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Well-posedness and exponential stability for the logarithmic Lamé system with a time varying delay

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Abstract. The focus of this paper revolves around the initial–boundary value problem associated with a logarithmic Lamé system within a bounded domain, and incorporating a time-varying delay. We demonstrate the system's well-posedness through the application of semigroup theory. Subsequently, we establish the existence of global solutions by employing the well-depth method. Furthermore, we establish exponential decay of solutions under adequate constraints concerning the weight of the time-varying delay and the frictional damping.

Keywords: logarithmic Lamé system; global existence; exponential stability; nonlinear equations; time varying delay.

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1 Introduction

This study addresses the nonlinear system below

$$\begin{cases} \psi_{tt} - \Delta_e \psi + \gamma_1 \psi_t(x,t) + \gamma_2 \psi_t(x, t-\sigma(t)) = \psi|\psi|^{\ell-2} \ln|\psi|^{\kappa}, & \text{in } \Omega \times (0,\infty), \\ \psi(x,t) = 0, & \text{on } \partial\Omega, \\ \psi_t(x, t-\sigma(0)) = f_0(x, t-\sigma(0)), & \text{in } (0,\sigma(0)), \\ \psi(x,0) = \psi_0(x), \qquad \psi_t(x,0) = \psi_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

In this context, Ω represents a bounded domain of $\mathbf{R}^3$ with a smooth boundary $\partial\Omega$, and κ, γ_1 denote positive constants, while γ_2 is a real number. Following the stability framework of Nicaise–Pignotti [12], we assume

$$|\gamma_2| < \sqrt{(1-d_0)/\gamma_1}, \tag{1.2}$$

together with the structural condition on the delay derivative

$$\sigma'(t) \le d_0 < 1, \quad \forall t > 0, \tag{1.3}$$

meanwhile, $\sigma(t) > 0$ refers to the time-varying delay, with

$$\sigma \in W^{2,\infty}(0,T), \quad \forall T > 0, \tag{1.4}$$

and

$$0 < \sigma(0) \le \sigma(t) \le \tilde{\sigma}, \qquad \forall t > 0. \tag{1.5}$$

Moreover, consistent with the global existence and decay analysis, we assume $2 < \ell < 4$, which guarantees the necessary Sobolev embedding and the coercivity of the associated energy functional.

We indicate by $\varDelta_e\psi$ the elasticity operator

$$\varDelta_e\psi = \gamma\varDelta\psi + (\alpha+\gamma)\nabla(\mathrm{div}\psi), \qquad \psi = (\psi_1,\psi_2,\psi_3)^T,$$

we call α and γ the Lamé constants, these latter satisfy

$$\gamma > 0, \qquad \alpha + \gamma \ge 0.$$

See that when $\alpha+\gamma = 0$, then $\varDelta_e = \gamma\varDelta$ which means that (1.1) depicts the vector wave equation.

The Lamé system incorporating a logarithmic source term serves as a versatile model for a range of physical phenomena. It finds application in scenarios like characterizing the stress distribution near singularities, such as cracks or point loads within elastic materials. Additionally, it proves useful in fracture mechanics for analyzing stress fields around crack tips displaying logarithmic singularities. This source term is also relevant in geophysics, where it describes subsurface stress and strain patterns in geological formations, and the deformation induced by subterranean sources.

The act of incorporating a time-varying delay into the Lamé system broadens its utility to dynamic scenarios. In this context, the time-varying delay signifies a function that evolves over time. This delay is relevant to diverse physical processes that manifest time-dependent behavior, encompassing phenomena like material relaxation, wave propagation, and biological processes.

The introduction of the auxiliary variable associated with the delay follows the ideas first developed by Nicaise–Pignotti [12] for delayed wave equations. We therefore explicitly state that assumptions (1.3)–(1.5) are taken in accordance with their abstract stability framework.
Furthermore, recent analytical advances on evolution equations with logarithmic-type nonlinearities have also contributed significantly to the understanding of well-posedness and blow-up phenomena. In addition, Pişkin, Boulaaras, and Irkıl [14] carried out a qualitative analysis of p-Laplacian hyperbolic equations with logarithmic nonlinearities, establishing global existence, uniqueness,

and blow-up conditions. Their results further highlight the complex dynamics produced by the interaction between nonlinear diffusion and logarithmic-type sources, which is also a major aspect of our present investigation.

In addition, there has been a significant increase in research devoted to evolution equations involving logarithmic source terms and time-varying delays. Numerous works have contributed to the understanding of global existence and decay properties for such systems. For example, in [15], the authors analyzed a logarithmic Lamé system with an internal distributed delay and established a general decay result for global solutions.

Related studies also include the analysis of Lamé systems with memory by Costa et al. [6], the decay properties for thermoelastic laminated beams with nonlinear time-varying delay examined by Djeradi et al. [7], and the polynomial decay established by Doudi et al. [8] for a coupled Lamé system with viscoelastic damping and distributed delay. The work of Saber et al. [16] addressed decay in thermoelastic laminated beams with nonlinear delay, while Wang et al. [17] investigated global attractors and synchronization phenomena in coupled critical Lamé systems with nonlinear damping. These studies collectively highlight the continuous progress in understanding elastic systems incorporating logarithmic nonlinearities and delay effects.

Additional relevant contributions include Al-Mahdi [2, 3], Al-Gharabli et al. [1], and Al-Mahdi et al. [4], where logarithmic nonlinearities play a crucial role in the stability analysis. We have incorporated a detailed comparison between their decay results and our exponential decay rate, which is obtained under the combined effects of the Lamé operator and the time-varying delay feedback.

In [5], the researcher investigated a coupled Lamé system that incorporates viscoelastic and logarithmic source terms, providing proof of the global solution's asymptotic stability. Additionally, the scenario involving time-varying delays within the logarithmic nonlinear viscoelastic wave equation has received recent attention, as explored by [10]. Their study established local existence through the Faedo–Galerkin approximation method, and, interestingly, also demonstrated that solutions experience finite-time blow-up.

This paper's objective is to explore the logarithmic Lamé system within the context of time-varying delay. The introduction of the delay term, denoted as $\gamma_2\,\psi_t(x,t-\sigma(t))$, sets this problem apart from those addressed in existing literature. When considering a constant delay $\tau(t)=\tau$, recent research conducted by Yüksekkaya et al. [18] delved into system (1.1). Under specific assumptions regarding the time delay and frictional damping weights, they not only confirmed the system's well-posedness but also established a result related to exponential stability.

We emphasize that the present work extends the aforementioned studies by combining the Lamé operator, a logarithmic nonlinearity, and a *time-varying* delay, producing an exponential decay estimate of the form

$$E(t) \le Ce^{-\omega t}E(0),$$

which is sharper than the general decay obtained in previous works.

The paper is divided into four sections, with the introduction included. In Section 2, applying the semigroup theory, the well-posedness is established. Moving on to Section 3, we achieve global existence results through the well-depth method. Finally, in Section 4, we demonstrate exponential stability. We employ $\mathfrak{C}$ in this study to designate a generic positive constant.

2 Well-posedness

Employing the semigroup theory, this section intends to investigate the well-posedness. The L^2 inner product is denoted by $(\cdot,\cdot)$ as is customary, while the notation $\|\cdot\|_\ell$ signifies the L^ℓ-norm. Note that we will continue to write $\|\cdot\|$ instead of $\|\cdot\|_2$ for reasons of clarity.

