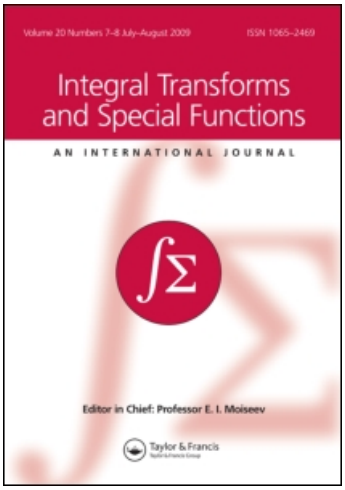

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THEOREMS ON *N*-DIMENSIONAL LAPLACE TRANSFORMATIONS INVOLVING THE SOLUTION OF WAVE EQUATIONS

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The primary objective of this article is to establish several new results for calculating Laplace transformation pairs of N-dimensions from known one-dimensional Laplace transformations. Next, we applied these results to a number of commonly used special functions to obtain a new Laplace transformation in two and N-dimensions. Finally, several one-dimensional wave equations involving special functions are explicitly solved by using two-dimensional Laplace transformations and applying some of the developed results in Section 2.

Keywords: Multi-dimensional transforms; Laplace transforms; Wave equation

MSC 1991: 44A30; 44A10; 35F05

1 INTRODUCTION AND NOTATION

The Laplace transform, it can be fairly said, stands first in importance among all integral transforms for which there are many specific examples in which other transforms prove more expedient. The Laplace transform is the most powerful in dealing with both initial-boundary-value problems (IBVPs) and transforms.

By the use of N-dimensional Laplace transformations, a partial differential equation (PDE) and its associated boundary conditions can be transformed into an algebraic equation in n independent complex variables. This algebraic equation can be solved for multiple transform of the solution of the original PDE. Multiple inversion of this transform then gives the desired solution [8], [9], [11], and [12]. This technique is used in Section 3 of this article to solve several one-dimensional Wave equations. For more results, see [2–7], [13], [15–20].

In what follows, we will use the following notations.

For any real n-dimensional variable $\bar{x} = (x_1, x_2, \ldots, x_n)$, and for any complex n-dimensional variable $\bar{s} = (s_1, s_2, \ldots, s_n)$, we denote $\overline{x^v} = (x_1^v, x_2^v, \ldots, x_n^v)$ and $\overline{s^v} = (s_1^v, s_2^v, \ldots, s_n^v)$ where v is any real number.

Let $p_k(\bar{x})$ or $p_k(\bar{s})$ be the kth symmetric polynomial in the components x_k or s_k, of x or s, respectively. Then for $\bar{s} = (s_1, s_2, \ldots, s_n)$, we denote

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(i) $p_1(\overline{s^v}) = s_1^v + s_2^v + \cdots + s_n^v = \sum_{j=1}^{n} s_j^v$.
(ii) $p_n(\overline{s^v}) = s_1^v \cdot s_2^v \cdots s_n^v = \prod_{j=1}^{n} s_j^v$.
In addition, we shall write
(iii) $\bar{x} \cdot \bar{s} = \sum_{j=1}^{n} x_j s_j$.
(iv) Re (s) = Real part of a complex number s.
(v) Re $[p_1(\overline{s^v})]$ = Real part of a complex number $(s_1^v + s_2^v + \cdots + s_n^v)$.

As usual, we denote by N the set of natural numbers, $N = \{1, 2, \ldots\}$ and $N_0 = N \cup \{0\}$. By R^n we mean the *n-dimensional Euclidean space*, $n \in N$. Analogously, we denote *unitary space* by C^n. By subsets R_+^n and $\overline{R}_+^n$ of R^n, we mean the following

$$R_+^n = \{\bar{x}\colon x \in R^n, \bar{x} > 0\},$$
$$\overline{R}_+^n = \{\bar{x}\colon \bar{x} \in R^n, \bar{x} \geq 0\}.$$

By $L_1[R_+^n; \exp(-\bar{a} \cdot \bar{x})]$, we mean the linear space of all measurable function f defined on R_+^n for which the following conditions hold:

(i) $\int_\omega |f(\bar{x})\, \mathrm{d}\bar{x}$ are finite.
(ii) $f(\bar{x})$ satisfies the condition

$$|f(\bar{x})| \leq M \exp(\bar{a} \cdot \bar{x}),$$

for all measurable functions f on R_+^n, where M and $\bar{a}$ are positive constants.

