat{\hat{X}}_{k+1|k}] \\ &= [I - K_{k+1}H_{k+1}][\hat{X}_{k+1|k} - \hat{\hat{X}}_{k+1|k}] \\ &= [I - K_{k+1}H_{k+1}]M_{k+1}U_{k+1} \end{aligned} \tag{23}$$

The equation, which relates $N_{k+1}$ and $M_{k+1}$ can be obtained by comparing both sides of the above equation;

$$N_{k+1} = [I - K_{k+1}H_{k+1}]M_{k+1} \tag{24}$$

Once the initial value of $M_k$ is chosen, $M_{k+1}$ can be recursively calculated by (18), and the $N_{k+1}$ can be obtained by (24) in each iteration. Note that the $K_{k+1}$ for the input filter is equal to $K_{k+1}$ for the input-free filter since the input term assumes to be nonrandom (for more detail see [15], [5]). In addition, both filters have the same covariance matrices. On the other words, the equality of filter gain for $X_k$ and $\overline{X}_k$ in these filters arises from the same error covariance matrices of $X_k$ and $\overline{X}_k$ denoted by $P^x_{k|k}$. The innovations $\tilde{\tilde{Z}}_{k+1}$ and $\tilde{Z}_{k+1}$ as the measurement residues of the input-free model and the input model are defined, respectively;

$$\tilde{\tilde{Z}}_{k+1} = Z_{k+1} - H_{k+1}\hat{\hat{X}}_{k+1|k} \tag{25}$$

$$\tilde{Z}_{k+1} = Z_{k+1} - H_{k+1}\hat{X}_{k+1|k} \tag{26}$$

The difference of these two innovations is defined as:

$$\Delta\tilde{Z}_{k+1} = \tilde{\tilde{Z}}_{k+1} - \tilde{Z}_{k+1} = H_{k+1}[\hat{X}_{k+1|k} - \hat{\hat{X}}_{k+1|k}] \tag{27}$$

Using (27) and (20), we represent $U_{k+1}$ in the equation, which relates the input state to the measurement residue as below:

$$\begin{aligned} \tilde{\tilde{Z}}_{k+1} &= H_{k+1}[\hat{X}_{k+1|k} - \hat{\hat{X}}_{k+1|k}] + \tilde{Z}_{k+1} \\ &= H_{k+1}M_{k+1}U_{k+1} + \tilde{Z}_{k+1} \end{aligned} \tag{28}$$

It is clear from the input-free filter (10)-(14) which is a standard Kalman filter, that the measurement residue $\tilde{\tilde{Z}}_{k+1|k}$ exists and easily obtainable while the measurement residue $\tilde{Z}_{k+1|k}$ is actually not available. Note that (28) is in the standard form $\mathrm{Z}_{k+1} = \mathrm{H}_{k+1}\chi_{k+1} + \nu_{k+1}$, and can be viewed as an observation model of $U_{k+1}$. In (28), the measurement residue $\tilde{\tilde{Z}}_{k+1|k}$ is a non-zero mean white random process since the input term introduces a bias in the innovation $\tilde{\tilde{Z}}_{k+1|k}$. The amount of this bias will supply the information about the existence of input value. In contrast, the measurement residue $\tilde{Z}_{k+1|k}$ is a zero mean white random process ($E[\tilde{Z}_{k+1}] = 0$) with covariance matrix $P^z_{k|k}$ [5], [16]. On the other words, we would like to estimate $U_{k+1}$ in order to minimize the error covariance matrix $P^u_{k+1}$ under the constraint $E[\tilde{Z}_{k+1}] = E[Z_{k+1} - H_{k+1}\hat{X}_{k+1|k}] = 0$, or $E[X_{k+1}] = \hat{X}_{k+1|k}$, yield unbiased estimation.

