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A General Framework for Constrained Linear–Quadratic Control under Parametric Uncertainty

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Abstract. This paper develops a discrete-time linear–quadratic control framework with box-constrained inputs under bounded parametric uncertainties. Using the discrete-time Pontryagin Maximum Principle, a closed-form state-feedback law is obtained via backward Riccati recursion, guaranteeing satisfaction of input constraints. Simulation results show that the closed-loop system remains stable and converges to the origin under moderate uncertainties, with control inputs staying within prescribed bounds. The proposed method offers a simple and practical solution without requiring a full robust control design.

1. Introduction

Designing constrained discrete-time optimal controllers for linear systems is a classical problem with wide-ranging applications. In practice, exact system parameters are often uncertain due to modeling errors or variations, which can degrade performance or destabilize the system. To address this, we propose a linear–quadratic framework with bounded inputs that effectively handles moderate parametric uncertainties without requiring a full robust control design.

Keywords: Necessary conditions; Parametric uncertainty; Pontryagin Minimum Principle; Feedback control.

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The framework uses the discrete-time Pontryagin Maximum Principle (PMP) to derive optimality conditions. For linear dynamics and quadratic costs, these conditions reduce to explicit algebraic expressions, allowing a saturated state-feedback law that guarantees input constraints and preserves stability. Bounded parametric deviations in system matrices are considered, and the framework ensures that the closed-loop system remains stable under small to moderate uncertainties.

Simulation results for a two-dimensional system show that the states converge to the origin and control inputs remain within bounds across all considered scenarios. Total cost analysis confirms acceptable performance under varying uncertainty levels, highlighting the framework's simplicity and practical effectiveness for real-world applications.

2. Problem Formulation

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ the set of real matrices. The transpose of X is $X^\top$, and $\|\cdot\|$ denotes the Euclidean norm. The discrete-time index is $k \in \mathbb{Z}$ with finite horizon $[k_0, k_f]$. We consider the discrete-time nonlinear system

$$\mathbf{x}_{k+1} = g_k(\mathbf{x}_k, \mathbf{u}_k), \qquad k = k_0, \ldots, k_f - 1, \tag{1}$$

where $g_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable. The initial condition $\mathbf{x}_{k_0} = \mathbf{x}_0$ is given. When unambiguous, the time index k is omitted. The control inputs satisfy componentwise bounds

$$\mathcal{U} = \{\mathbf{u}_k \mid \underline{\mathbf{u}} \le \mathbf{u}_k \le \overline{\mathbf{u}},\ k = k_0, \ldots, k_f - 1\}, \tag{2}$$

where $\underline{\mathbf{u}}, \overline{\mathbf{u}} \in \mathbb{R}^m \cup \{\pm\infty\}$. Unless stated otherwise, $\mathcal{U}$ is assumed to be time-invariant, convex, and compact. The finite-horizon cost functional is

$$J = \sum_{k=k_0}^{k_f-1} f_k(\mathbf{x}_k, \mathbf{u}_k) + \phi(\mathbf{x}_{k_f}), \tag{3}$$

with continuously differentiable stage cost f_k and terminal cost ϕ. For the linear–quadratic case used in explicit Riccati recursions, we adopt

$$f_k(\mathbf{x}_k, \mathbf{u}_k) = \tfrac{1}{2}\mathbf{x}_k^\top \mathbf{Q}_k \mathbf{x}_k + \tfrac{1}{2}\mathbf{u}_k^\top \mathbf{R}_k \mathbf{u}_k, \tag{4}$$

$$\phi(\mathbf{x}_{k_f}) = \tfrac{1}{2}\mathbf{x}_{k_f}^\top \mathbf{F}_{k_f} \mathbf{x}_{k_f}, \tag{5}$$

where $\mathbf{Q}_k \succeq 0$, $\mathbf{R}_k \succ 0$, and $\mathbf{F}_{k_f} \succeq 0$. For linear dynamics

$$g_k(\mathbf{x}_k, \mathbf{u}_k) = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k \tag{6}$$

the pair $(\mathbf{A}_k, \mathbf{B}_k)$ is stabilizable.

3. Optimality Condition

Applying the discrete-time PMP, we introduce adjoint variables $\{\boldsymbol{\lambda}_k\}$ associated with the system dynamics. These yield the backward costate recursion and the pointwise optimality condition under bounded control inputs. For completeness, the discrete-time PMP is restated in the notation adopted in this paper.