By $E_{\bar{a}}$, we mean the set of all functions from R^n into C with the following properties: There exists a point $\bar{a} \in R^n$ such that $f \in L_1[R_+^n; \exp(-\bar{a} \cdot \bar{x})]$ and

$$f(\bar{x}) = 0, \quad \bar{x} \in R_+^n \backslash R_+^{\bar{n}},$$

$E_{\bar{a}}$ is equipped with the norm $\|f\|_{E_{\bar{a}}} = \|f\|_{L_1[R_+^n; \exp(-\bar{a} \cdot \bar{x})]}$.

In the present work, the class Ω is taken to consist of all complex valued functions f that are piecewise continuous in the range $x \geq 0$ and of exponential order as $x \to \infty$.

If $u(x, y)$ be a function of two variables x and y, we denote

$$\frac{\partial u(x, y)}{\partial x} = u_x, \quad \frac{\partial u(x, y)}{\partial y} = u_y,$$
$$\frac{\partial^2 (x, y)}{\partial x \partial y} = u_{xy}, \quad \frac{\partial^2 u(x, y)}{\partial x^2} = u_{xx}, \quad \frac{\partial^2 u(x, y)}{\partial y^2} = u_{yy}.$$

We denote

$$u(x, 0) = \alpha(x), \quad u(0, y) = \beta(y),$$
$$u_y(x, 0) = \theta(x), \quad u_x(0, y) = \delta(y)$$

and assume $\alpha(x)$, $\beta(y)$, $\theta(x)$, and $\delta(y)$ are Laplace transformable.

The n-dimensional Laplace transform of a function from R_+^n into C is defined by means of

$$F(\bar{s}) = \mathbb{L}_n\{f(\bar{x}); \bar{s}\} = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \exp[-\bar{s}\cdot\bar{x}] f(\bar{x})\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n$$
$$= \int_{R_+^n} \exp[-\bar{s}\cdot\bar{x}] f(\bar{x})\,\mathrm{d}\bar{x}. \tag{1}$$

The domain of definition of F is the set of all points $\bar{s} \in C^n$ such that the integral in Eq. (1) is convergent.

Instead of the n-dimensional Laplace transform (1), one sometimes calculates the so-called n-dimensional Laplace–Carson transform:

$$F(\bar{s}) = p_n(\bar{s}) \int_{R_+^n} \exp(-\bar{s}\cdot\bar{x}) f(\bar{x})\,\mathrm{d}\bar{x}. \tag{2}$$

Symbolically, we denote the pair $F(\bar{s})$ and $f(\bar{x})$ with the operational relation

$$F(\bar{s}) \overset{n}{\underset{n}{=}} f(\bar{x}) \quad \text{or} \quad f(\bar{x}) \overset{n}{\underset{n}{=}} F(\bar{s}). \tag{3}$$

In this notation, some formulas become more simple.

We denote Eq. (3) in one-dimensional case by the following

$$F(s) \doteqdot f(x).$$

Let $f \in E_{\bar{a}}, \bar{a} \in R_+^n$ and let there exist $(\partial/\partial x_j) f = D_j f$, $j = 1, 2, \ldots n$, $D^1 f = \partial^n f/(\partial x_1 \partial x_2 \ldots \partial x_n)$ and $D_j f \in C(R_+^n)$, $D^1 f \in E_{\bar{a}}$. Then at each point of continuity of f, the so-called complex inversion formula holds

$$\mathbb{L}_n^{-1}\{F(\bar{s}); \bar{x}\} = f(\bar{x}) = (2\pi i)^{-n} \int_{(\bar{\alpha})} \exp(\bar{s}\cdot\bar{x}) F(\bar{s})\,\mathrm{d}\bar{s}, \quad \bar{\alpha} \in R_+^n, \ \bar{\alpha} > \alpha. \tag{4}$$

Here, the integral has to be understood in the sense of the principal value of Cauchy, *i.e.*,

$$\int_{(\bar{\alpha})} \ldots \mathrm{d}\bar{s} = \lim_{\substack{\beta_j \to \infty \\ j=1,2,\ldots,n}} \int_{\alpha_1 - i\beta_1}^{\alpha_1 + i\beta_1} \int_{\alpha_2 - i\beta_2}^{\alpha_2 + i\beta_2} \cdots \int_{\alpha_n - i\beta_n}^{\alpha_n + i\beta_n} \cdots \mathrm{d}s_1\,\mathrm{d}s_2 \ldots \mathrm{d}s_n.$$

The integral at the right side of Eq. (4) is called the n-dimensional inverse Laplace transformation of F.