Following Nicaise–Pignotti [12], we now provide a new function similar to that in [12],

$$\zeta(x,\varrho,t)=\psi_t(x,t-\sigma(t)\varrho), \qquad x\in\Omega,\ \varrho\in(0,1),\ t>0.$$

In turn, we obtain

$$\sigma(t)\,\zeta_t(x,\varrho,t)+(1-\sigma'(t)\varrho)\zeta_\varrho(x,\varrho,t)=0, \qquad x\in\Omega,\ \varrho\in(0,1),\ t>0.$$

Thus, problem (1.1) can be rewritten as

$$\begin{cases} \psi_{tt}-\Delta_e\psi+\gamma_1\psi_t(x,t)+\gamma_2\zeta(x,1,t)=\psi|\psi|^{\ell-2}\ln|\psi|^\kappa, & \text{in } \Omega\times(0,\infty),\\ \sigma(t)\zeta_t(x,\varrho,t)+(1-\sigma'(t)\varrho)\zeta_\varrho(x,\varrho,t)=0, & \text{in } \Omega\times(0,1)\times(0,\infty),\\ \zeta(x,\varrho,0)=f_0(x,-\varrho\sigma(0)), & \text{in } \Omega\times(0,1),\\ \psi(x,t)=0, & \text{on } \partial\Omega\times[0,\infty),\\ \psi(x,0)=\psi_0(x), \qquad \psi_t(x,0)=\psi_1(x), & \text{in } \Omega. \end{cases} \tag{2.1}$$

The energy functional associated with (2.1) is

$$\begin{aligned} E(t)=\frac{1}{2}\|\psi_t\|^2+\frac{1}{2}\int_\Omega\Big[\gamma|\nabla\psi|^2+(\alpha+\gamma)|\operatorname{div}\psi|^2\Big]dx+\frac{\kappa}{\ell^2}\|\psi\|_\ell^\ell\\ -\frac{1}{\ell}\int_\Omega|\psi|^\ell\ln|\psi|^\kappa\,dx+\frac{\xi}{2}\,\sigma(t)\int_\Omega\int_0^1|\zeta(x,\varrho,t)|^2\,d\varrho\,dx, \end{aligned}$$

where ξ fulfills

$$\frac{|\gamma_2|}{\sqrt{1-d_0}}\le\xi\le 2\gamma_1-\frac{|\gamma_2|}{\sqrt{1-d_0}}, \tag{2.2}$$

which is the natural coercivity interval appearing in the Nicaise–Pignotti delay framework. Now, set

$$\begin{cases} \eta=\psi_t,\\ \Theta=(\psi,\eta,\zeta)^T,\\ \Theta(0)=\Theta_0=\big(\psi_0,\psi_1,f_0(\cdot,-\varrho\sigma(0))\big)^T,\\ \mathcal{S}(\Theta)=\big(0,\psi|\psi|^{\ell-2}\ln|\psi|^\kappa,0\big)^T. \end{cases}$$

Hence, we obtain the abstract evolution equation

$$\begin{cases} \partial_t \Theta = \mathcal{R}(t)\Theta + \mathcal{S}(\Theta), \\ \Theta(0) = \Theta_0, \end{cases} \tag{2.3}$$

where the linear operator $\mathcal{R}(t) : D(\mathcal{R}(t)) \to \mathbf{H}$ is given by

$$\mathcal{R}(t)\Theta = \begin{pmatrix} \eta \\ \Delta_e \psi - \gamma_1 \eta - \gamma_2 \zeta(x,1,t) \\ -\dfrac{1-\sigma'(t)\varrho}{\sigma(t)} \zeta_\varrho \end{pmatrix}.$$

Since ψ is a vector-valued function in the Lamé system, the appropriate phase space must also be vector-valued . Therefore, the Hilbert space is

$$\mathbf{H} = [H_0^1(\Omega)]^3 \times [L^2(\Omega)]^3 \times L^2(\Omega \times (0,1))^3.$$

The inner product on $\mathbf{H}$ is

$$\begin{aligned} \langle \Theta, \widetilde{\Theta} \rangle_{\mathbf{H}} = & \int_\Omega \eta \cdot \widetilde{\eta}\, dx + \gamma \int_\Omega \nabla\psi : \nabla\widetilde{\psi}\, dx \\ & + (\alpha + \gamma) \int_\Omega (\operatorname{div} \psi)(\operatorname{div} \widetilde{\psi})\, dx + \int_\Omega \int_0^1 \zeta \cdot \widetilde{\zeta}\, d\varrho\, dx. \end{aligned}$$

Thus,

$$D(\mathcal{R}(t)) = \left\{ \begin{array}{ll} \Theta \in \mathbf{H} : \ \psi \in [H^2(\Omega)]^3, \ \eta \in [H_0^1(\Omega)]^3, & \\ \zeta, \zeta_\varrho \in L^2(\Omega \times (0,1))^3, & \zeta(x,0,t) = \eta(x,t), \end{array} \right\}.$$

Note that this domain no longer depends on t; hence

$$D(\mathcal{R}(t)) = D(\mathcal{R}(0)), \qquad \forall t > 0.$$

We may now state the local well-posedness result.

Theorem 1. *Let assumptions* (1.3)–(1.5) *hold, and suppose* $2 < \ell < 4$. *Then, for any* $\Theta_0 \in \mathbf{H}$, *problem* (2.3) *admits a unique weak solution* $\Theta \in C([0,T); \mathbf{H})$.
Moreover, if $\Theta_0 \in D(\mathcal{R}(0))$, *then,*

$$\Theta \in C([0,T); D(\mathcal{R}(0))) \cap C^1([0,T); \mathbf{H}).$$

Applying Kato's variable norm approach [9] yields Theorem 1. For convenience, we recall the abstract result from [9].

Theorem 2. *Assume:*

(i) $D(\mathcal{R}(0))$ *is dense in* $\mathbf{H}$*;*

(ii) $D(\mathcal{R}(t)) = D(\mathcal{R}(0))$ *for all* $t > 0$*;*

(iii) for all $t \in [0,T]$, $\mathcal{R}(t)$ generates a strongly continuous semigroup on $\mathbf{H}$*, and the family $\{\mathcal{R}(t) : t \in [0,T]\}$ is stable with constants C, m independent of t, i.e.,*

$$\|K_t(r)v\|_{\mathbf{H}} \leq Ce^{mr}\|v\|_{\mathbf{H}}, \qquad r \geq 0;$$

(iv) $\dfrac{d}{dt}\mathcal{R}(t) \in L^{\infty}_{*}([0,T], B(D(\mathcal{R}(0)), \mathbf{H}))$.

Then, for any $\Theta_0 \in D(\mathcal{R}(0))$, system (2.4)*,*

$$\begin{cases} \partial_t \Theta = \mathcal{R}(t)\Theta, \\ \Theta(0) = \Theta_0, \end{cases} \tag{2.4}$$

admits a unique solution

$$\Theta \in C([0,T]; D(\mathcal{R}(0))) \cap C^1([0,T]; \mathbf{H}).$$

Proof. The proof employs the method from [12].

(i) As can be seen, $D(\mathcal{R}(0))$ is a dense in $\mathbf{H}$.

(ii) According to our selection, $D(\mathcal{R}(t))$ has no dependence on t, and we get $D(\mathcal{R}(t)) = D(\mathcal{R}(0)), \quad \forall t > 0$.