Theorem 3.1. *Consider the minimization of the discrete-time optimal control problem (3), subject to the system dynamics (1) and the bounded control constraints (2). Let*$(\mathbf{x}_k^*, \mathbf{u}_k^*)$ *denote an optimal state control trajectory. Then there exists a corresponding adjoint sequence* $\{\lambda_k\}_{k_0}^{k_f}$ *satisfying the backward recursion*

$$\lambda_k = f_{k x_k}(\mathbf{x}_k^*, \mathbf{u}_k^*) + \lambda_{k+1} g_{k x_k}(\mathbf{x}_k^*, \mathbf{u}_k^*), \quad k = k_0, \ldots, k_f - 1, \tag{7}$$

with the terminal condition

$$\lambda_{k_f}^* = \left.\frac{\partial \phi(\mathbf{x}_{k_f})}{\partial \mathbf{x}_{k_f}}\right|_* . \tag{8}$$

Furthermore, the following pointwise optimality condition holds for all $k \in \{k_0, \ldots, k_f - 1\}$*:*

$$\begin{cases} \mathbf{u}_k^* = \underline{\mathbf{u}}, & \text{if } \dfrac{\partial H}{\partial \mathbf{u}_k} > 0, \\ \mathbf{u}_k^* \in \mathcal{U}, & \text{if } \dfrac{\partial H}{\partial \mathbf{u}_k} = 0, \\ \mathbf{u}_k^* = \overline{\mathbf{u}}, & \text{if } \dfrac{\partial H}{\partial \mathbf{u}_k} < 0, \end{cases} \tag{9}$$

where H *is the Hamiltonian*

$$H(k, \mathbf{x}_k, \mathbf{u}_k, \lambda_{k+1}) = f_k(\mathbf{x}_k, \mathbf{u}_k) + \lambda_{k+1} g_k(\mathbf{x}_k, \mathbf{u}_k).$$

3.1. Quadratic Performance Index and Linear Dynamics

A widely studied special case of Problem (3) arises when the system dynamics are linear and the cost functional is quadratic. Although this structure is classical, it plays a central role in constrained optimal control because the PMP conditions derived in Theorem 3.1 reduce to simple algebraic expressions that can be evaluated efficiently in real time. We therefore summarize only the elements required for the subsequent development, without re-deriving standard LQR results.

Consider the cost functional (3) with that utilized in (4) – (5) and dynamical system (1) that is in linear form Under this structure, the costate equation (7) becomes the linear backward recursion

$$\boldsymbol{\lambda}_k = \mathbf{Q}_k \mathbf{x}_k + \mathbf{A}_k^\top \boldsymbol{\lambda}_{k+1}, \tag{10}$$

and the optimality condition of Theorem 3.1 yields the unconstrained minimizer

$$\mathbf{u}_k^{\mathrm{unc}} = -\mathbf{R}_k^{-1}\mathbf{B}_k^\top \boldsymbol{\lambda}_{k+1}.$$

When subject to the box constraint (2), the optimal control becomes the projection of the unconstrained minimizer onto the admissible interval:

$$\mathbf{u}_k^* = \min\Big\{\max\{u_k^{\mathrm{unc}},\ \underline{\mathbf{u}}\},\ \overline{\mathbf{u}}\Big\}. \tag{11}$$

which is exactly the sign pattern established in Theorem 3.1 for a minimization problem.

The closed-loop dynamics induced by (11) are therefore piecewise affine, with the interior-region dynamics given by

$$\mathbf{x}_{k+1}^* = \mathbf{A}_k\mathbf{x}_k^* - \mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^\top \boldsymbol{\lambda}_{k+1}^*,$$

and bounded modes replacing the control term by $\mathbf{B}_k\underline{\mathbf{u}}$ or $\mathbf{B}_k\overline{\mathbf{u}}$ when the bounds are active.

Crucially, the purpose of this subsection is not to re-derive the classical LQR framework, but to highlight that the general PMP conditions of Theorem 3.1 collapse to explicit algebraic expressions in the quadratic-linear case.

3.2. Closed-Loop Control for Free Final State

In this section, $\mathbf{P}_k$ denotes the discrete-time Riccati matrix associated with problems having a free terminal state and defines the linear state–costate relation $\boldsymbol{\lambda}_k = \mathbf{P}_k\mathbf{x}_k$. For fixed-terminal-state problems, the corresponding inverse Riccati matrices are denoted by $\mathbf{M}_k$. These matrices encode the state–costate relationship and enable closed-form expressions for the optimal control. Under linear–quadratic dynamics with box-constrained inputs, the optimal control admits a closed-form solution, and the first-order conditions are sufficient for global optimality due to the strict convexity induced by $\mathbf{R}_k \succ 0$.