If we denote

$$u(x, 0) = f(x), u(0, y) = g(y),$$
$$u_y(x, y)|_{y=0} = u_y(x, 0) = f_1(x), \quad u_x(x, y)|_{x=0} = u_x(0, y) = g_1(y),$$

and their one-dimensional Laplace transformations are denoted by $G(s_2)$, $F_1(s_1)$, and $G_1(s_2)$, respectively, then

$$\mathbb{L}_2\{u(x, y); s_1, s_2\} = \int_0^\infty \int_0^\infty \exp(-s_1 x - s_2 y) u(x, y)\, \mathrm{d}x\, \mathrm{d}y = U(s_1, s_2), \tag{5}$$

$$\mathbb{L}_2\{u_{xx}; s_1, s_2\} = s_1^2 U(s_1, s_2) - s_1 G(s_2) - G_1(s_2), \tag{6}$$

$$\mathbb{L}_2\{u_{yy}; s_1, s_2\} = s_2^2 U(s_1, s_2) - s_2 F(s_1) - F_1(s_1). \tag{7}$$

It is assumed that the integrals involved exist. The existence conditions for two-dimensions are given in Ditkin and Prudnikov [10, p. 4] and for similar conditions hold for N-dimensions, we refer to Brychkov *et al.* [1; ch. 2].

2 THEOREMS ON N-DIMENSIONAL LAPLACE TRANSFORMATION

THEOREM 2.1 *Let*

(i) $\mathbb{L}\{f(x); s\} = \phi(s)$, *and*
(ii) $\mathbb{L}\{x^{-1/2} f(x^2); s\} = F(s)$, *assuming that* $f(x^2)$ *is a function of class* Ω.
(iii) $\mathbb{L}\{x^{-3/2}\phi\left(\frac{1}{x}\right); s\} = \xi(s)$,
(iv) $\mathbb{L}\{x^{-3/2}\xi\left(\frac{1}{x^2}\right); s\} = \eta(s)$,
where $x^{-3/2}\phi(1/x)$ *and* $x^{-3/2}\xi(1/x^2)$ *are also functions of class* Ω, *and* $x^{-3/2}\exp(-sx - u/x) f(u)$ *and* $u^{-1/2}x^{-3/2}\exp(-sx - 2u^{1/2}/x) f(u)$ *belong to* $L_1[(0, \infty) \times (0, \infty)]$.
Then

$$\mathbb{L}_n\left\{\frac{1}{p_n^{(\overline{x^{1/2}})}} F[2p_1(\overline{x^{-1}})]; \bar{s}\right\} = \frac{\pi^{(n-2)/2}}{2^{1/2} p_n(\overline{s^{1/2}})}\, \eta[(p_1(\overline{s^{1/2}}))^2], \tag{8}$$

where $n = 2, 3, \ldots, N$ *and* $\mathrm{Re}\,[p_1(\overline{s^{1/2}})] > c$ *and* c *is a constant.*

Proof From (i), we obtain $\phi(s) = \int_0^\infty \exp(-st) f(t)\, \mathrm{d}t$ for $\mathrm{Re}\, s > c_0$, where c_0 is a constant, so that

$$x^{-3/2}\phi\left(\frac{1}{x}\right) = x^{-3/2} \int_0^\infty \exp\left(-\frac{u}{x}\right) f(u)\, \mathrm{d}u. \tag{9}$$

Multiplying both sides of Eq. (9) by $\exp(-sx)$ and integrating from 0 to ∞, we obtain

$$\int_0^\infty \exp(-sx) x^{-3/2}\phi\left(\frac{1}{x}\right) \mathrm{d}x = \int_0^\infty \left[\int_0^\infty x^{-3/2} \exp\left(-sx - \frac{u}{x}\right) f(u)\, \mathrm{d}u\right] \mathrm{d}x.$$

The integrand $x^{-3/2}\exp(-sx - u/x) f(u)$ belongs to $L_1[(0, \infty) \times (0, \infty)]$, that, by Fubini's Theorem, interchanging the order of the integral on the right side of Eq. (9) is permissible. By

using (iii) on the left and interchanging the order of integration on the right side of Eq. (9), we obtain

$$\xi(s) = \int_0^\infty f(u) \left[\int_0^\infty x^{-3/2} \exp\left(-sx - \frac{u}{x}\right) \mathrm{d}x \right] \mathrm{d}u, \tag{10}$$

where Re $s > \lambda_1$ and λ_1 is a constant.
A result in Roberts and Kaufman [14] regarding the inner integral in Eq. (10) be used to evaluate this integral as

$$\int_0^\infty x^{-3/2} \exp\left(-sx - \frac{u}{x}\right) \mathrm{d}x = \pi^{1/2} u^{-1/2} \exp\left(-2u^{1/2} s^{1/2}\right).$$