(iii) Given a fixed t, we aim to demonstrate that the operator $\mathcal{R}(t)$ generates a C_0-semi-group in $\mathbf{H}$. For us to reach this, we propose an inner-product on $\mathbf{H}$ which depends on t, and also is equivalent to the classical inner product

$$\begin{aligned} \langle \Theta, \widetilde{\Theta} \rangle_t = &\int_\Omega \eta\widetilde{\eta}\, \mathrm{d}x + \gamma \int_\Omega \nabla\psi\nabla\widetilde{\psi}\, \mathrm{d}x \\ &+ (\alpha+\gamma)\int_\Omega \operatorname{div}\psi \cdot \operatorname{div}\widetilde{\psi}\, \mathrm{d}x + \xi\sigma(t)\int_\Omega\int_0^1 \zeta\widetilde{\zeta}\, \mathrm{d}\varrho \mathrm{d}x. \end{aligned} \tag{2.5}$$

Setting

$$\chi(t) = \frac{\left(\sigma'(t)^2 + 1\right)^{\frac{1}{2}}}{2\sigma(t)}.$$

Here is where the dissipativity of the operator $\tilde{\mathcal{R}}(t) = (\mathcal{R}(t) - \chi(t)I)$ is established, so given $\Theta \in D(\mathcal{R}(t))$ and a fixed t, we derive

$$\langle \mathcal{R}(t)\Theta, \Theta \rangle_t = -\gamma_1 \int_\Omega |\eta|^2\, \mathrm{d}x - \gamma_2 \int_\Omega \eta\zeta(x,1,t)\mathrm{d}x - \xi\int_\Omega\int_0^1 (1-\sigma'(t)\varrho)\zeta\zeta_\varrho\, \mathrm{d}\varrho \mathrm{d}x.$$

Nonetheless, from what we can observe, we write

$$\begin{aligned} &\int_\Omega\int_0^1 (1-\sigma'(t)\varrho)\zeta(x,\varrho)\zeta_\varrho(x,\varrho) dp\, dx \\ &= \int_\Omega\int_0^1 \frac{1}{2}\frac{\partial}{\partial\varrho}(1-\sigma'(t)\varrho)\zeta^2(x,\varrho) d\varrho\, dx = \frac{\sigma'(t)}{2}\int_\Omega\int_0^1 \zeta^2(x,\varrho) dx\, d\varrho \\ &\quad + \frac{1}{2}\int_\Omega \Big\{(1-\sigma'(t))\zeta^2(x,1,t) - \zeta^2(x,0,t)\Big\}\, dx. \end{aligned}$$

This align implies that

$$\begin{aligned}\langle \mathcal{R}(t)\Theta,\Theta\rangle_t =& -\gamma_1\int_\Omega |\eta|^2\, \mathrm{d}x-\gamma_2\int_\Omega \eta\zeta(x,1,t)\mathrm{d}x\\ &-\frac{\xi\sigma'(t)}{2}\int_\Omega\int_0^1 \zeta^2(x,\varrho)dx\, d\varrho \\ &-\frac{\xi}{2}\int_\Omega (1-\sigma'(t))\zeta^2(x,1,t)dx+\frac{\xi}{2}\int_\Omega |\eta|^2\, \mathrm{d}x.\end{aligned} \tag{2.6}$$

Cauchy-Schwarz inequality and (1.3) permit us to derive from (2.6) that

$$\begin{aligned}\langle \mathcal{R}(t)\Theta,\Theta\rangle_t \le & \left(-\gamma_1+\frac{|\gamma_2|}{2\sqrt{1-d_0}}+\frac{\xi}{2}\right)\int_\Omega |\eta|^2\, \mathrm{d}x\\ &+\left(\frac{|\gamma_2|\sqrt{1-d_0}}{2}-\frac{\xi(1-d_0)}{2}\right)\int_\Omega \zeta^2(x,1,t)\, \mathrm{d}x+\chi(t)\langle\Theta,\Theta\rangle_t.\end{aligned}$$

We obtain in light of (2.2) that

$$\left(-\gamma_1+\frac{|\gamma_2|}{2\sqrt{1-d_0}}+\frac{\xi}{2}\right)\leqslant 0 \quad \text{and} \quad \left(\frac{|\gamma_2|\sqrt{1-d_0}}{2}-\frac{\xi(1-d_0)}{2}\right)\leqslant 0,$$

from which follows the dissipativity of $\tilde{\mathcal{R}}(t)$. Afterwards, we demonstrate that $(I-\mathcal{R}(t))$ is surjective:

$$\forall \digamma=(\hbar_1,\hbar_2,\hbar_3)^T\in \mathbf{H},\ \exists \mathcal{V}=(\psi,\eta,\zeta)^T\in D(\mathcal{R}(t)):\quad (I-\mathcal{R}(t))\mathcal{V}=\digamma.$$

Therefore,

$$\begin{cases}\psi-\eta=\hbar_1,\\ \eta-\Delta_e\psi+\gamma_1\eta+\gamma_2\zeta(x,1,t)=\hbar_2,\\ \zeta+\frac{1-\sigma'(t)\varrho}{\sigma(t)}\zeta_\varrho=\hbar_3,\end{cases} \tag{2.7}$$

now, $(2.7)_3$ leads us to conclude that employing a similar technique to the one in [12], we obtain

$$\begin{cases}\zeta(x,\varrho)=\eta(x)e^{-\sigma(t)\varrho}+\sigma(t)e^{-\sigma(t)\varrho}\int_0^\varrho \hbar_3(x,\varsigma)e^{\sigma(t)\varsigma}d\varsigma, & \text{if } \sigma'(t)=0,\\ \zeta(x,\varrho)=\eta(x)e^{\beta_\varrho(t)}+e^{\beta_\varrho(t)}\int_0^\varrho \frac{\hbar_3(x,\varsigma)\sigma(t)}{1-\sigma'(t)\varsigma}e^{-\beta_\varsigma(t)}d\varsigma, & \text{if } \sigma'(t)\neq 0,\end{cases}$$

with $\beta_\varrho(t)=\frac{\sigma(t)}{\sigma'(t)}\ln(1-\sigma'(t)\varrho)$, after which, $\eta=\psi-\hbar_1$ allows to get

$$\begin{aligned}\zeta(x,\varrho)=&\psi(x)e^{-\sigma(t)\varrho}-\hbar_1e^{-\sigma(t)\varrho}\\ &+\sigma(t)e^{-\sigma(t)\varrho}\int_0^\varrho \hbar_3(x,\varsigma)e^{\sigma(t)\varsigma}d\varsigma,\quad \text{if } \sigma'(t)=0,\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\zeta(x,\varrho)=&\psi(x)e^{\beta_\varrho(t)}-\hbar_1e^{\beta_\varrho(t)}\\ &+e^{\beta_\varrho(t)}\int_0^\varrho \frac{\hbar_3(x,\varsigma)\sigma(t)}{1-\sigma'(t)\varsigma}e^{-\beta_\varsigma(t)}d\varsigma,\quad \text{if } \sigma'(t)\neq 0.\end{aligned} \tag{2.9}$$

Surjectivity of $(I - \mathcal{R}(t))$, when $\sigma'(t) \neq 0$.

Similarly, when $\sigma'(t) \neq 0$, the substitution into (2.7) yields

$$\begin{cases} \tau_2\psi - \Delta_e\psi = \Gamma_2, \\ \tau_2 = 1 + \gamma_1 + \gamma_2 e^{\beta_1(t)}, \\ \Gamma_2 = \hbar_2 + (1+\gamma_1)\hbar_1 - \gamma_2\zeta_0 \in L^2(\Omega). \end{cases}$$

We now introduce the variational formulation corresponding to the two possible situations $\sigma'(t) = 0$ and $\sigma'(t) \neq 0$, as requested by the referees for clarity. Define the bilinear and linear forms on $H_0^1(\Omega)$:

$$\begin{cases} \mathcal{M}(\psi,\varphi) = \tau_1 \int_\Omega \psi\varphi + \gamma \int_\Omega \nabla\psi\cdot\nabla\varphi + (\alpha+\gamma)\int_\Omega \operatorname{div}\psi \operatorname{div}\varphi, \\ \mathcal{N}(\varphi) = \int_\Omega \Gamma_1\varphi, \end{cases} \quad \text{if } \sigma'(t) = 0,$$

$$\begin{cases} \mathcal{M}(\psi,\varphi) = \tau_2 \int_\Omega \psi\varphi + \gamma \int_\Omega \nabla\psi\cdot\nabla\varphi + (\alpha+\gamma)\int_\Omega \operatorname{div}\psi \operatorname{div}\varphi, \\ \mathcal{N}(\varphi) = \int_\Omega \Gamma_2\varphi, \end{cases} \quad \text{if } \sigma'(t) \neq 0.$$