Theorem 3.2. *Consider the discrete-time optimal control problem with dynamics* (6) *and cost functional* (3)*, subject to the control constraints* (2)*. Assume that* $\mathbf{R}_k \succ 0$ *for all* k*, and that the pairs* $(\mathbf{A}_k, \mathbf{B}_k)$ *are stabilizable and* $(\mathbf{Q}_k^{1/2}, \mathbf{A}_k)$ *are detectable in the standard LQR sense. Then there exists a symmetric sequence* $\{\mathbf{P}_k\}$ *satisfying the backward Riccati recursion*

$$\boldsymbol{\lambda}_k = \mathbf{Q}_k\mathbf{x}_k + \mathbf{A}_k^\top \boldsymbol{\lambda}_{k+1}, \tag{12}$$

together with the terminal condition (8)*. In particular, the matrices* $\mathbf{P}_k$ *satisfy*

$$\mathbf{P}_k\mathbf{x}_k = \mathbf{Q}_k\mathbf{x}_k + \mathbf{A}_k^\top \mathbf{P}_{k+1}\mathbf{x}_{k+1}, \tag{13}$$

and, for the unconstrained problem, the optimal feedback control law is

$$\mathbf{u}_k^{\mathrm{unc}} = -\mathbf{K}_k\mathbf{x}_k, \quad \mathbf{K}_k = \big(\mathbf{R}_k + \mathbf{B}_k^\top \mathbf{P}_{k+1}\mathbf{B}_k\big)^{-1}\mathbf{B}_k^\top \mathbf{P}_{k+1}\mathbf{A}_k. \tag{14}$$

Equivalently, the Riccati matrices satisfy the standard recursion

$$\mathbf{P}_k = \mathbf{Q}_k + \mathbf{A}_k^\top \mathbf{P}_{k+1}\mathbf{A}_k - \mathbf{A}_k^\top \mathbf{P}_{k+1}\mathbf{B}_k \big(\mathbf{R}_k + \mathbf{B}_k^\top \mathbf{P}_{k+1}\mathbf{B}_k\big)^{-1}\mathbf{B}_k^\top \mathbf{P}_{k+1}\mathbf{A}_k, \tag{15}$$

with terminal condition $\mathbf{P}_{k_f} = \mathbf{F}_{k_f}$. *Whenever* $\mathbf{P}_{k+1}$ *is nonsingular,* (15) *is equivalently expressed as*

$$\mathbf{P}_k = \mathbf{Q}_k + \mathbf{A}_k^\top \big(\mathbf{P}_{k+1}^{-1} + \mathbf{E}_k\big)^{-1}\mathbf{A}_k, \tag{16}$$

where $\mathbf{E}_k = \mathbf{B}_k\mathbf{R}_k^{-1}\mathbf{B}_k^\top$. *Under the bounded control constraint* (2)*, the optimal control law is given by Theorem 3.1. In particular, the corresponding closed-loop control takes the form*

$$\mathbf{u}_k^* = \min\Big\{\max\{-\mathbf{K}_k\mathbf{x}_k^*,\ \underline{\mathbf{u}}\},\ \overline{\mathbf{u}}\Big\}. \tag{17}$$

3.3. Parametric Uncertainty

In practical applications, the exact values of the system matrices $\mathbf{A}_k$ and $\mathbf{B}_k$ are often not perfectly known due to modeling errors or parameter variations. To account for such effects, we consider the discrete-time system with **parametric uncertainty**

$$\mathbf{x}_{k+1} = (\mathbf{A}_k + \Delta\mathbf{A}_k)\mathbf{x}_k + (\mathbf{B}_k + \Delta\mathbf{B}_k)\mathbf{u}_k, \tag{18}$$

where $\Delta\mathbf{A}_k \in \mathbb{R}^{n\times n}$ and $\Delta\mathbf{B}_k \in \mathbb{R}^{n\times m}$ are unknown but **bounded perturbations**, satisfying

$$\|\Delta\mathbf{A}_k\| \le \alpha_k, \qquad \|\Delta\mathbf{B}_k\| \le \beta_k,$$

for given nonnegative scalars α_k and β_k that quantify the magnitude of the parametric uncertainty [7].