Therefore, Eq. (10) can be written as

$$\xi(s) = \pi^{1/2} \int_0^\infty u^{-1/2} \exp\left(-2u^{1/2} s^{1/2}\right) f(u)\, \mathrm{d}u. \tag{11}$$

Using Eq. (2.4) in (iv), we arrive at

$$\eta(s) = \pi^{1/2} \int_0^\infty \left[\int_0^\infty s^{-3/2} u^{-1/2} \exp\left(-sx - \frac{2u^{1/2}}{x}\right) f(u)\, \mathrm{d}u \right] \mathrm{d}x, \tag{12}$$

where Re $s > \lambda_2$ for some constant λ_2.

Again, because $x^{-3/2} u^{1/2} \exp(-sx - 2u^{1/2}/x) f(u)$ belongs to $L_1[(0, \infty) \times (0, \infty)]$, by Fubini's Theorem, we can interchange the order of integration on the right side of Eq. (12) to obtain

$$\eta(s) = \pi^{1/2} \int_0^\infty u^{-1/2} f(u) \left[\int_0^\infty x^{-3/2} \exp\left(-sx - \frac{(2^{3/2} u^{1/4})^2}{4x}\right) \mathrm{d}x \right] \mathrm{d}u. \tag{13}$$

Using the previously mentioned result in Roberts and Kaufman [14] on the inner integral in Eq. (10), we can evaluate the integral inside the brackets.

Consequently, Eq. (13) becomes

$$\eta(s) = 2^{-1/2} \pi \int_0^\infty u^{-3/4} f(u) \exp(-2^{3/2} u^{1/4} s^{1/2})\, \mathrm{d}u. \tag{14}$$

By substituting $u = v^2$ in Eq. (14), we obtain

$$\eta(s) = 2^{1/2} \pi \int_0^\infty v^{-1/2} f(v^2) \exp(-2^{3/2} v^{1/2} s^{1/2})\, \mathrm{d}v. \tag{15}$$

Replacing s by $[p_1(\overline{s^{1/2}})]^2$ and multiplying both sides of Eq. (15) by $p_n(\overline{s^{1/2}})$, we arrive at

$$p_n(\overline{s^{1/2}})\eta[(p_1(\overline{s^{1/2}}))^2] = 2^{1/2}x\int_0^\infty v^{-1/2}f(v^2)p_n(\overline{s^{1/2}})\exp(-2^{3/2}v^{1/2}p_1(\overline{s^{1/2}}))\,\mathrm{d}v. \tag{16}$$

Now, we use the operational relation given in Ditkin and Prudnikov [10]:

$$s_i^{1/2}\exp(-as_i^{1/2}) \doteqdot (\pi x_i)^{-1/2}\exp\left(-\frac{a^2}{4x_i}\right) \quad \text{for } i = 1, 2, \ldots, n. \tag{17}$$

Equation (16) can be rewritten as

$$p_n(\overline{s^{1/2}})\eta[(p_1(\overline{s^{1/2}}))^2] \overset{n}{\underset{n}{=}} \frac{2^{1/2}}{\pi^{(n-2)/2}p_n(\overline{x^{1/2}})}\int_0^\infty v^{-1/2}f(v^2)\exp(-2vp_1(\overline{x^{-1}}))\,\mathrm{d}v \tag{18}$$

Using (ii) and Eq. (18), we obtain

$$p_n(\overline{s^{1/2}})\eta[(p_1(\overline{s^{1/2}}))^2] \overset{n}{\underset{n}{=}} \frac{2^{1/2}}{\pi^{(n-2)/2}}\cdot\frac{1}{p_n(\overline{x^{1/2}})}F[2p_1(\overline{x^{-1}})].$$

Hence,

$$\mathbb{L}_n\left\{\frac{1}{p_n(\overline{x^{1/2}})}F[2p_1(\overline{x^{-1}})];\,\bar{s}\right\} = \frac{\pi^{(n-2)/2}}{2^{1/2}p_n(\overline{s^{1/2}})}\eta[(p_1(\overline{s^{1/2}}))^2],$$

where $n = 2, 3, \ldots, N$.

