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ON CONTRACTIBILITY OF MATRIX ALGEBRAS

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Abstract. We show first that for each C*-algebra A, contractibility of A implies contractibility of $M_n(A)$.
We next prove that an incidence algebra A of upper triangular matrices, defined by a partially ordered set Ω on $\{1, 2, \ldots, n\}$ satisfying $(p, q) \in \Omega \Rightarrow p \leq q$, is a contractible Banach algebra iff there is no discordant couple of D-transitive triples of elements of Ω.

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1. Introduction. Contractible Banach algebras are one of the homologically best algebras. A Banach algebra A is said to be contractible if it has the following (equivalent) properties [2]:

(i) $H^1(A, X) = 0$ for any Banach A-bimodule X, i.e. each derivation of A with values in any Banach A-bimodule is inner.

(ii) Every (one-sided or two-sided) Banach A-bimodule is projective.

(iii) A is unital and as a Banach A-bimodule is projective.

(iv) A is unital and the canonical morphism $\pi : A \overset{\wedge}{\otimes} A \longrightarrow A$ defined by $\pi(a \otimes b) = ab$ is a retraction, i.e. has a right inverse, in the category $\mathbf{A} - \mathbf{mod} - \mathbf{A}$ of Banach A-bimodules.

The reader is referred to [3] for details, undefined terms and notations on homology of Banach algebras.

2. Contractibility of square matrices with entries in a C^*-algebra. Throughout this section, A is a unital C*-algebra and $M_n(A)$ denotes the C*-algebra of $n \times n$ matrices with entries in A. Note that $M_n(A)$ is *-isomorphic to $A \otimes M_n(\mathbf{C})$. If $a \in A$ and $[\lambda_{ij}] \in M_n(\mathbf{C})$, then $a \otimes [\lambda_{ij}]$ is identified to the matrix $[\lambda_{ij} a]$ in $M_n(A)$; in particular if $\{E_{ij} \ ; \ i,j = 1, \ldots, n\}$ is the standard system of matrix units for $M_n(\mathbf{C})$, we denote the corresponding element to $a \otimes E_{ij}$ by aE_{ij}. We have

$$\| a_{ij} \| \leq \| [a_{ij}] \| \leq \sum_{1 \leq i,j \leq n} \| a_{ij} \| \quad \text{for all} \quad 1 \leq i,j \leq n \quad [4].$$

Theorem 2.1. *Let A be contractible, then so is $M_n(A)$.*

Proof. Suppose that $\rho : A \longrightarrow A \overset{\wedge}{\otimes} A$ is a right inverse morphism to the canonical morphism $\pi : A \overset{\wedge}{\otimes} A \longrightarrow A$. Put $\tilde{\rho}(aE_{ij}) = \lim_n \sum_{k=1}^{m_n} u_k^n E_{ij} \otimes v_k^n E_{ij}$, in which $\rho(a) = \lim_n \sum_{k=1}^{m_n} u_k^n \otimes v_k^n$; and extend it by linearity to a mapping $\tilde{\rho}$ from $M_n(A)$ to $M_n(A) \widehat{\otimes} M_n(A)$:

$$\tilde{\rho}([a_{ij}]) = \tilde{\rho}\left(\sum_{1 \leq i,j \leq n} a_{ij} E_{ij} \right) = \sum_{1 \leq i,j \leq n} \tilde{\rho}(a_{ij} E_{ij}).$$

If $\sum_{k=1}^{m} u_k \otimes v_k = 0$, then by [4, proposition 11.1.8] there exist an $m \times m$ complex matrix $[c_{jk}]$ such that $\sum_{j=1}^{m} c_{jk} u_j = 0$ $(k = 1, \ldots, m)$ and $\sum_{k=1}^{m} c_{jk} v_k = v_j$ $(j = 1, \ldots, m)$. Hence, $\sum_{k=1}^{m} u_k E_{ij} \otimes v_k E_{ij} = 0$. Therefore, the map

$$\omega : \sum_{k=1}^{m} u_k \otimes v_k \longmapsto \sum_{k=1}^{m} u_k E_{ij} \otimes v_k E_{ij}$$

is well-defined and furthermore,

$$\begin{aligned} \| \sum_{k=1}^{m} u_k E_{ij} \otimes v_k E_{ij} \| &\leq \inf \left\{ \sum_{k=1}^{m} \| u'_k \| \| v'_k \| : \sum_{k=1}^{m} u_k \otimes v_k = \sum_{k=1}^{m} u'_k \otimes v'_k \right\} \\ &= \| \sum_{k=1}^{m} u_k \otimes v_k \| . \end{aligned}$$

