Some Notes on the Baer-invariant of a Nilpotent Product of Groups*

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Abstract

W.Haebich (Bull. Austral. Math. Soc., 7, 1972, 279-296) presented a formula for the Schur multiplier of a regular product of groups. In this paper first, it is shown that the Baer-invariant of a nilpotent product of groups with respect to the variety of nilpotent groups has a homomorphic image and in finite case a subgroup of Haebich's type. Second a formula will be presented for the Baer-invariant

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of a nilpotent product of cyclic groups with respect to the variety of nilpotent groups.

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1.INTRODUCTION

In 1907, I.Schur [ ], using representation method, showed that if G is the direct product of A and B, then the Schur multiplier of G has the following form :

$$M(G) = M(A \times B) \cong M(A) \times M(B) \times (A_{ab} \otimes B_{ab}).$$

Also, J.Wiegold [ ] in 1971 obtained the above result by some properties of covering groups:

$$M(A \times B) \cong M(A) \oplus M(B) \oplus \frac{[A,B]}{[A,B,A*B]}\ ,\ where\ \frac{[A,B]}{[A,B,A*B]} \cong A_{ab} \otimes B_{ab}\ .$$

In 1979, M.R.R.Moghaddam [ ] found a formula for the Baer-invariant (the generalization of the Schur multiplier with respect to the variety of groups.) of a direct product of two groups with respect to the variety of nilpotent groups of class at most c, $\mathcal{N}_c$, where $c+1$ is a prime number or 4. This result generalized somehow the work of Schur and Wiegold.

In 1997, we [ ] presented an explicit formula for the Baer-invariant of a finite abelian group with respect to the variety of nilpotent groups for every $c \geq 1$. Since a finite abelian group is a direct product of some cyclic groups, so this result is somehow a generalization of Schur-wiegold's result and also Moghaddam's result.

It is known that nilpotent product is a generalization of direct product. Therefore it is interesting to find a formula for the Schur multiplier or the Baer-invariant of a nilpotent product.

In 1972, W.Haebich [ ] (Theorem 2.14) presented a formula for the Schur multiplier of a regular product of a family of groups. We know that regular product is a generalization of nilpotent product, so this result is an interesting generalization of the Schur-Wiegold's result. Also, M.R.R.Moghaddam [ ] (Theorem 2.7) in 1979 gave a formula similar to Haebich's formula for the Schur multiplier of a nilpotent product. His approach was different from W.Haebich.

Finally, N.D.Gupta and M.R.R.Moghaddam [ ] (Theorem 2.15) in 1992 have calculated the Baer-invariant of the nilpotent dihedral group of class n

$$G_n =< x, y, |x^2, y^2, [x, y]^{2^{n-1}} >$$

with respect to the variety of nilpotent groups, $\mathcal{N}_c$. It is routine to verify that G_n is isomorphic to the nth nilpotent product of two cyclic groups i.e $G_n = \mathbf{Z}_2 \overset{n}{*} \mathbf{Z}_2$.

2.NOTATION AND PRELIMINERIES

It is supposed that the reader is familiar with the notions of variety of groups, verbal and marginal subgroups, Schur multiplier and Baer-invariant. In particular if G is a group with a free presentation

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1 \ ,$$

then the Baer-invariant of G with respect to the variety of nilpotent groups of class at most $c \geq 1$,$\mathcal{N}_c$, denoted by $calN_cM(G)$, is defined to be

$$\mathcal{N}_cM(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,\ _cF]}$$

(See C.R.Leedham-Green and S.McKay [ ], from where our notation have been taken and H.Neumann [ ] for the notion of variety of groups.)

Clearly, if $c = 1$, then $\mathcal{N}_c = \mathcal{A}$ is the variety of abelian groups and

$$\mathcal{N}_cM(G) = M(G) \cong \frac{R \cap F'}{[R, F]}$$

is the *Schur multiplier* of G.

Definition 2.1

A variety $\mathcal{V}$ is said to be a *Schur-Baer variety* if for any group G for which the marginal factor group, $G/V^*(G)$, is finite, then the verbal subgroup,$V(G)$, is also finite and $|V(G)|$ divides a power of $|G/V^*(G)|$. I.Schur in [11] proved that the variety of abelian groups,$\mathcal{A}$, is a Schur-Baer variety. Also R.Baer in [1] proved that the variety defined by some outer commutator words has the above property. The following theorem tells us a very important property of Schur-Baer varieties.

Theorem 2.2(C.R.Leedham-Green and S.McKay [6])

The following conditions on the variety $\mathcal{V}$ are equivalent:

(i) $\mathcal{V}$ is a Schur-Baer variety.