Example 2.1′ Let $f(x) = x^{v/4}$. Then

$$\phi(s) = \frac{\Gamma((v/4)+1)}{s^{((v/4)+1)}}, \quad \operatorname{Re} s > 0, \ \operatorname{Re} v > -4,$$
$$\xi(s) = \frac{\Gamma((v/4)+1)\Gamma((v/4)+(1/2))}{s^{(v/4)+(1/2)}}, \quad \operatorname{Re} v > -2, \ \operatorname{Re} s > 0$$

and

$$\eta(s) = \frac{\Gamma((v/4)+1)\Gamma((v/4)+(1/2))\Gamma((v+1)/2)}{s^{(v+1)/2}}, \quad \operatorname{Re} v > -1, \ \operatorname{Re} s > 0.$$
$$F(s) = \frac{\Gamma((v+1)/2)}{s^{(v+1)/2}}.$$

Therefore,

$$\mathbb{L}_n\left\{\frac{1}{p_n(\overline{x^{1/2}})[p_1(\overline{x^{-1}})]^{(v+1)/2}};\,\bar{s}\right\} = \pi^{(n-1)/2}\Gamma\left(\frac{v}{2}+1\right)\cdot\frac{1}{p_n(\overline{s^{1/2}})[p_1(\overline{s^{1/2}})]^{v+1}}, \tag{19}$$

where $\operatorname{Re} v > -1$, $\operatorname{Re}[p_1(\overline{s^{1/2}})] > 0$, and $n = 2, 3, \ldots, N$.

Example 2.2 Assume that $f(x) = \sin(ax^{1/2})$. Then

$$\phi(s) = \frac{\alpha\pi^{1/2}}{2} s^{-3/2} \exp\left(\frac{a^2}{4s}\right), \quad \operatorname{Re} s > 0$$

and

$$\xi(s) = \frac{2\alpha\pi^{1/2}}{4s + \alpha^2}, \quad \operatorname{Re} s > -\operatorname{Re} \frac{a^2}{4},$$

$$\eta(s) = \frac{\pi}{\alpha} s^{1/2} S_{-1,1/2}\left(\frac{2s}{\alpha}\right), \quad \operatorname{Re} a > 0, \ \operatorname{Re} s > 0,$$

$$F(s) = \frac{\pi^{1/2}}{(s^2 + \alpha^2)^{1/4}} \sin\left[\frac{1}{4} \tan^{-1}\left(\frac{\alpha}{s}\right)\right], \quad \operatorname{Re} s > |\operatorname{Im} a|.$$

where $\operatorname{Re} a > 0$, $\operatorname{Re} s > 0$.

Hence, we obtain

$$\mathbb{L}_n \left\{ \frac{\sin[(1/4)\tan^{-1}(\alpha/(2p_1(\overline{x^{-1}})))]}{p_n(\overline{x^{1/2}})[4p_1^2(\overline{x^{-1}}) + \alpha^2]^{1/4}}; \bar{s} \right\} = \left(\frac{\pi^{n+1}}{2}\right)^{1/2} \cdot \frac{p_1(\overline{s^{1/2}})}{\alpha p_n(\overline{s^{1/2}})} S_{-1,1/2}\left[\frac{2}{\alpha} p_1^2(\overline{s^{1/2}})\right], \tag{20}$$

where $\operatorname{Re} \alpha > 0$, $\operatorname{Re}[p_1(\overline{s^{1/2}})] > 0$.

Example 2.3 Consider $f(x) = x^\tau p^F q \left[\begin{matrix} (a)_p; \\ (b)_q; \end{matrix} \quad kx \right]$. Then

$$\phi(s) = \frac{\Gamma(\tau + 1)}{s^{\tau+1}} p + 1^F q \left[\begin{matrix} (a)_p, \tau + 1; \\ (b)_q; \end{matrix} \quad ks \right],$$

where $p \le q$, $\operatorname{Re} \tau > -1$ and $\operatorname{Re} s > \begin{cases} 0 & \text{if } p < q \\ \operatorname{Re} k & \text{if } p = q. \end{cases}$

$$F(s) = \frac{\Gamma((4\tau + 1)/2)}{s^{(4\tau+1)/2}} p + 2^F q \left[\begin{matrix} (a)_p, \tau + 1/4, \tau + 3/4; \\ (b)_q; \end{matrix} \quad \frac{4k}{s^2} \right],$$

where $p \le q - 1$, $\operatorname{Re} \tau > -1/4$; $\operatorname{Re} s > 0$ if $p \le q - 1$; and $\operatorname{Re}\ (s + 2k \cos \pi r) > 0 (r = 0, 1)$ if $p = q - 1$.