The desired form of the filtering solution for estimating the unknown vector $U_{k+1}$ is a difference equation expressing $\hat{U}_{k+1|k+1}$ in terms of $\hat{U}_{k|k}$ and $\tilde{\tilde{Z}}_{k+1}$. In the following, we derive a recursive algorithm to estimate $U_{k+1}$ in order to minimize the error covariance matrix of the input vector or $P^u_{k+1}$. Suppose that $\tilde{U}_{k+1|k}$ denotes the input state residue;

$$\tilde{U}_{k+1|k} = U_{k+1} - \hat{U}_{k+1|k} \tag{29}$$

Using (28) and zero-mean property of $\tilde{Z}_{K+1}$, we would like to obtain a recursive algorithm in the form of Kalman filter as below:

$$\begin{aligned} \hat{U}_{k+1|k+1} &= \hat{U}_{k+1|k} + K^u_{k+1}[\tilde{\tilde{Z}}_{k+1} - \hat{\tilde{\tilde{Z}}}_{k+1|k}] \\ &= \hat{U}_{k+1|k} + K^u_{k+1}[\tilde{\tilde{Z}}_{k+1} - H_{k+1}M_{k+1}\hat{U}_{k+1|k}] \end{aligned} \tag{30}$$

Using the residue of $U_{k+1}$ defined in (29) and using (30) and (28), gives

$$\begin{aligned} &\tilde{U}_{k+1|k+1} = U_{k+1} - \hat{U}_{k+1|k+1} = \\ &U_{k+1} - \hat{U}_{k+1|k} - K^u_{k+1}[\tilde{\tilde{Z}}_{k+1} - H_{k+1}M_{k+1}\hat{U}_{k+1|k}] \\ &= \tilde{U}_{k+1|k} - K^u_{k+1}[H_{k+1}M_{k+1}U_{k+1} + \tilde{Z}_{k+1} - H_{k+1}M_{k+1}\hat{U}_{k+1|k}] \\ &= \tilde{U}_{k+1|k} - K^u_{k+1}H_{k+1}M_{k+1}\tilde{U}_{k+1|k} - K^u_{k+1}\tilde{Z}_{k+1} \end{aligned} \tag{31}$$

The covariance matrix of $\widetilde{U}_{k+1|k+1}$ defined in (31) gives

$$\begin{aligned} P^u_{k+1|k+1} &= P^u_{k+1|k} + K^u_{k+1}H_{k+1}M_{k+1}P^u_{k+1|k}M'_{k+1}H'_{k+1}(K^u_{k+1})' \\ &+ K^u_{k+1}P^z_{k+1|k}(K^u_{k+1})' - P^u_{k+1|k}M'_{k+1}H'_{k+1}(K^u_{k+1})' \\ &- P^{uz}_{k+1|k}(K^u_{k+1})' - K^u_{k+1}H_{k+1}M_{k+1}P^u_{k+1|k} \\ &+ K^u_{k+1}H_{k+1}M_{k+1}P^{uz}_{k+1|k}(K^u_{k+1})' - K^u_{k+1}P^{zu}_{k+1|k} \\ &+ K^u_{k+1}P^{zu}_{k+1|k}M'_{k+1}H'_{k+1}(K^u_{k+1})' \end{aligned} \tag{32}$$

Where the covariance matrix of $\widetilde{Z}_{k+1}$, denoted by $P^z_{k+1|k}$ must be calculated. From (26):

$$P^z_{k+1|k} = H_{k+1}P^x_{k+1|k}H'_{k+1} + R_{k+1} \tag{33}$$

It is seen that the input state residue $\widetilde{U}_{k+1|k}$ is time-correlated with the measurement residue of the input model $\widetilde{Z}_{k+1}$, denoted by $P^{zu}_{k+1|k}$.