Thus,

$$\| \tilde{\rho}(aE_{ij}) \| = \lim_{n\to\infty} \| \sum_{k=1}^{m_n} u_k^n E_{ij} \otimes v_k^n E_{ij} \|$$
$$\leq \lim_{n\to\infty} \| \sum_{k=1}^{m_n} u_k^n \otimes v_k^n \| = \| \lim_n \sum_{k=1}^{m_n} u_k^n \otimes v_k^n \| \leq \| \rho \| \| a \| .$$

It follows that $\tilde{\rho}$ is well-defined and $\| \tilde{\rho} \| \leq n^2 \| \rho \|$.
$\tilde{\rho}$ is a morphism in $\mathbf{M_n(A)} - \mathbf{mod} - \mathbf{M_n(A)}$. In fact for each $[a_{ij}], [b_{ij}] \in M_n(A)$, by virtue of $[c_{ij}] = \sum_{1\leq i,j\leq n} c_{ij}E_{ij}$, we have

$$\tilde{\rho}(b_{pq}[a_{ij}]) = \sum_{1\leq i,j\leq n} \tilde{\rho}([b_{pq}].a_{ij}E_{ij}) = \sum_{1\leq i,j\leq n} \tilde{\rho}(\sum_{p=1}^{n} b_{pi}a_{ij}E_{pj})$$
$$= \sum_{1\leq i,j,p\leq n} \tilde{\rho}(b_{pi}a_{ij}E_{pj}) = \sum_{1\leq i,j,p\leq n} (\sum_{q=1}^{n} b_{pq}E_{pq})\tilde{\rho}(a_{ij}E_{ij})$$
$$= \sum_{1\leq i,j\leq n} [b_{pq}]\tilde{\rho}(a_{ij}E_{ij}) = [b_{pq}]\tilde{\rho}([a_{ij}]).$$

Similarly, $\tilde{\rho}([a_{ij}][b_{pq}]) = \tilde{\rho}([a_{ij}])[b_{pq}]$.

It is straightforward to show that $\tilde{\rho}$ is a right inverse to the canonical morphism $\tilde{\pi} : M_n(A) \overset{\wedge}{\otimes} M_n(A) \longrightarrow M_n(A)$. It follows that $M_n(A)$ is contractible. □

COROLLARY 2.2. *$M_n(\mathbf{C})$ is contractible.*

Proof. $\mathbf{C}$ is contractible, since the canonical morphism $\pi : \mathbf{C} \overset{\wedge}{\otimes} \mathbf{C} \longrightarrow \mathbf{C}$ is an isomorphism. □

COROLLARY 2.3. *If A has discrete primitive spectrum and all its irreducible representations are finite dimentional, then the same is true for $M_n(A)$.*

Proof. A unital C*-algebra is contractible iff its primitive spectrum is discrete and each of its irreducible representations is finite dimensional [1, Assertion IV.5.15]. □

The following is a good question in the argued context:

QUESTION 2.4. Is the biprojectivity of $M_n(A)$ implies the biprojectivity of A?

3. Contractibility of incedence algebras of upper triangular matrices. In this section we characterize a class of contractible Banach algebras among matrix algebras. We shall set up our notation.