(ii) For every finite group G, its Baer-invariant, $\mathcal{V}M(G)$, is of order dividing a power of $|G|$.

Definition 2.3

Consider the following map

$$\varphi^* : \prod_{i\in I}^{*} A_i \longrightarrow \prod_{i\in I}^{\times} A_i$$

$$a_1 a_2 \dots a_n \longmapsto (a_1, a_2, \dots, a_n)$$

which is a natural map from the free product of $\{A_i\}_{i\in I}$ onto the direct product of $\{A_i\}_{i\in I}$. Clearly its kernel is the normal closure of

$$< [A_i, A_j] \mid i, j \in I, i \neq j >$$

in the free product $\prod_{i\in I}^{*} A_i$ and denoted by $[A_i]^*$ and called *the cartesian subgroup* of the free product.

For the properties of cartesian subgroup see H.Neumann [ ].

Definition 2.4

Let $\mathcal{V}$ be a variety of groups defined by a set of laws V then *the verbal product* of a family of groups $\{A_i\}_{i\in I}$ associated with the variety $\mathcal{V}$ is defined to be

$$\mathcal{V}\prod_{i\in I} A_i = \frac{\prod_i^* A_i}{V(A) \cap [A_i]^*} \ .$$

The verbal product is also known as *varietal product* or simply *$\mathcal{V}$_product.* If $\mathcal{V}$ is the variety of all groups, then the correspoding verbal product is the *free product*; if $\mathcal{V} = \mathcal{A}$ is the variety of all abelian groups, then the verbal product is the *direct product.* We denote $\mathcal{V}\prod_i A_i$ for the $\mathcal{V}$_product of the family of groups A_i, $i \in I$.

The $\mathcal{V}$-product of the $A_i's$ is 'between' the free product and the direct product in the sense that the natural epimorphism of the free product onto the direct product can be factored through the $\mathcal{V}$-product, i.e

$$\prod_i{}^* A_i \xrightarrow{\nu} \mathcal{V}\prod_i A_i \xrightarrow{\varphi(\nu)} \prod_i{}^\times A_i \quad .$$

The kernel of the natural epimorphism $\varphi(\nu)$ of the $\mathcal{V}$-product onto the direct product of the $A_i's$ is the cartesian subgroup of $\mathcal{V}\prod_i A_i$, that is the normal closure of $< [A_i, A_j] \mid i \neq j, i, j \in I >$, in the $\mathcal{V}$-product, which is denoted by $[A_i]^V$.

Lemma 2.5

With the above notation, we have

(i) $Ker\varphi(\nu) = [A_i]^V = \nu([A_i]^*)$ where $\varphi^* = \varphi(\nu) \circ \nu$ as given in 2.7 .

(ii) If $a \neq 1$ is an element of $\mathcal{V}\prod_i A_i$, then

$$a = a_{i_1} \dots a_{i_m} c \ \ , \ \ 1 \neq a_{i_j} \in A_{i_j} \ \ , i_1 < i_2 < \dots < i_m \ \ , \ \ c \in [A_i]^V \ ,$$

and the a_{i_j} and c are uniquely determined by a and chosen order of I .

Proof. See [ ] .

Definition 2.6

Let $\{A_i\}_{i\in I}$ be a family of groups and $\mathcal{N}_c$ be the variety of nilpotent groups of class at most $c \geq 1$. Then the $\mathcal{N}_c$-product, $\mathcal{N}_c\prod_i A_i$, is called the *cth nilpotent product* of A_i's. In particular, if A and B are two groups, then the cth nilpotent product of A and B, denoted by $A \overset{c}{*} B$, is as follows:

$$A \overset{c}{*} B = \frac{A * B}{[A, B, \ _{c-1}A * B]} \ .$$

Theorem 2.7 (M.R.R.Moghaddam [9])

Let A and B be two groups and $A \overset{n}{*} B$ be the nth nilpotent product of A and B. Then for $n \geq 1$

$$M(A \overset{n}{*} B) \cong M(A) \oplus M(B) \oplus \frac{[A, B, \ _{(n-1)}A * B]}{[A, B, \ _{n}A * B]} \ .$$

Definition 2.8

Let $\{A_i\}_{i\in I}$ be a family of subgroups of an arbitrary group G. We say that G is a *regular product* of its subgroups $A_i{}'s, i \in I$, where I is an ordered set if the following two conditions hold:

(i) $G =< A_i | i \in I >$,

(ii) $A_i \cap \hat{A}_i = 1$ for all $i \in I$, where $\hat{A}_i = \prod_{j\neq i, j\in I} A_j^G$

The subgroups $A_i{}'s, i \in I$ will be referred to as the *regular factors* of the group G. The direct product, the free product, and the verbal product of an arbitrary set of groups are examples of regular products.