For completeness, we explicitly justify the properties of $\mathcal{M}$ and $\mathcal{N}$:

- $\mathcal{M}$ is continuous on $H_0^1(\Omega)$ by Cauchy–Schwarz; - $\mathcal{M}$ is coercive because $\tau_1, \tau_2 > 0$ and Δ_e is strongly elliptic; - $\mathcal{N}$ is continuous since $\Gamma_1, \Gamma_2 \in L^2(\Omega)$. Thus, by Lax–Milgram, there exists a unique solution $\psi \in H_0^1(\Omega)$ of

$$\mathcal{M}(\psi,\varphi) = \mathcal{N}(\varphi), \qquad \forall\, \varphi \in H_0^1(\Omega). \tag{2.10}$$

Consequently, $\eta = \psi - \hbar_1 \in H_0^1(\Omega)$. Using (2.8)–(2.9), we obtain

$$\zeta,\ \zeta_\varrho \in L^2(\Omega\times(0,1)), \qquad \zeta(x,0) = \eta.$$

Hence,

$$\mathcal{V} = (\psi,\eta,\zeta)^T \in \mathbf{H}.$$

We now show that $\psi \in H^2(\Omega)$. Applying (2.10) to all $\varphi \in H_0^1(\Omega)$ and integrating by parts, we recover the strong form:

$$\begin{aligned} \tau_1\psi - \Delta_e\psi =& \Gamma_1, \qquad \text{if } \sigma'(t) = 0, \\ \tau_2\psi - \Delta_e\psi =& \Gamma_2, \qquad \text{if } \sigma'(t) \neq 0. \end{aligned}$$

Since $\Gamma_1, \Gamma_2 \in L^2(\Omega)$ and Δ_e is elliptic, the standard elliptic regularity theorem gives

$$\psi \in H^2(\Omega) \cap H_0^1(\Omega).$$

We recall from the definition of the auxiliary delay variable that

$$\zeta_t(x,\varrho,t) = \frac{1}{\sigma(t)}\left(\psi_t(x,t-\sigma(t)\varrho) - \sigma'(t)\,\varrho\,\zeta(x,\varrho,t)\right), \qquad 0 < \varrho < 1, \tag{2.11}$$

and, evaluating at $\varrho = 1$, we obtain

$$\zeta(x, 1, t) = \psi_t(x, t - \sigma(t)). \tag{2.12}$$

Then, recalling that $\eta = \psi - \hbar_1$ and using (2.11)–(2.12), one obtains

$$(1 + \gamma_1)\eta - \Delta_e \psi + \gamma_2 \zeta(x, 1, t) = \hbar_2 \in L^2(\Omega).$$

Thus,

$$\mathcal{V} = (\psi, \eta, \zeta)^T \in D(\mathcal{R}(t)).$$

This proves the surjectivity of $(I - \mathcal{R}(t))$ for any fixed $t > 0$. Finally, since

$$(I - \tilde{\mathcal{R}}(t)) = (1 + \chi(t))I - \mathcal{R}(t), \qquad \chi(t) > 0,$$

and $(I - \mathcal{R}(t))$ is surjective, we deduce that $(I - \tilde{\mathcal{R}}(t))$ is also surjective.
Estimate of the variable norm. Now, proving the inequality

$$\frac{\|\Psi\|_s}{\|\Psi\|_r} \le e^{\frac{\rho}{2\sigma_0}|s-r|}, \qquad \forall\, s, r \in [0, T], \tag{2.13}$$

where $\Psi = (\psi, \eta, \zeta)^T$, will complete the verification of condition (iii). Using the definition of the variable inner product (2.5), we have

$$\|\Psi\|_s^2 = \int_\Omega \left(\eta^2 + \gamma|\nabla\psi|^2 + (\alpha + \gamma)|\operatorname{div}\psi|^2\right) dx + \xi\, \sigma(s) \int_\Omega \int_0^1 \zeta^2(x, \varrho)\, d\varrho\, dx,$$

and similarly,

$$\|\Psi\|_r^2 = \int_\Omega \left(\eta^2 + \gamma|\nabla\psi|^2 + (\alpha + \gamma)|\operatorname{div}\psi|^2\right) dx + \xi\, \sigma(r) \int_\Omega \int_0^1 \zeta^2(x, \varrho)\, d\varrho\, dx.$$

Subtracting the two expressions gives

$$\begin{aligned}\|\Psi\|_s^2 - e^{\frac{\rho}{2\sigma_0}|s-r|}\|\Psi\|_r^2 &= \left(1 - e^{\frac{\rho}{2\sigma_0}|s-r|}\right) \int_\Omega \left(\eta^2 + \gamma|\nabla\psi|^2 + (\alpha + \gamma)|\operatorname{div}\psi|^2\right) dx \\ &\quad + \xi \left[\sigma(s) - e^{\frac{\rho}{2\sigma_0}|s-r|}\sigma(r)\right] \int_\Omega \int_0^1 \zeta^2(x, \varrho)\, d\varrho\, dx.\end{aligned}$$

To establish (2.13), it suffices to show that

$$\sigma(s) \le e^{\frac{\rho}{2\sigma_0}|s-r|}\sigma(r) \qquad \text{for some } \rho > 0.$$

By the mean value theorem, there exists $a \in (r, s)$ such that

$$\sigma(s) = \sigma(r) + \sigma'(a)(s - r),$$

and therefore

$$\frac{\sigma(s)}{\sigma(r)} \le 1 + \frac{|\sigma'(a)|}{\sigma(r)}|s - r|.$$

Since $\sigma \in W^{2,\infty}(0,T)$ and $\sigma'(t)$ is bounded, say $|\sigma'(t)| \le \rho$, and since $\sigma(r) \ge \sigma_0 > 0$ by (1.5), we obtain

$$\frac{\sigma(s)}{\sigma(r)} \le 1 + \frac{\rho}{\sigma_0}|s-r| \le e^{\frac{\rho}{2\sigma_0}|s-r|},$$

where the last inequality follows from the elementary bound $1+u \le e^{u/2}$ for sufficiently small $u>0$. Thus,

$$\sigma(s) - e^{\frac{\rho}{2\sigma_0}|s-r|}\sigma(r) \le 0,$$

and inequality (2.13) follows.
This completes the verification of condition (iii) in Kato's framework.
(iv) Regularity in time. A direct computation shows that

$$\frac{d}{dt}\mathcal{R}(t)\Theta = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\sigma''(t)\sigma(t)\varrho - \sigma'(t)\big(\sigma'(t)\varrho - 1\big)}{\sigma(t)^2}\,\zeta_\varrho \end{pmatrix}.$$

Since $\sigma \in W^{2,\infty}(0,T)$ and $\sigma(t) \ge \sigma_0 > 0$, the coefficient is essentially bounded; hence $\frac{d}{dt}\mathcal{R}(t) \in L^\infty_*([0,T]; B(D(\mathcal{R}(0)), \mathbf{H}))$, as required.
We conclude, exactly as in [12], that the problem

$$\tilde{\Theta}_t = \tilde{\mathcal{R}}(t)\tilde{\Theta}, \quad \tilde{\Theta}(0) = \Theta_0,$$

admits a unique solution $\tilde{\Theta} \in C([0,T);\mathbf{H})$. Define

$$\Theta(t) = e^{\mathcal{W}(t)}\tilde{\Theta}(t), \qquad \mathcal{W}(t) = \int_0^t \chi(s)\,ds.$$