Let Ω be a partially ordered set on $\{1, 2, \ldots, n\}$ such that $(p, q) \in \Omega$ implies $p \leq q$, and $A(\Omega)$ be the subalgebra of $M_n(\mathbf{C})$ of all matrices (a_{ij}) such that $a_{ij} = 0$ unless$(i, j) \in \Omega$. Note that the condition $(p, q) \in \Omega \Rightarrow p \leq q$ ensures that each element of $A(\Omega)$ is upper triangular and the transitivity guarantees that $A(\Omega)$ is closed under multiplication. $A(\Omega)$ is a Banach algebra with respect to the usual norm which inheriting from $M_n(\mathbf{C})$. Applying the terminology of [1], $A(\Omega)$ is called an incidence algebra of upper triangular matrices (the other names are digraph algebra and finite dimensional CSL-algebra). We shall write A for $A(\Omega)$. E_{pq} stands as already for the matrix having zero everywhere except the (p, q)th place and the (p, q)th place has the entry 1. The multiplication in A is determined by the products $E_{ij}E_{kl} = \delta_{jk}E_{il}$ where δ denotes the Kronecker symbol. Evidently $\{E_{pq}|(p, q \in \Omega\}$is a linear basis for A.

A transitive triple is an ordered triple of the form $((p, q), (q, r), (p, r))$ of elements in Ω; if in addition $p \neq q$ and $q \neq r$, it said to be D-transitive. Two distinct D-transitive triple are called discordant if their third components are equal or there exists a pair (i, j) occuring in different components of them.

Lemma 3.1. *Let $\rho : A \longrightarrow A\widehat{\otimes}A$ be a right inverse morphism to the canonical morphism in $\mathbf{A} - \mathbf{mod} - \mathbf{A}$* and *$((p, q), (q, r), (p, r))$ be a D-transitive triple. Then*

$$\rho(E_{pq}) = E_{pq} \otimes E_{qq} + \sum_{i,j,k,l} \alpha_{ij}^{kl} E_{ij} \otimes E_{kl} \tag{1}$$

$$\rho(E_{qr}) = E_{qq} \otimes E_{qr} + \sum_{i,j,k,l} \beta_{ij}^{kl} E_{ij} \otimes E_{kl} \tag{2}$$

$$\rho(E_{pr}) = E_{pq} \otimes E_{qr} + \sum_{i,j,k,l} \gamma_{ij}^{kl} E_{ij} \otimes E_{kl} \tag{3}$$

in which the summations are taken over all i, j, k, l such that $((i, j), (k, l), (i, l))$ is not a transitive triple.

Proof. Notice first that $\{E_{ij} \otimes E_{kl}|(i, j)\ ,\ (k, l) \in \Omega\}$ is a linear basis for $A\widehat{\otimes}A$.

Since $\pi \circ \rho = 1_A$, we have

$$\rho(E_{pq}) = a_0 E_{pp} \otimes E_{pq} + a_1 E_{p(p+1)} \otimes E_{(p+1)q} + \cdots \\ \cdots + a_{q-p} E_{pq} \otimes E_{qq} + \sum_{i,j,k,l} \alpha_{ij}^{kl} E_{ij} \otimes E_{kl} \tag{4}$$

$$\rho(E_{qr}) = b_0 E_{qq} \otimes E_{qr} + b_1 E_{q(q+1)} \otimes E_{(q+1)r} + \cdots \\ \cdots + b_{r-q} E_{qr} \otimes E_{rr} + \sum_{i,j,k,l} \beta_{ij}^{kl} E_{ij} \otimes E_{kl}, \tag{5}$$

$$\rho(E_{pr}) = c_0 E_{pp} \otimes E_{pr} + c_1 E_{p(p+1)} \otimes E_{(p+1)r} + \cdots \\ \cdots + c_{q-p} E_{pq} \otimes E_{qr} + \ldots \\ + c_{r-p} E_{pr} \otimes E_{rr} + \sum_{i,j,k,l} \gamma_{ij}^{kl} E_{ij} \otimes E_{kl}. \quad (6)$$