$$P^{zu}_{k+1|k} = E\{\widetilde{Z}_{k+1}\widetilde{U}'_{k+1|k}\} \tag{34}$$

In view of this fact, the algorithm has been proposed in [6] is a sub-optimal algorithm where $E\{\widetilde{Z}_{k+1}\widetilde{U}'_{k+1|k}\} = 0$. The extra computation of this cross covariance matrix $P^{zu}_{k+1|k}$ (which relates to $P^{xu}_{k+1|k}$) is the reason for the computational complexity in the augmented state methods. Therefore, if this term can be eliminated, one can reduce the complexity of computational effort. In the following, we propose an expression which relates $P^{zu}_{k+1|k}$ to $P^u_{k+1|k}$. Since the magnitude of the input term $U_{k+1}$ in (20) is unknown, we can only use the estimation of $\hat{U}_{k+1}$ to modify the state vector of the input-free model $\overline{X}_{k+1|k}$ to obtain the state vector of the input model $X_{k+1|k}$. Therefore, we rewrite (20):

$$\hat{X}_{k+1|k} = \hat{\overline{X}}_{k+1|k} + M_{k+1}\hat{U}_{k+1|k} \tag{35}$$

Using the equations (17), (3), (26) and (35), gives:

$$\begin{aligned} \widetilde{Z}_{k+1} &= Z_{k+1} - H_{k+1}\hat{X}_{k+1|k} = H_{k+1}[X_{k+1} - \hat{X}_{k+1|k}] + V_{k+1} \\ &= H_{k+1}\widetilde{X}_{k+1|k} + V_{k+1} = H_{k+1}[\overline{X}_{k+1} + M_{k+1}U_{k+1} - \sum_{i=0}^{k}\omega_i W^u_i] - \\ &H_{k+1}[\hat{\overline{X}}_{k+1|k} + M_{k+1}\hat{U}_{k+1|k}] + V_{k+1} \\ &= H_{k+1}[\overline{X}_{k+1} - \hat{\overline{X}}_{k+1|k}] + H_{k+1}M_{k+1}[U_{k+1} - \hat{U}_{k+1|k}] \\ &- H_{k+1}\sum_{i=0}^{k}\omega_i W^u_i + V_{k+1} \\ &= H_{k+1}\widetilde{\overline{X}}_{k+1|k} + H_{k+1}M_{k+1}\widetilde{U}_{k+1|k} - H_{k+1}\sum_{i=0}^{k}\omega_i W^u_i + V_{k+1} \end{aligned} \tag{36}$$

$$\begin{aligned} \widetilde{Z}_{k+1|k} &= H_{k+1}\widetilde{\overline{X}}_{k+1|k} + H_{k+1}M_{k+1}\widetilde{U}_{k+1|k} \\ &- H_{k+1}\sum_{i=0}^{k}\omega_i W^u_i + V_{k+1} \end{aligned} \tag{37}$$

It should be noted that the term $H_{k+1}\widetilde{\overline{X}}_{k+1|k} + V_{k+1}$ in (37) is not equal to $\widetilde{\overline{Z}}_{k+1}$ in (16). Using (37), the cross covariance matrix $P^{zu}_{k+1|k}$ can be calculated:

$$\begin{aligned} P^{zu}_{k+1|k} &= E[\widetilde{Z}_{k+1}\widetilde{U}'_{k+1|k}] = E[(H_{k+1}\widetilde{\overline{X}}_{k+1|k} + H_{k+1}M_{k+1}\widetilde{U}_{k+1|k} \\ &- H_{k+1}\sum_{i=0}^{k}\omega_i W^u_i + V_{k+1})\widetilde{U}'_{k+1|k}] = H_{k+1}M_{k+1}E\{\widetilde{U}_{k+1|k}\widetilde{U}'_{k+1|k}\} \\ &= H_{k+1}M_{k+1}P^u_{k+1|k} \end{aligned} \tag{38}$$

Where $E[\widetilde{\overline{X}}_{k+1|k}\widetilde{U}'_{k+1|k}]$, $E[\sum_{i=0}^{k}\omega_i W^u_i \widetilde{U}'_{k+1|k}] = 0$ and $E[V_{k+1}\widetilde{U}'_{k+1|k}] = 0$ are equal to zero. One important property of the optimal estimation of $\hat{U}_{k+1}$ is that the input residue $\widetilde{U}_{k+1|k}$ must be orthogonal to $Z_{k+1}$, $V_{k+1}$ and $W^u_k$ or any linear function of $Z_{k+1}$.