Lemma 2.9

Let A be a subgroup of a group G, and N be a normal subgroup of G and $\{M_i | i \in I\}$ be a family of normal subgroups of G. Then

$$[A \prod_i M_i, N] = [A, N] \prod_i [M_i, N] \ .$$

The following theorem gives a characterization of regular product .

Theorem 2.10 (O.N.Golovin [2])

Suppose that a group G is generated by a family $\{A_i | i \in I\}$ of its subgroups, where I is an ordered set. Then G is a regular product of the $A_i{}'s$ if and only if every element of G can be written uniquely as a product

$$a_{\lambda_1} a_{\lambda_2} \dots a_{\lambda_m} u \ ,$$

where

$$1 \neq a_{\lambda_i} \in A_{\lambda_i}, \lambda_i < \lambda_2 < \dots < \lambda_m$$

and

$$u \in [A_i^G] =< [A_\lambda^G, A_\mu^G] | \lambda \neq \mu, \lambda, \mu \in I > \ .$$

Proof. See [2,5] .

Definition 2.11

Let G be a regular product of $A_i{}'s, i \in I$. A homomorphism $f : G \longrightarrow H$ of G to a group H is called a *regular homomorphism* if

$$Ker\, f \subseteq [A_i^G] \ .$$

The groups $Im\, f$ and $G/Ker\, f$ are called a *regular homomorphic image* and a *regular quotient group*, respectively.

Theorem 2.12

If G is a regular product of the subgroups $A_i{}'s, i \in I$, and $f : G \longrightarrow H$ is a regular homomorphism, then f restricted to A_i is an *isomorphism* for each $i \in I$ and $f(G)$ is a regular product of $f(A_i)$.

Proof. See [10] .

Theorem 2.13

Let G be generated by the $A_i{}'s, i \in I$, and let

$$\psi : \prod_{i \in I}{}^{*} A_i \longrightarrow G$$

be the homomorphism induced by the identity map on each A_i. Then G is a regular product of the A_i's if and only if ψ is regular.

Proof. See [ ].

Theorem 2.14 (W.Haebich [ ])

Let G be a regular product of a family of its subgroups $\{A_i | i \in I\}$ and let H be the kernel of the natural homomorphism

$$A = \prod_{i}{}^{*} A_i \longrightarrow G \quad ,$$

induced by the identity map on each A_i , $i \in I$. Then

$$M(G) \cong (\prod_{i\in I}^{\times} M(A_i)) \oplus \frac{H}{[H,A]} \ .$$

Theorem 2.15(N.D.Gupta and M.R.R.Moghaddam [ ])

Let G_n be the nilpotent dihedral group of class n, i.e $G_n = \mathbf{Z}_2 \overset{n}{*} \mathbf{Z}_2$, then (i) If $c < n$, then $\mathcal{N}_c M(G_n)$ is the cyclic group of order 2^c. (ii) If $c \geq n$, then $\mathcal{N}_c M(G_n)$ is an abelian extension of a cyclic group of order 2^{n-1} by an elementary abelian 2-group of rank $r(c+1)$, the number of basic commutators of wieght $c+1$ on two letters. In particular, for $c \geq n$,

$$\mathcal{N}_c M(G_n) \cong \mathbf{Z}_{2^n} \oplus \mathbf{Z}_2 \oplus \ldots \oplus \mathbf{Z}_2 \ (\ r(c+1) - 1 \ \textit{copies of} \ \mathbf{Z}_2 \ ) \ .$$

Proof. See [ ] Theorem .

Lemma 2.16

If H is a subgroup of a finite abelian group G, then G has a subgroup isomorphic to G/H .

3. A SUBGROUP OF HAEBICH'S TYPE

Let $\{A_i | i \in I\}$ be a family of groups (we can consider I as an ordered set) and for each $i \in I$, F_i denote a fixed free group such that the following exact sequence be a free presentation for A_i

$$1 \longrightarrow R_i \longrightarrow F_i \xrightarrow{\nu_i} A_i \longrightarrow 1 \quad .$$

We denote by ν the natural homomorphism from the free product $F = \prod_{i\in I}^{*} F_i$ onto $A = \prod_{i\in I}^{*} A_i$ induced by the ν_i's.Also, the group G_n will be assumed to be the nth nilpotent product of the A_i's, $i \in I$. If ψ_n is the natural homomorphism from A onto G_n induced by the identity map on each A_i ;

$$F = \prod_{i\in I}^{*} F_i \xrightarrow{\nu} A = \prod_{i\in I}^{*} A_i \xrightarrow{\psi_n} G_n = \prod_{i\in I}^{\overset{n}{*}} A_i \longrightarrow 1 \ ,$$

then we denote by H_n be the kernel of ψ_n and L_n the inverse image of H_n in F under ν ; $H_n = ker\psi_n$ & $L_n = \nu^{-1}(H_n)$.