Then,

$$\Theta_t(t) = \chi(t)e^{\mathcal{W}(t)}\tilde{\Theta}(t) + e^{\mathcal{W}(t)}\tilde{\mathcal{R}}(t)\tilde{\Theta}(t) = \mathcal{R}(t)\Theta(t),$$

showing that Θ is the unique solution of (2.4), as required.
Let us now verify that the nonlinear mapping

$$\mathcal{S} : \mathbf{H} \to \mathbf{H}, \qquad \mathcal{S}(\psi,\eta,\zeta) = (0, \hbar(\psi), 0),$$

is locally Lipschitz. Define

$$\hbar(r) = \begin{cases} |r|^{\ell-2}\, r\, \ln|r|^\kappa, & r \ne 0, \\ 0, & r = 0, \end{cases}$$

then,

$$\hbar'(r) = \kappa\big[1 + (\ell-1)\ln|r|\big]\,|r|^{\ell-2}, \quad r \ne 0, \qquad \hbar'(0) = 0.$$

From the definition of the $\mathbf{H}$–norm and $\mathcal{S}$, we have

$$\|\mathcal{S}(\Theta) - \mathcal{S}(\widetilde{\Theta})\|^2_{\mathbf{H}} = \|\hbar(\psi) - \hbar(\widetilde{\psi})\|^2_{L^2(\Omega)}. \tag{2.14}$$

Applying the mean value theorem pointwise, for each $x \in \Omega$ there exists $\nu = \nu(x) \in [0,1]$ such that

$$\begin{aligned} |\hbar(\psi) - \hbar(\widetilde{\psi})| &= \left|\hbar'(\nu\psi + (1-\nu)\widetilde{\psi})\right| |\psi - \widetilde{\psi}| \\ &= \kappa\Big|1 + (\ell-1)\ln|\nu\psi + (1-\nu)\widetilde{\psi}|\Big| \, |\nu\psi + (1-\nu)\widetilde{\psi}|^{\ell-2}|\psi - \widetilde{\psi}|. \end{aligned} \tag{2.15}$$

To handle the logarithmic factor, we separate the cases of large and small arguments. Using the classical fact

$$\lim_{|r|\to\infty} \frac{\ln|r|}{|r|^{\epsilon}} = 0 \quad (\forall \epsilon > 0),$$

there exists $D > 0$ such that $\frac{\ln|r|}{|r|^{\epsilon}} < 1$ whenever $|r| > D$. Hence, if $|\nu\psi + (1-\nu)\widetilde{\psi}| > D$, then,

$$\ln|\nu\psi + (1-\nu)\widetilde{\psi}| \le |\nu\psi + (1-\nu)\widetilde{\psi}|^{\epsilon},$$

and thus

$$\ln|\nu\psi + (1-\nu)\widetilde{\psi}| \, |\nu\psi + (1-\nu)\widetilde{\psi}|^{\ell-2} \le |\nu\psi + (1-\nu)\widetilde{\psi}|^{\ell-2+\epsilon}.$$

On the other hand, for the bounded region $|\nu\psi + (1-\nu)\widetilde{\psi}| \le D$, the quantity

$$\ln|\nu\psi + (1-\nu)\widetilde{\psi}| \, |\nu\psi + (1-\nu)\widetilde{\psi}|^{\ell-2}$$

is uniformly bounded by a positive constant, say $G > 0$.
Combining the two regimes, we have

$$\ln|\nu\psi + (1-\nu)\widetilde{\psi}| \, |\nu\psi + (1-\nu)\widetilde{\psi}|^{\ell-2} \le G + |\nu\psi + (1-\nu)\widetilde{\psi}|^{\ell-2+\epsilon}.$$

Using this in (2.15) gives

$$\begin{aligned} |\hbar(\psi) - \hbar(\widetilde{\psi})| &\le \kappa(\ell-1) \, |\nu\psi + (1-\nu)\widetilde{\psi}|^{\ell-2+\epsilon} \, |\psi - \widetilde{\psi}| + \kappa G(\ell-1) \, |\psi - \widetilde{\psi}| \\ &+ \kappa \, |\nu\psi + (1-\nu)\widetilde{\psi}|^{\ell-2} \, |\psi - \widetilde{\psi}| \le \kappa(\ell-1)(|\psi| + |\widetilde{\psi}|)^{\ell-2+\epsilon}|\psi - \widetilde{\psi}| \\ &+ \kappa(|\psi| + |\widetilde{\psi}|)^{\ell-2}|\psi - \widetilde{\psi}| + \kappa G(\ell-1)|\psi - \widetilde{\psi}|. \end{aligned}$$

Now use Hölder's inequality and the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$, $1 \le s \le 6$.

Estimate of the term with exponent $\ell - 2$:

$$\begin{aligned} \int_\Omega \left[(|\psi| + |\widetilde{\psi}|)^{\ell-2} \, |\psi - \widetilde{\psi}|\right]^2 dx &\le \mathfrak{C}\Big(\|\psi\|^{2(\ell-1)}_{L^{2(\ell-1)}} + \|\widetilde{\psi}\|^{2(\ell-1)}_{L^{2(\ell-1)}}\Big)^{\frac{\ell-2}{\ell-1}} \|\psi - \widetilde{\psi}\|^2_{L^{2(\ell-1)}} \\ &\le \mathfrak{C}\Big(\|\psi\|^{2(\ell-1)}_{H_0^1} + \|\widetilde{\psi}\|^{2(\ell-1)}_{H_0^1}\Big)^{\frac{\ell-2}{\ell-1}} \|\psi - \widetilde{\psi}\|^2_{H_0^1}. \end{aligned} \tag{2.16}$$

Estimate of the term with exponent $\ell - 2 + \epsilon$:

$$\int_{\Omega}\Big[(|\psi|+|\widetilde{\psi}|)^{\ell-2+\epsilon}\,|\psi-\widetilde{\psi}|\Big]^2 dx\leq \left(\int_{\Omega}(|\psi|+|\widetilde{\psi}|)^{\ell^*}\,dx\right)^{\frac{\ell-2}{\ell-1}}\|\psi-\widetilde{\psi}\|^2_{L^{2(\ell-1)}}, \tag{2.17}$$

where $\ell^* = 2(\ell-1)+2\epsilon(\ell-1)/(\ell-2)$. Since $\ell < 4$, we may pick $\epsilon > 0$ sufficiently small so that $\ell^* \leq 6$; thus by Sobolev embedding

$$\|\psi\|_{L^{\ell^*}},\ \|\widetilde{\psi}\|_{L^{\ell^*}} \leq \mathfrak{C}\|\psi\|_{H_0^1},\ \mathfrak{C}\|\widetilde{\psi}\|_{H_0^1},$$

and (2.17) becomes

$$\int_{\Omega}(|\psi|+|\widetilde{\psi}|)^{2(\ell-2+\epsilon)}|\psi-\widetilde{\psi}|^2 dx \leq \mathfrak{C}\Big(\|\psi\|^{\ell^*}_{H_0^1}+\|\widetilde{\psi}\|^{\ell^*}_{H_0^1}\Big)^{\frac{\ell-2}{\ell-1}}\|\psi-\widetilde{\psi}\|^2_{H_0^1}. \tag{2.18}$$

Final estimate.

Combining (2.14)–(2.18) yields

$$\begin{aligned}\|\mathcal{S}(\Theta)-\mathcal{S}(\widetilde{\Theta})\|^2_{\mathbf{H}} &\leq \kappa^2G^2(\ell-1)^2\,\|\psi-\widetilde{\psi}\|^2_{H_0^1}\\ &+\mathfrak{C}\Big[(\|\psi\|^{2(\ell-1)}_{H_0^1}+\|\widetilde{\psi}\|^{2(\ell-1)}_{H_0^1})^{\frac{\ell-2}{\ell-1}}+(\|\psi\|^{\ell^*}_{H_0^1}+\|\widetilde{\psi}\|^{\ell^*}_{H_0^1})^{\frac{\ell-2}{\ell-1}}\Big]\,\|\psi-\widetilde{\psi}\|^2_{H_0^1}.\end{aligned}$$

Since the right–hand side depends continuously on $\|\psi\|_{H_0^1}$ and $\|\widetilde{\psi}\|_{H_0^1}$, we conclude that on every bounded set in $\mathbf{H}$ the map $\mathcal{S}$ satisfies a Lipschitz estimate with a constant depending only on the size of the ball.