$$P^{zu}_{k+1|k} = H_{k+1}M_{k+1}P^u_{k+1|k} \tag{39}$$

Substitute $P^{zu}_{k+1|k}$ in (32) the covariance matrix of $\hat{U}_{k+1|k+1}$ becomes

$$\begin{aligned} P^u_{k+1|k+1} &= P^u_{k+1|k} + K^u_{k+1}H_{k+1}M_{k+1}P^u_{k+1|k}M'_{k+1}H'_{k+1}(K^u_{k+1})' \\ &+ K^u_{k+1}P^z_{k+1|k}(K^u_{k+1})' - P^u_{k+1|k}M'_{k+1}H'_{k+1}(K^u_{k+1})' \\ &- P^u_{k+1|k}M'_{k+1}H'_{k+1}(K^u_{k+1})' - K^u_{k+1}H_{k+1}M_{k+1}P^u_{k+1|k} \\ &+ K^u_{k+1}H_{k+1}M_{k+1}P^u_{k+1|k}M'_{k+1}H'_{k+1}(K^u_{k+1})' \\ &- K^u_{k+1}H_{k+1}M_{k+1}P^u_{k+1|k} + \\ &K^u_{k+1}H_{k+1}M_{k+1}P^u_{k+1|k}M'_{k+1}H'_{k+1}(K^u_{k+1})' \end{aligned} \tag{40}$$

$$\begin{aligned} P^u_{k+1|k+1} &= P^u_{k+1|k} + 3K^u_{k+1}H_{k+1}M_{k+1}P^u_{k+1|k}M'_{k+1}H'_{k+1}(K^u_{k+1})' \\ &+ K^u_{k+1}P^z_{k+1|k}(K^u_{k+1})' - 2P^u_{k+1|k}M'_{k+1}H'_{k+1}(K^u_{k+1})' \\ &- 2K^u_{k+1}H_{k+1}M_{k+1}P^u_{k+1|k} \end{aligned} \tag{41}$$

The second and the third terms of the above equation can be considered as $K^u_{k+1}WW'(K^u_{k+1})'$, where

$$WW' = 3H_{k+1}M_{k+1}P^u_{k+1|k}M'_{k+1}H'_{k+1} + P^z_{k+1|k} \tag{42}$$

Since the covariance matrices $P^u_{k+1|k}$ and $P^z_{k+1|k}$ are symmetric, we can find a decomposition in the form of $WW'$, for appropriate matrix $W$, then

$$P^u_{k+1|k+1} = P^u_{k+1|k} + [K^u_{k+1}W - D][K^u_{k+1}W - D]' - DD' \tag{43}$$

Comparing (41) and (43), the term $D$ is defined as below:

$$D = 2P^u_{k+1|k}M'_{k+1}H'_{k+1}(W')^{-1} \tag{44}$$

The minimum $P^u_{k+1|k+1}$ is obtained by setting $K^u_{k+1}W$ equal to $D$. Therefore,

$$\begin{aligned} K^u_{k+1} &= 2P^u_{k+1|k+1}M'_{k+1}H'_{k+1}(W')W^{-1} = 2P^u_{k+1|k}M'_{k+1}H'_{k+1} \\ &\times[3H_{k+1}M_{k+1}P^u_{k+1|k}M'_{k+1}H'_{k+1} + P^z_{k+1|k}]^{-1} \end{aligned} \tag{45}$$