Importantly, the closed-loop control law previously defined in (17) can still be applied directly to the uncertain system. Specifically, the bounded state-feedback

$$\mathbf{u}_k^* = \min\Big\{\max\{-\mathbf{K}_k\mathbf{x}_k^*,\ \underline{\mathbf{u}}\},\ \overline{\mathbf{u}}\Big\}$$

remains well-defined for all admissible $(\Delta\mathbf{A}_k, \Delta\mathbf{B}_k)$ within the prescribed bounds. For small to moderate uncertainties, the nominal LQR gain $\mathbf{K}_k$ ensures that the interior dynamics retain stability, and the box constraints naturally limit the control action, preventing excessive deviation caused by parameter variations [4].

Although the proposed control law is not a robust controller in the classical sense (i.e., it is not specifically designed to compensate for all possible parametric variations), it is nevertheless capable of maintaining stability and satisfactory performance for bounded uncertainties of moderate size. The inherent structure of the bounded state-feedback limits the control action and prevents large deviations caused by the unknown but bounded perturbations $\Delta\mathbf{A}_k$ and $\Delta\mathbf{B}_k$. In this way, the framework provides a simple yet effective approach for handling parametric uncertainties while respecting input constraints, without requiring a full robust control design [4].

4. Simulation

In this section, we illustrate the performance of the proposed framework under bounded parametric uncertainties in the system matrices. We consider the discrete-time linear system (18), where the nominal system matrices are chosen as

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and the cost functional is given by (4)-(5) with $\mathbf{Q} = \mathrm{diag}(10, 1)$, $R = 1$, and the control bounded by $|u| \leq 1$. Three scenarios of parametric uncertainties (α, β) are considered, corresponding to increasing deviations from the nominal dynamics. The bounded control law defined in (17) is applied.

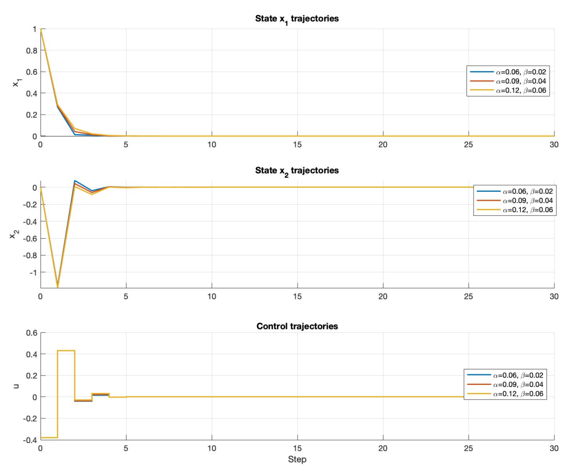

FIGURE 1. Trajectories of states x_1, x_2 and control input u_k for three scenarios of parametric uncertainty (α, β).

Figures 1show the state and control trajectories for all three scenarios over $N = 30$ simulation steps. As expected, the trajectories remain bounded due to input, and the state converges to the origin in all cases. Increasing uncertainty levels result in slightly faster convergence, which is a consequence of the particular choice of $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$. Notably, despite the bounded uncertainties, the control input never exceeds its prescribed limits.The total cost is computed for each scenario and summarized in Table 1. The results highlight that, while the control law handles input bounded effectively, the total cost does not necessarily increase monotonically with the uncertainty levels. This observation reflects that the proposed method is not robust in the classical sense but remains a simple and practical approach for bounded parametric deviations.

TABLE 1. Total cost for each parametric uncertainty scenario.

α	β	Total Cost J
0.06	0.02	12.249
0.09	0.04	12.352
0.12	0.06	12.481

The simulation results indicate that the proposed framework guarantees input constraints, ensures convergence to the origin, and provides acceptable performance under moderate parametric uncertainties.

5. Conclusion

This paper presented a discrete-time linear–quadratic framework with bounded state-feedback to handle input constraints and moderate parametric uncertainties. By leveraging the discrete-time PMP, the optimality conditions were reduced to explicit algebraic expressions, enabling a simple and computationally efficient saturated control law. Simulation results demonstrated that the proposed framework guarantees convergence to the origin, respects control limits, and provides acceptable performance under varying levels of uncertainty. While not a fully robust design, the method offers a practical and effective approach for real-world applications where moderate parameter variations are present. Future work may extend this framework to more complex nonlinear systems or to formally incorporate robustness against larger uncertainties.

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