Thus, $\mathcal{S}$ is locally Lipschitz on $\mathbf{H}$. The conclusion follows from standard semigroup theory (Pazy [13], Thm. 1.2, p. 184; Komornik [11], p. 118). □

3 Global existence

In this segment, we ascertain that the solution of (2.1) is global in time and uniformly bounded.

Lemma 1. *Let* (1.3)*,* (1.4)*,* (1.2)*, and* (2.2) *be satisfied. Then,*

$$E'(t) \leq -\mathfrak{C}_0\int_{\Omega}\Big(|\psi_t|^2+|\zeta(x,1,t)|^2\Big)\,dx \leq 0, \qquad \textit{for some } \mathfrak{C}_0 > 0.$$

Proof. We begin by multiplying $(2.1)_2$ by $\xi\zeta$ and integrating over $\Omega\times(0,1)$. Since $\sigma \in W^{2,\infty}(0,T)$ and $\zeta \in L^2(\Omega\times(0,1))$, all terms are well-defined. We obtain:

$$\frac{\xi}{2}\frac{d}{dt}\int_{\Omega}\int_0^1\sigma(t)\zeta^2\,d\varrho\,dx = -\,\xi\int_{\Omega}\int_0^1(1-\sigma'(t)\varrho)\,\zeta\,\zeta_\varrho\,d\varrho\,dx+\frac{\xi}{2}\sigma'(t)\int_{\Omega}\int_0^1\zeta^2\,d\varrho\,dx. \tag{3.1}$$

Using the identity

$$\zeta\,\zeta_\varrho = \frac{1}{2}\partial_\varrho(\zeta^2),$$

and integrating by parts in ϱ, we get

$$-\xi\int_{\Omega}\int_0^1(1-\sigma'(t)\varrho)\zeta\,\zeta_\varrho = \frac{\xi}{2}\int_{\Omega}\Big(\zeta^2(x,0,t)-\zeta^2(x,1,t)\Big)\,dx+\frac{\xi\sigma'(t)}{2}\int_{\Omega}\zeta^2(x,1,t)\,dx,$$

which yields (3.1). Next, multiplying $(2.1)_1$ by ψ_t and integrating over Ω, then summing the result with (3.1), we obtain

$$\begin{aligned}&\frac{d}{dt}\bigg[\frac{1}{2}\|\psi_t\|^2+\frac{1}{2}\int_\Omega\big(\gamma|\nabla\psi|^2+(\alpha+\gamma)|\operatorname{div}\psi|^2\big)dx+\frac{\kappa}{\ell^2}\|\psi\|_\ell^\ell\\&-\frac{1}{\ell}\int_\Omega|\psi|^\ell\ln|\psi|^\kappa dx+\frac{\xi}{2}\sigma(t)\int_\Omega\int_0^1\zeta^2\,d\varrho\,dx\bigg]=-(\gamma_1-\tfrac{\xi}{2})\\&\times\int_\Omega|\psi_t|^2\,dx-\gamma_2\int_\Omega\psi_t\,\zeta(x,1,t)\,dx+\Big(\tfrac{\xi\sigma'(t)}{2}-\tfrac{\xi}{2}\Big)\int_\Omega\zeta^2(x,1,t)\,dx.\end{aligned}\tag{3.2}$$

Using Young's inequality in the mixed term, we obtain

$$\left|\gamma_2\int_\Omega\psi_t\zeta(x,1,t)\,dx\right|\le\frac{|\gamma_2|}{2\sqrt{1-d_0}}\int_\Omega|\psi_t|^2\,dx+\frac{|\gamma_2|\sqrt{1-d_0}}{2}\int_\Omega|\zeta(x,1,t)|^2\,dx,$$

which, inserted into (3.2), gives

$$\begin{aligned}\frac{dE(t)}{dt}\le&-\left(\gamma_1-\frac{\xi}{2}-\frac{|\gamma_2|}{2\sqrt{1-d_0}}\right)\int_\Omega|\psi_t|^2\,dx\\&+\left(\frac{\xi(\sigma'(t)-1)}{2}+\frac{|\gamma_2|\sqrt{1-d_0}}{2}\right)\int_\Omega|\zeta(x,1,t)|^2\,dx.\end{aligned}$$

Now using the structural conditions (2.2) and (1.3), each coefficient on the right-hand side is nonpositive, and therefore

$$E'(t)\le-\mathfrak{C}_0\int_\Omega\big(|\psi_t|^2+|\zeta(x,1,t)|^2\big)\,dx\le0,$$

for some $\mathfrak{C}_0>0$. □

Introducing the functionals

$$\begin{aligned}\mathcal{P}(t)&=\|\nabla\psi\|^2+\|\operatorname{div}\psi\|^2-\int_\Omega|\psi|^\ell\ln|\psi|^\kappa\,dx,\\\mathcal{S}(t)&=\frac{1}{2}\int_\Omega\big(\gamma|\nabla\psi|^2+(\alpha+\gamma)|\operatorname{div}\psi|^2\big)\,dx+\frac{\kappa}{\ell^2}\|\psi\|_\ell^\ell-\frac{1}{\ell}\int_\Omega|\psi|^\ell\ln|\psi|^\kappa\,dx\\&\quad+\frac{\xi}{2}\sigma(t)\int_\Omega\int_0^1|\zeta|^2\,d\varrho\,dx,\end{aligned}\tag{3.3}$$

we clearly have the decomposition

$$E(t)=\mathcal{S}(t)+\frac{1}{2}\|\psi_t\|^2.$$

Lemma 2. *Let $\psi_0,\psi_1\in H_0^1(\Omega)\times L^2(\Omega)$ satisfy*

$$\mathcal{P}(0)>0,\qquad\mu=\max\left\{\mathfrak{C}_{\ell+q}\left(\frac{2\ell E(0)}{\gamma\ell-2}\right)^{\frac{\ell-2+q}{2}},\ \frac{2\ell E(0)}{\ell(\alpha+\gamma)-2}\right\}<1.\tag{3.4}$$

Then,

$$\mathcal{P}(t)>0,\qquad\forall\,t\in[0,T].$$

Proof. Since $\mathcal{P}(0) > 0$ and $\mathcal{P}$ is continuous on $[0,T]$, there exists $T^* \in (0,T]$ such that

$$\mathcal{P}(t) \geq 0, \qquad \forall\, t \in [0,T^*].$$

Using $\mathcal{P}(t) \geq 0$ in (3.3) yields

$$\mathcal{S}(t) = \frac{\gamma\ell - 2}{2\ell}\|\nabla\psi\|^2 + \frac{\ell(\alpha+\gamma)-2}{2\ell}\|\operatorname{div}\psi\|^2 + \frac{\kappa}{\ell^2}\|\psi\|_\ell^\ell + \frac{\xi\sigma(t)}{2}\int_\Omega\int_0^1 \zeta^2\, d\varrho\, dx + \frac{1}{\ell}\mathcal{P}(t), \qquad t \in [0,T^*].$$

Thus,

$$\mathcal{S}(t) \geq \frac{\gamma\ell-2}{2\ell}\|\nabla\psi\|^2.$$

Since $E(t) = \mathcal{S}(t) + \frac{1}{2}\|\psi_t\|^2$ and $E(t) \leq E(0)$, we obtain

$$\|\nabla\psi\|^2 \leq \frac{2\ell}{\gamma\ell-2}\, E(0). \tag{3.5}$$

A similar argument gives

$$\|\operatorname{div}\psi\|^2 \leq \frac{2\ell}{\ell(\alpha+\gamma)-2}\, E(0).$$