$$\xi(s) = \frac{\Gamma(\tau + 1)\Gamma(\tau + 1/2)}{s^{\tau+1/2}} p + 2^F q \left[\begin{matrix} (a)_p, \tau + 1, \tau + \frac{1}{2}; \\ (b)_q; \end{matrix} \quad ks \right],$$

where $p \le q - 1$, $\operatorname{Re} \tau > -1/2$ and $\operatorname{Re} s > \begin{cases} 0 & \text{if } p < q \\ \operatorname{Re} k & \text{if } p = q. \end{cases}$

$$\eta(s) = \frac{\Gamma(\tau + 1)\Gamma(\tau + 1/2)\Gamma(2\tau + 1/2)}{s^{2\tau+1/2}} p + 4^F q$$

$$\left[\begin{matrix}(a)_p, \tau+1, \tau+\frac{1}{2}, \tau+\frac{1}{4}, \tau+\frac{3}{4}; \\ (b)_q;\end{matrix} \quad \frac{4k^2}{s^2}\right],$$

where $p \le q-3$, Re $\tau > -1/4$; Re $s > 0$ if $p \le q-4$; and Re $(s+2k\cos\pi r) > 0 (r = 0, 1)$ if $p = q-3$.

Therefore, we have

$$\mathbb{L}_n\left\{\frac{1}{p_n(\overline{x^{1/2}})[p_1(\overline{x^{-1}})]^{2\tau+1/2}} p+2^F q\left[\begin{matrix}(a)_p, \frac{4\tau+1}{4}, \frac{4\tau+3}{4}; \\ (b)_q;\end{matrix} \quad \frac{k}{p_1^2(\overline{x^{-1}})}\right]; \bar{s}\right\}$$

$$= \frac{\pi^{(n-1)/2}\Gamma(2\tau+1)}{p_n(\overline{s^{1/2}})[p_1(\overline{s^{1/2}})]^{4\tau+1}} p+4^F q\left[\begin{matrix}(a)_p, \tau+1, \tau+\frac{1}{2}, \tau+\frac{1}{4}, \tau+\frac{3}{4}; \\ (b)_q;\end{matrix} \quad \frac{4k}{p_1^4(\overline{s^{-1}})}\right] \tag{21}$$

where $p \le q-3$, Re $\tau > -1/4$; Re $[p_1(\overline{s^{1/2}})] > 0$ if $p \le q-4$; and Re $[p_1(\overline{s^{1/2}}) + 2k\cos\pi r] > 0$ $(r = 0, 1)$ if $p = q-3$.

Example 2.4 Suppose that

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{64} \\ 0 & \text{if } x > \frac{1}{64}. \end{cases}$$

Then,

$$\phi(s) = \frac{1-\exp(-s/64)}{s},$$

In addition, we obtain

$$\xi(s) = \left(\frac{\pi}{s}\right)^{1/2}\left[1-\exp\left(-\frac{s^{1/2}}{4}\right)\right], \quad \text{Re } s > 0,$$

$$\eta(s) = \frac{\pi}{s^{1/2}}[1-\exp(-s^{1/2})], \quad \text{Re } s > 0,$$

$$F(s) = \frac{\gamma(1/2, s/8)}{s^{1/2}}, \quad \text{Re } s > -\infty.$$

Therefore, using Theorem 2.1, we obtain the following result:

$$\mathbb{L}_n\left\{\frac{1}{p_n(\overline{x^{1/2}})[p_1(\overline{x^{-1}})]^{\frac{1}{2}}}\text{Erf}\left[\left(\frac{p_1(\overline{x^{-1}})}{2}\right)^{1/2}\right]; \bar{s}\right\}$$

$$= \frac{\pi^{(n-1)/2}}{p_n(\overline{s^{1/2}})p_1(\overline{s^{1/2}})}[1-\exp(-p_1(\overline{s^{1/2}}))]. \tag{22}$$

where Re $[p_1(\overline{s^{1/2}})] > 0$, $n = 2, 3, \ldots, N$.

Example 2.1′ Let $n = 2$, from Example 2.1, we arrive at the following well-known result in two-dimensions.

$$\mathbb{L}_2\left\{\frac{(xy)^{v/2}}{(x+y)^{(v+1)/2}}; s_1, s_2\right\} = \pi^{1/2}\Gamma\left(\frac{v}{2}+1\right)\cdot\frac{2}{(s_1s_2)^{1/2}(s_1^{1/2}+s_2^{1/2})^{v+1}}, \tag{23}$$

where Re $v > -1$, Re $[s_1^{1/2}+s_2^{1/2}] > 0$.