The minimum error covariance $P^u_{k+1|k+1}$ is obtained:

$$
\begin{aligned}
P^u_{k+1|k+1} &= P^u_{k+1|k} - DD' = P^u_{k+1|k} - \\
&[2P^u_{k+1|k}M^{'}_{k+1}H^{'}_{k+1}](WW')^{-1}[2P^u_{k+1|k}M^{'}_{k+1}H^{'}_{k+1}]' \\
&= P^u_{k+1|k} - 4P^u_{k+1|k}M^{'}_{k+1}H^{'}_{k+1}[3H_{k+1}M_{k+1}P^u_{k+1|k} \\
&\times M^{'}_{k+1}H^{'}_{k+1} + P^z_{k+1|k}]^{-1}H_{k+1}M_{k+1}P^u_{k+1|k} \\
&= [I - 2K^u_{k+1}H_{k+1}M_{k+1}]P^u_{k+1|k}
\end{aligned} \tag{46}
$$

Based on (2), we have

$$P^u_{k+1|k} = C_k P^u_{k|k} C^{'}_k + Q^u_k \tag{47}$$

If the value of $K^u_{k+1}$ given by (45) substitute in (30), the estimation of $\hat{U}_{k+1}$ leads to a minimum error covariance.

## 4. CONCLUSIONS

The proposed scheme is based on a new partitioned dynamic modeling and intends to overcome the computational expensiveness drawbacks of the other works are based on the augmented methods. The proposed OPSKE provides the optimal state estimate, which is equivalent to that of the AUSKE.

## 5. SIMULATION RESULTS

To evaluate the proposed algorithm, an example of maneuvering target tracking problem which turns, in two-dimensional space is simulated such as a ship or an aircraft with constant elevation. In this simulation example, the performance of the OPSKE for the maneuvering target tracking has been compared with the work suggested in [2] as an example of the AUSKE method. As mentioned before in the augmented state method the state vector includes the input vector i.e., acceleration and jerk parameter in maneuvering target tracking problem. The sampling interval is T=0.01 (sec) and target maneuver is applied at 9th second (900th sample). The initial conditions are selected similar for the AUSKE as well as the OPSKE. The state vectors are

$X_k = [x_k \quad v^x_k \quad y_k \quad v^y_k]'$, $U_k = [u^x_k \quad j^x_k \quad u^y_k \quad j^y_k]'$,

$X^{Aug}_k = [x_k \quad v^x_k \quad y_k \quad v^y_k \quad u^x_k \quad j^x_k \quad u^y_k \quad j^y_k]'$

where $x_k$, $v^x_k$, $u^x_k$ and $j^x_k$ denote the position, velocity, acceleration and jerk of the target along the $x$ axis, respectively. We consider the target initial conditions for the state and the acceleration vectors as below:

$X_0 = [2165\,m \quad -80\,m/s \quad 1250\,m \quad 25\,m/s]'$ ,

$U_0 = [0\,g \quad 0\,g/\sec \quad 0\,g \quad 0\,g/\sec]'$

$X^{Aug}_0 = [2165m \quad -80m/s \quad 1250m \quad 25m/s \quad 0g \quad 0g/\sec \quad 0g \quad 0g/\sec]'$

The target begins to maneuver as $U_{900} = [0g \quad -0.7\,g/\sec \quad 0g \quad 0.4\,g/\sec]'$ for $9\ (\sec) \le t \le 90\,(\sec)$. The system matrices are given by

$$A_k = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, B_k = \begin{bmatrix} T^2/2 & T^3/6 & 0 & 0 \\ T & T^2/2 & 0 & 0 \\ 0 & 0 & T^2/2 & T^3/6 \\ 0 & 0 & T & T^2/2 \end{bmatrix}, C_k = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$H_k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}'$$

$$Q^u_k = 2\alpha\sigma_j \begin{bmatrix} T^3/3 & T^2/2 & 0 & 0 \\ T^2/2 & T & 0 & 0 \\ 0 & 0 & T^3/3 & T^2/2 \\ 0 & 0 & T^2/2 & T \end{bmatrix},$$

$$Q^x_k = 2\alpha\sigma_j \begin{bmatrix} T^7/252 & T^6/72 & 0 & 0 \\ T^6/72 & T^5/20 & 0 & 0 \\ 0 & 0 & T^7/252 & T^6/72 \\ 0 & 0 & T^6/72 & T^5/20 \end{bmatrix}$$