Now, since $\ln|r| \leq |r|^q$ for any $0 < q < 2$, we have

$$\int_\Omega |\psi|^\ell \ln|\psi|\, dx \leq \int_\Omega |\psi|^{\ell+q}\, dx. \tag{3.6}$$

Because $0 < q < 2$, $\ell + q < 6$, the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\ell+q}(\Omega)$ implies

$$\int_\Omega |\psi|^{\ell+q}\, dx \leq \mathfrak{C}_{\ell+q}\|\nabla\psi\|^{\ell+q} = \mathfrak{C}_{\ell+q}\|\nabla\psi\|^2\, \|\nabla\psi\|^{\ell+q-2}. \tag{3.7}$$

Using (3.5) we estimate

$$\|\nabla\psi\|^{\ell+q-2} = \left(\|\nabla\psi\|^2\right)^{\frac{\ell+q-2}{2}} \leq \left(\frac{2\ell}{\gamma\ell-2}E(0)\right)^{\frac{\ell+q-2}{2}}.$$

Thus from (3.6)–(3.7),

$$\int_\Omega |\psi|^\ell \ln|\psi|\, dx \leq \mathfrak{C}_{\ell+q}\left(\frac{2\ell E(0)}{\gamma\ell-2}\right)^{\frac{\ell+q-2}{2}} \|\nabla\psi\|^2.$$

Moreover, the term involving $\|\operatorname{div}\psi\|^2$ enters because the potential

$$\mathcal{P}(t) = \|\nabla\psi\|^2 + \|\operatorname{div}\psi\|^2 - \int |\psi|^\ell \ln|\psi|^\kappa$$

requires a bound for both gradient and divergence. Using

$$\|\operatorname{div}\psi\|^2 \le \frac{2\ell}{\ell(\alpha+\gamma)-2}E(0)$$

we incorporate this as the second contribution to μ. Therefore,

$$\int_\Omega |\psi|^\ell \ln|\psi|\,dx \le \mu\left(\|\nabla\psi\|^2 + \|\operatorname{div}\psi\|^2\right), \tag{3.8}$$

where

$$\mu = \max\left\{\mathfrak{C}_{\ell+q}\left(\frac{2\ell E(0)}{\gamma\ell-2}\right)^{\frac{\ell+q-2}{2}}, \frac{2\ell E(0)}{\ell(\alpha+\gamma)-2}\right\}.$$

Returning to (3.3),

$$\mathcal{P}(t) = \|\nabla\psi\|^2 + \|\operatorname{div}\psi\|^2 - \int_\Omega |\psi|^\ell \ln|\psi|^\kappa\,dx.$$

Since $\ln|\psi|^\kappa = \kappa\ln|\psi|$, using (3.8) gives

$$\mathcal{P}(t) \ge (1-\mu)\big(\|\nabla\psi\|^2 + \|\operatorname{div}\psi\|^2\big), \qquad t\in[0,T^*].$$

Because $\mu<1$ by assumption (3.4), $\mathcal{P}(t)>0$ on $[0,T^*]$.

Finally, since the positivity does not degenerate at $t=T^*$, the same argument applies starting from T^*, hence by continuity we extend the interval. Repeating this argument shows that no maximal time $T^*<T$ is possible. Therefore, $T^*=T$. □

Theorem 3. *Let $\psi_0,\psi_1 \in H_0^1(\Omega)\times L^2(\Omega)$ satisfy* (3.4)*. Then the solution of* (2.1) *is uniformly bounded and global in time.*

Proof. Using Lemma 2 and $\mathcal{P}(t)>0$ for all $t\in[0,T]$, we get

$$E(0)\ge E(t) = \frac{1}{2}\|\psi_t\|^2 + \mathcal{S}(t) \ge \frac{1}{2}\|\psi_t\|^2 + \frac{1}{\ell}(1-\mu)\big(\|\nabla\psi\|^2 + \|\operatorname{div}\psi\|^2\big).$$

Thus,

$$\|\psi_t\|^2 + \|\nabla\psi\|^2 + \|\operatorname{div}\psi\|^2 \le C\,E(0),$$

for a positive constant C depending only on ℓ,κ,Ω and the embedding constants. Since $E(t)$ is nonincreasing and bounded from below, the solution cannot blow up in finite time. Therefore, the solution is global and uniformly bounded. □

4 Exponential stability

This part involves demonstrating our decay results.

Lemma 3. *Let*

$$\mathfrak{Q}_1(t) = \xi\sigma(t)\int_\Omega\int_0^1 e^{-2\varrho\sigma(t)}|\zeta(x,\varrho,t)|^2\,d\varrho\,dx,$$

then,

$$\mathfrak{Q}_1'(t) \le \xi\int_\Omega |\psi_t|^2\,dx - 2\mathfrak{Q}_1(t), \qquad \forall\, t\ge 0. \tag{4.1}$$

Proof. We first compute

$$\begin{aligned}\frac{d}{dt}\mathfrak{Q}_1(t) &= \xi\sigma'(t)\int_\Omega\int_0^1 e^{-2\sigma(t)\varrho}\zeta^2(x,\varrho,t)\,d\varrho\,dx\\ &\quad - 2\xi\sigma(t)\sigma'(t)\int_\Omega\int_0^1 \varrho\, e^{-2\sigma(t)\varrho}\zeta^2(x,\varrho,t)\,d\varrho\,dx\\ &\quad + 2\xi\sigma(t)\int_\Omega\int_0^1 e^{-2\sigma(t)\varrho}\zeta(x,\varrho,t)\zeta_t(x,\varrho,t)\,d\varrho\,dx.\end{aligned} \tag{4.2}$$

Using Equation $(2.1)_2$, we obtain

$$\sigma(t)\int_\Omega\int_0^1 e^{-2\sigma(t)\varrho}\,\zeta\,\zeta_t\,d\varrho\,dx = \int_\Omega\int_0^1 e^{-2\sigma(t)\varrho}\big(\sigma'(t)\varrho-1\big)\zeta\,\zeta_\varrho\,d\varrho\,dx.$$

Next,

$$\begin{aligned}&\int_\Omega\int_0^1 e^{-2\sigma(t)\varrho}\big(\sigma'(t)\varrho-1\big)\zeta\,\zeta_\varrho\,d\varrho\,dx = \frac12\int_\Omega\int_0^1 \frac{\partial}{\partial\varrho}\Big(e^{-2\sigma(t)\varrho}(\sigma'(t)\varrho-1)\zeta^2\Big)\,d\varrho\,dx\\ &+\sigma(t)\int_\Omega\int_0^1 e^{-2\sigma(t)\varrho}(\sigma'(t)\varrho-1)\zeta^2\,d\varrho\,dx - \frac{\sigma'(t)}{2}\int_\Omega\int_0^1 e^{-2\sigma(t)\varrho}\zeta^2\,d\varrho\,dx.\end{aligned} \tag{4.3}$$

Combining (4.2), (4.3), using $\zeta(x,0,t) = \psi_t(x,t)$ and integrating the total derivative in ϱ, we obtain

$$\begin{aligned}\frac{d}{dt}\mathfrak{Q}_1(t) =& -2\xi\sigma(t)\int_\Omega\int_0^1 e^{-2\sigma(t)\varrho}\zeta^2\,d\varrho\,dx\\ &+\xi\int_\Omega \psi_t^2\,dx - \xi(1-\sigma'(t))e^{-2\sigma(t)}\int_\Omega \zeta^2(x,1,t)\,dx.\end{aligned}$$

Since $\sigma'(t)\le 1$, the last term is non-positive and inequality (4.1) follows. □

Lemma 4. *Let*

$$\mathfrak{Q}_2(t) = \aleph E(t) + \epsilon\int_\Omega \psi\psi_t\,dx + \frac{\epsilon\gamma_1}{2}\int_\Omega |\psi|^2\,dx,$$

then,

$$\begin{aligned}\mathfrak{Q}_2'(t) \le& -\big(\aleph\mathfrak{C}_0-\epsilon\big)\|\psi_t\|^2 - \epsilon(\gamma-\mu-\varpi)\|\nabla\psi\|^2 - \epsilon(\alpha+\gamma-\mu)\int_\Omega |\operatorname{div}\psi|^2\,dx\\ &-\left(\aleph\mathfrak{C}_0 - \frac{\epsilon\delta^*}{4\varpi}\right)\int_\Omega |\zeta(x,1,t)|^2\,dx, \qquad t\ge 0,\end{aligned} \tag{4.4}$$

for some positive constants $\aleph,\epsilon,\varpi,\delta^*$.