If we let $v = 0$ in Eq. (23), we obtain

$$\mathbb{L}_2\left\{\frac{1}{(x+y)^{1/2}}, s_1, s_2\right\} = \frac{\pi^{1/2}}{(s_1s_2)^{1/2}(s_1^{1/2}+s_1^{1/2})}. \tag{24}$$

From Eq. (24) and the following relations

$$\mathbb{L}_2\{xF(x,y); s_1, s_2\} = -\frac{\partial}{\partial s_1}f(s_1, s_2),$$
$$\mathbb{L}_2\{yF(x,y); s_1, s_2\} = \frac{\partial}{\partial s_2}f(s_1, s_2),$$

where $f(s_1, s_2) = \mathbb{L}_2\{F(x,y); s_1, s_2\}$.
We deduce that

$$\mathbb{L}_2\left\{(x+y)^{1/2}; s_1, s_2\right\} = \frac{\pi^{1/2}(s_1+s_2+s_1^{1/2}s_2^{1/2})}{2(s_1s_2)^{3/2}(s_1^{1/2}+s_2^{1/2})}. \tag{25}$$

THEOREM 2.2 *Suppose f and $x^{-3/2}\phi(1/x)$ belong to class Ω, where ϕ is the one-dimensional Laplace transform of f.*
Let

(i) $\mathbb{L}\{x^{j-1/2}f(x); s\} = F_j(s)$ *for* $j = 0, 1$, *and let* $d/ds\{s^{-v}\zeta(1/s^2)\}$ *exist for* Re $s > c_0$, *where c_0 is a constant.*
If $x^{-3/2}\exp(-sx - u/x)$ *belongs to* $L_1[(0,\infty)\times(0,\infty)]$ *and the following conditions hold*:
(ii) $\mathbb{L}\{x^{-3/2}\phi(1/x); s\} = \zeta(s)$,
(iii) $-d/ds\{s^{-v}\zeta(1/s^2)\} = \eta(s)$.

Then

$$\mathbb{L}_n\left\{\frac{vF_0[p_1(\overline{x^{-1}})] - 2p_1(\overline{x^{-1}})F_1[p_1(\overline{x^{-1}})]}{p_n(\overline{x^{1/2}})}; \bar{s}\right\} = \frac{\pi^{(n-1)/2}}{p_n(\overline{s^{1/2}})[p_1(\overline{s^{1/2}})]^{v+1}}\eta[(p_1(\overline{s^{1/2}}))^{-1}], \tag{26}$$

where Re $[p_1(\overline{s^{1/2}})] > c$, *$c$ is a constant. It is assumed that the integrals involved exist for* $n = 2, 3, \ldots, N$.

Proof By the hypothesis and (ii), we get

$$\zeta(s) = \int_0^\infty x^{-3/2}\phi\left(\frac{1}{x}\right)\exp(-sx)\,\mathrm{d}x = \int_0^\infty x^{-3/2}\exp(-sx)\left[\int_0^\infty f(u)\exp\left(-\frac{u}{x}\right)\mathrm{d}u\right]\mathrm{d}x,$$

where Re $s > c_0$ for some constant c_0 which leads to

$$\zeta(s) = \int_0^\infty \left[\int_0^\infty x^{-3/2} \exp\left(-sx - \frac{u}{x}\right) f(u)\, \mathrm{d}u \right] \mathrm{d}x. \tag{27}$$

Next we wish to interchange the order of the integral on the right side of Eq. (27). The interchange is permissible, according to Fubini's Theorem.
Therefore,

$$\zeta(s) = \int_0^\infty f(u) \left[\int_0^\infty x^{-3/2} \exp\left(-sx - \frac{u}{x}\right) \mathrm{d}x \right] \mathrm{d}u, \quad \text{where Re } s > c_0. \tag{28}$$

A result in Roberts and Kaufman [14] regarding the inner integral in Eq. (28) is used to evaluate the integral as

$$\zeta(s) = \pi^{1/2} \int_0^\infty u^{-1/2} f(u) \exp(-2u^{1/2} s^{1/2})\, \mathrm{d}u. \tag{29}$$

Equation (29) together with (iii) shows that

$$\eta(s) = \pi^{1/2} s^{-v-1} \left[v \int_0^\infty u^{-1/2} f(u) \exp\left(-\frac{2u^{1/2}}{s}\right) \mathrm{d}u - 2s^{-1} \int_0^\infty f(u) \exp\left(-\frac{2u^{1/2}}{s}\right) \mathrm{d}u \right],$$

so that

$$s^{v+1}\eta(s) = \pi^{1/2} \left[v \int_0^\infty u^{-1/2} f(u) \exp\left(-\frac{2u^{1/2}}{s}\right) \mathrm{d}u - 2s^{-1} \int_0^\infty f(u) \exp\left(-\frac{2u^{1/2}}{s}\right) \mathrm{d}u \right], \tag{30}$$

where Re $s > c_0$.