Now we have the following lemma.

Lemma 3.1

By the above notation and assumption, we have

(i) $H_n = \nu(K_n)$, where $K_n = \gamma_{n+1}(F) \cap [F_i]^*$

(ii) $G_n \cong F/L_n$, where $L_n = ker(\psi_n \circ \nu) = (\prod_{i\in I} R_i^F)K_n$

Proof.

(i) By definition of nth nilpotent product, we have $H_n = \gamma_{n+1}(A) \cap [A_i]^*$, where $[A_i]^*$ is the cartesian subgroup of A , so clearly $\nu(K_n) = H_n$.

(ii) Clearly $Ker\nu = \prod_{i\in I} R_i^F$, and composition $\psi_n \circ \nu : F \longrightarrow G_n$ is surjective homomorphism. Since $H_n = Ker\psi_n = \nu(K_n)$, it follows that

$$L_n = Ker(\psi_n \circ \nu) = (Ker\nu)K_n = (\prod_{i\in I} R_i^F)K_n \quad .$$

Hence $G_n \cong F/L_n$. □

Lemma 3.2

If $D_1 = \prod_{i\neq j\ ,\ i,j\in I}[R_i, F_j]^F$, then $\prod_{i\in I} R_i^F = (\prod_{i\in I} R_i)D_1$.

Proof.

See [ ] Lemma .

Notation 3.3

We define (i) $D_c = \prod_{\exists j, \mu_j \neq i} [R_i, F_{\mu_1}, \ldots, F_{\mu_c}]^F$,
(ii) $E_c = D_1 \cap \gamma_{c+1}(F)$.
Clearly $D_c \subseteq E_c$ for every $c \geq 1$.

Now we prove the following important lemma.

Lemma 3.4

By the above notation and assumotion, we have
(i) $[L_n, \ _cF] = (\prod_{i\in I}[R_i, \ _cF_i])D_c[K_n, \ _cF]$.
(ii) If $n \leq c$, then $L_n \cap \gamma_{c+1}(F) = (\prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))E_cK_c$.
(iii) If $n > c$, then $L_n \cap \gamma_{c+1}(F) = (\prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))E_cK_n$.

Proof.

(i) By definition of $C_{(n)}$, we have

$$[C_{(n)}, \ _cB] = [(\prod_{i\in I} C_i^B)K_n, \ _cB] = [(\prod_{i\in I} C_i^B), \ _cB][K_n, \ _cB] \qquad (\textit{by Lemma 2.18})$$
$$= (\prod_{i\in I}[C_i, \ _cB]^B)[K_n, \ _cB] \qquad (\textit{by Lemma 2.18})$$
$$\subseteq (\prod_{i\in I}[C_i, \ _cB_i])D_c[K_n, \ _cB] \qquad (\textit{by definition of } D_c)$$

The reverse inclusion is obviously true, so the result holds.

(ii) Let $n \leq c$, and $g \in C_{(n)} \cap \gamma_{c+1}(B)$, by Lemmas 3.1 and 3.2 we have

$$g = c_{\lambda_1} \ldots c_{\lambda_t} dk \quad ,$$

where $c_{\lambda_i} \in C_{\lambda_i}$, $d \in D_1$, $k \in K_n$. Now consider the following homomorphism

$$B = \prod_{i\in I}{}^{*} B_i \xrightarrow{nat} \prod_{i\in I}{}^{\times} B_i \quad .$$

Since $g \in \gamma_{c+1}(B)$, the image of g under the above natural homomorphism is

$$(c_{\lambda_1}, \ldots, c_{\lambda_t}) \in \gamma_{c+1}(\prod{}^{\times} B_i) = \prod{}^{\times} \gamma_{c+1}(B_i) \quad .$$

Therefore $c_{\lambda_i} \in \gamma_{c+1}(B_{\lambda_i}) \cap C_{\lambda_i}$ and so $dk \in \gamma_{c+1}(B) \cap [B_i]^* = K_c$.
The reverse inclusion can be seen easily, and so the result holds.

(iii) It is clear. $\square$

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