The summations are taken over all i, j, k, l such that $((i,j),(k,l),(i,l))$ is not transitive triple. Note also that $\sum_{u=0}^{q-p} a_u = \sum_{u=0}^{r-q} b_u = \sum_{u=0}^{r-p} c_u = 1$. Since ρ is A-bimodule morphism, (4) and (5) imply:

$$\rho(E_{pr}) = \rho(E_{pq}.E_{qr}) = \rho(E_{pq}).E_{qr} = a_0 E_{pp} \otimes E_{pr} + a_1 E_{p(p+1)} \otimes E_{(p+1)r} + \cdots \\ \cdots + a_{q-p} E_{pq} \otimes E_{qr} + \cdots \quad (7)$$

$$\rho(E_{pr}) = \rho(E_{pq}.E_{qr}) = E_{pq}.\rho(E_{qr}) = b_0 E_{pq} \otimes E_{qr} + b_1 E_{p(q+1)} \otimes E_{(q+1)r} + \cdots \\ \cdots + b_{r-q} E_{pr} \otimes E_{rr} + \cdots \quad (8)$$

Now comparing (6), (7) and (8) we can conclude the following:

$$a_0 = c_0 = 0,\ a_1 = c_1 = 0, \ldots, a_{q-p-1} = c_{q-p-1} = 0,$$

and then

$$b_1 = c_{q-p+1},\ b_2 = c_{q-p+2}, \ldots, b_{r-q} = c_{r-p} = 0.$$

Thus, $a_{q-p} = b_0 = c_{q-p} = 1$ and

$$\rho(E_{pq}) = E_{pq} \otimes E_{qq} + \sum_{i,j,k,l} \alpha_{ij}^{kl} E_{ij} \otimes E_{kl}$$

$$\rho(E_{qr}) = E_{qq} \otimes E_{qr} + \sum_{i,j,k,l} \beta_{ij}^{kl} E_{ij} \otimes E_{kl}$$

$$\rho(E_{pr}) = E_{pq} \otimes E_{qr} + \sum_{i,j,k,l} \gamma_{ij}^{kl} E_{ij} \otimes E_{kl}.$$

□

We are now ready to prove our main result:

Theorem 3.2. *An incidence algebra $A(\Omega)$ of upper triangular matrices is contractible iff there is no discordant couple of D-transitive triples of elements of Ω.*

Proof. Suppose that there are no discordant couples of D-transitive triples and let $(m,n) \in \Omega$. We may consider two cases:

(i) Suppose (m,n) is a component of a D-transitive triple:

(i - 1) If the triple is of the form $((m,n),(n,r),(m,r))$, we define $\rho(E_{mn}) = E_{mn} \otimes E_{nn}$ according to (1).

(i - 2) If the triple is of the form $((s,m),(m,n),(s,n))$, we define $\rho(E_{mn}) = E_{mm} \otimes E_{mn}$ according to (2).

(i - 3) If the triple is of the form $((m,t),(t,n),(m,n))$, we define $\rho(E_{mn}) = E_{mt} \otimes E_{tn}$ according to (3).

(ii) If (m,n) is not a component of any D-transitive triple, we put $\rho(E_{mn}) = E_{mm} \otimes E_{mn}$.

Then ρ, being defined on a basis of A, is a well-defined A-bimodule morphism and furthermore $\pi \circ \rho = 1_A$. Hence, A is contractible.

If there is a discordant couple of D-transitive triples, Lemma 1 clearly yields a contradiction; e.g., if there exist $((i,j),(j,k),(i,k))$ and $((h,i),(i,j),(h,j))$ it follows from the lemma that $\rho(E_{ij}) = E_{ij} \otimes E_{jj} + \ldots = E_{ii} \otimes E_{ij} + \ldots$ which is contradictory to the linear independence of the E_{pq}'s. $\square$

Example 3.3. The incidence algebra of all 4×4 matrices of the form

$$\begin{pmatrix} * & 0 & * & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix},$$

where the stars indicate arbitrary complex numbers, is contractible; but the incidence algebra of all upper triangular 4×4 matrices is not contractible.

Question 3.4. What can we say about the contractibility of incidence algebra $A(\Omega)$ without the condition $(p,q) \in \Omega \Rightarrow p \leq q$ on Ω?

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