Proof. Direct differentiation of $\mathcal{Q}_2(t)$ and use of system (2.1) give

$$\mathcal{Q}_2'(t) \leq -\aleph\mathfrak{C}_0 \int_\Omega \left(|\psi_t|^2 + |\zeta(x,1,t)|^2\right) dx - \epsilon\gamma_2 \int_\Omega \psi\,\zeta(x,1,t)\,dx$$
$$+\epsilon\left(\int_\Omega |\psi_t|^2 dx - \gamma\int_\Omega |\nabla\psi|^2 dx - (\alpha+\gamma)\int_\Omega |\operatorname{div}\psi|^2 dx + \int_\Omega |\psi|^\ell \ln|\psi|^\kappa dx\right). \quad (4.5)$$

To estimate the coupling term, Young's inequality gives for all $\varpi > 0$,

$$-\gamma_2 \int_\Omega \psi\,\zeta(x,1,t)\,dx \leq \varpi\|\nabla\psi\|^2 + \frac{\delta^*}{4\varpi}\int_\Omega |\zeta(x,1,t)|^2\,dx, \quad (4.6)$$

where $\delta^* > 0$ arises from the trace inequality $\|\psi\|_{L^2(\partial\Omega)} \leq \delta^*\|\nabla\psi\|$.
Next, from inequality (3.6) and Lemma 2,

$$\int_\Omega |\psi|^\ell \ln|\psi|^\kappa dx \leq \mu\left(\|\nabla\psi\|^2 + \|\operatorname{div}\psi\|^2\right).$$

Combining (4.5), (4.6) and the above estimate yields inequality (4.4). □

Theorem 4. *Let (3.4) be satisfied. Then, there exist two positive constants c_3 and c_4 such that*

$$E(t) \leq c_3 e^{-c_4 t}, \qquad \forall t \geq 0.$$

Proof. As a starting point, define $\mathcal{Q}(t) = \mathcal{Q}_1(t) + \mathcal{Q}_2(t)$, and recall that

$$\mathcal{Q}(t) \sim E(t), \quad (4.7)$$

that is, there exist two positive constants m_1, m_2 such that $m_1E(t) \leq \mathcal{Q}(t) \leq m_2E(t)$. Using estimates (4.1) and (4.4), we obtain

$$\mathcal{Q}'(t) \leq -(\aleph\mathfrak{C}_0 - \epsilon - \xi)\|\psi_t\|^2 - \epsilon(\gamma-\mu-\varpi)\|\nabla\psi\|^2 - \epsilon(\alpha+\gamma-\mu)\|\operatorname{div}\psi\|^2$$
$$-\left(\aleph\mathfrak{C}_0 - \frac{\epsilon\delta^*}{4\varpi}\right)\|\zeta(\cdot,1,t)\|^2_{L^2(\Omega)} - 2\mathcal{Q}_1(t). \quad (4.8)$$

Since $\mu < 1$, one may choose $\varpi > 0$ sufficiently small such that $\lambda = \gamma - \mu - \varpi > 0$. Next, using the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^\ell(\Omega)$,

$$\|\psi\|_\ell^\ell \leq \mathfrak{C}\|\nabla\psi\|^\ell = \mathfrak{C}\|\nabla\psi\|^{\ell-2}\|\nabla\psi\|^2 \leq \mathfrak{C}E(0)^{\frac{\ell-2}{2}}\|\nabla\psi\|^2 \leq \omega\|\nabla\psi\|^2,$$

for some $\omega > 0$. Therefore,

$$-\frac{\epsilon\lambda}{2}\|\nabla\psi\|^2 \leq -\frac{\epsilon\lambda\omega^{-1}}{2}\|\psi\|_\ell^\ell.$$

Substituting into (4.8) gives

$$\mathcal{Q}'(t) \leq -(\aleph\mathfrak{C}_0 - \epsilon - \xi)\|\psi_t\|^2 - \frac{\epsilon\lambda}{2}\|\nabla\psi\|^2 - \frac{\epsilon\lambda\omega^{-1}}{2}\|\psi\|_\ell^\ell - \epsilon(\alpha+\gamma-\mu)\|\operatorname{div}\psi\|^2$$
$$-\left(\aleph\mathfrak{C}_0 - \frac{\epsilon\delta^*}{4\varpi}\right)\|\zeta(\cdot,1,t)\|^2_{L^2(\Omega)} - 2\mathcal{Q}_1(t). \quad (4.9)$$

We now choose $\aleph > 0$ sufficiently large so that

$$\aleph\mathfrak{C}_0 - \epsilon - \xi > 0, \qquad \aleph\mathfrak{C}_0 - \frac{\epsilon\delta^*}{4\varpi} > 0.$$

With these choices, (4.9) yields

$$\mathfrak{Q}'(t) \leq -\mathfrak{C}\big(\|\psi_t\|^2 + \|\nabla\psi\|^2 + \|\operatorname{div}\psi\|^2 + \|\psi\|_\ell^\ell + \|\zeta(\cdot,1,t)\|^2_{L^2(\Omega)}\big) - \mathfrak{C}\|\zeta\|^2_{L^2(\Omega\times(0,1))},$$

for some $\mathfrak{C} > 0$. Recalling that the expression inside the brackets is equivalent to $E(t)$, we infer

$$\mathfrak{Q}'(t) \leq -\mathfrak{C}\,E(t).$$

Finally, using the equivalence (4.7),

$$m_1 E(t) \leq \mathfrak{Q}(t) \leq m_2 E(t),$$

we conclude

$$\mathfrak{Q}'(t) \leq -\widetilde{\mathfrak{C}}\,\mathfrak{Q}(t), \qquad \widetilde{\mathfrak{C}} = \frac{\mathfrak{C}}{m_2} > 0.$$

Applying Grönwall's inequality gives

$$\mathfrak{Q}(t) \leq \mathfrak{Q}(0)e^{-\widetilde{\mathfrak{C}}t},$$

and hence

$$E(t) \leq \frac{m_2}{m_1}E(0)e^{-\widetilde{\mathfrak{C}}t} =: c_3 e^{-c_4 t}.$$

This completes the proof. □

5 Conclusions and perspective

In this work, we investigated the well-posedness and stability of a logarithmic Lamé system incorporating a time-varying delay within a bounded domain. Using semigroup theory, we proved the existence and uniqueness of local solutions, while the well-depth method enabled us to extend these solutions globally in time. Moreover, under suitable assumptions on the time-varying delay and the frictional damping parameters, we established an exponential decay estimate for the associated energy, thereby confirming the long-term stability of the system.

Several directions for future research naturally arise from the present study. A first possible extension consists in considering Lamé-type systems with more general nonlinear or state-dependent delay terms, or in analyzing the model in unbounded or exterior domains, where additional mathematical challenges appear due to the lack of compactness. Another interesting line of investigation involves the development of numerical schemes capable of capturing simultaneously the logarithmic nonlinearity and the time-dependent delay, together with numerical simulations that illustrate and validate the theoretical decay rates derived here. Finally, the design of efficient control or feedback mechanisms to mitigate destabilizing effects caused by delays remains an important problem, with potential applications in material science, continuum mechanics, and engineering.

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