$$Q^{xu}_k = 2\alpha\sigma_j \begin{bmatrix} T^5/30 & T^4/24 & 0 & 0 \\ T^4/8 & T^3/6 & 0 & 0 \\ 0 & 0 & T^5/30 & T^4/24 \\ 0 & 0 & T^4/8 & T^3/6 \end{bmatrix}, \quad P^x_0 = 10 I_{4\times4},$$

$$P^u_o = 0.1 I_{4\times4},\ P^{xu}_0 = I_{4\times4},\ H^{Aug}_k = \begin{bmatrix} H_k \\ 0_{2\times4} \end{bmatrix}'.$$

where $\sigma_j = 0.09 (ms^{-3})$ is the variance of the target jerk and $\alpha = 0.0123\,(s^{-1})$ is the reciprocal of the jerk time constant $\tau = 1/\alpha$. The measurement standard deviations of $x$ and $y$ target positions are: $\sigma_x = 10\sqrt{10}\,(m)$, $\sigma_y = 20\,(m)$. Thus, the measurement covariance matrix is $R_k = \begin{bmatrix} 1000 & 0 \\ 0 & 400 \end{bmatrix}$ for both methods. The Root Mean Square Error (RMSE) index is used for the results evaluation.

Fig. 1 shows the actual value and the estimation of $x$ and $y$ and RMS errors of $x$ and $y$ positions estimations by the proposed OPSKE and the AUSKE.

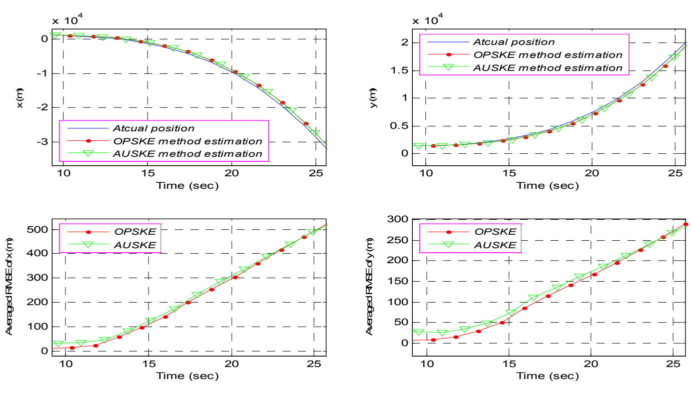


*Fig. 1. The actual value and the estimation of the x, y positions and RMS errors estimations by the OPSKE and the AUSKE methods.*

Fig. 2 shows the actual value and the estimations of $v^x$ , $v^y$ and the RMS errors of the $x$ and $y$ velocities estimations by the proposed method compared with the augmented method.

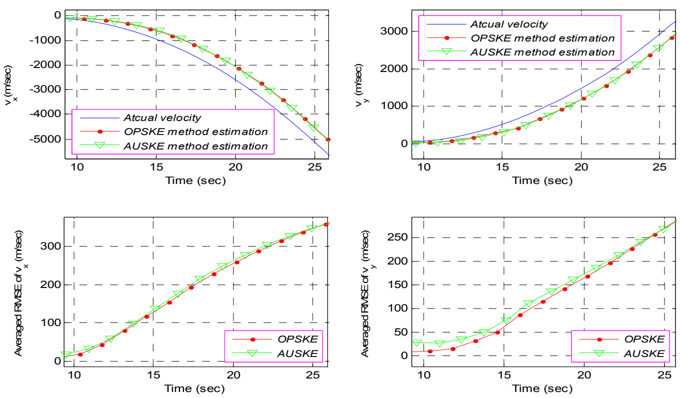


*Fig. 2. The actual value and the estimation of $v^x, v^y$ and RMS errors of x and y velocities estimations by the OPSKE and the AUSKE methods.*

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