Next, we replace s by $[p_1(\overline{s^{1/2}})]^{-1}$ and, then, multiply both sides of Eq. (30) by $p_n(\overline{s^{1/2}})$, to get

$$\begin{aligned} &p_n(\overline{s^{1/2}})[p_1(\overline{s^{1/2}})]^{-v-1}\eta[(p_1(\overline{s^{1/2}}))^{-1}] \\ &\quad = \pi^{1/2} v \int_0^\infty u^{-1/2} p_n(\overline{s^{1/2}}) f(u) \exp[-2u^{1/2} p_1(\overline{s^{1/2}})]\, \mathrm{d}u \\ &\qquad - 2\pi^{1/2} \int_0^\infty p_n(\overline{s^{1/2}}) p_1(\overline{s^{1/2}}) f(u) \exp[-2u^{1/2} p_1(\overline{s^{1/2}})]\, \mathrm{d}u. \end{aligned} \tag{31}$$

Now, we use the following operational relations given in Ditkin and Prudnikov [10]:

$$s_i^{1/2} \exp(-\alpha s_i^{1/2}) \doteqdot (\pi x_i)^{-1/2} \exp\left(-\frac{\alpha^2}{4x_i}\right) \quad \text{for } i = 1, 2, \ldots, n,$$

$$s_i \exp(-\alpha s_i^{1/2}) \doteqdot \frac{\alpha}{2} \pi^{-1/2} x_i^{-3/2} \exp\left(-\frac{\alpha^2}{4x_i}\right) \quad \text{for } i = 1, 2, \ldots, n.$$

Equation (31) can be rewritten as

$$\begin{aligned} &p_n(\overline{s^{1/2}})[p_1(\overline{s^{1/2}})]^{-v-1}\eta[(p_1(\overline{s^{1/2}}))^{-1}] \\ &\quad \overline{\overline{n}}\,\frac{v}{\pi^{(n-1)/2}p_n(\overline{x^{1/2}})}\int_0^\infty u^{-1/2}f(u)\exp[-2u^{1/2}p_1(\overline{x^{-1}})]\,\mathrm{d}u \\ &\quad -\frac{2p_1(\overline{x^{-1}})}{\pi^{(n-1)/2}p_n(\overline{x^{1/2}})}\int_0^\infty u^{1/2}f(u)\exp[-2u^{1/2}p_1(\overline{x^{-1}})]\,\mathrm{d}u. \end{aligned} \tag{32}$$

Equation (32) with (i) for $j = 0, 1$, yields to

$$\begin{aligned} &p_n(\overline{s^{1/2}})[p_1(\overline{s^{1/2}})]^{-v-1}\eta[(p_1(\overline{s^{1/2}}))^{-1}] \\ &\quad \overline{\overline{n}}\,\frac{1}{\pi^{(n-1)/2}p_n(\overline{x^{1/2}})}\{vF_0[p_1(\overline{x^{-1}})] - 2(p_1(\overline{x^{-1}})F_1[p_1(\overline{x^{-1}})])\}. \end{aligned}$$

Thus,

$$\mathbb{L}_n\left\{\frac{vF_0[p_1(\overline{x^{-1}})] - 2p_1(\overline{x^{-1}})F_1[p_1(\overline{x^{-1}})]}{p_n(\overline{x^{1/2}})};\bar{s}\right\} = \frac{\pi^{(n-1)/2}}{p_n(\overline{s^{1/2}})[p_n(\overline{s^{1/2}})]^{v+1}}\eta[(p_1(\overline{s^{1/2}}))^{-1}],$$

where $\mathrm{Re}[p_1(\overline{s^{1/2}})] > c$ and $n = 2, 3, \ldots, N$. ■

The following examples will illustrate the applications of Theorem 2.2. We shall consider the function f to be an elementary or some special function to construct certain functions with n variables, and we calculate their Laplace transforms using Theorem 2.2. We will use the two-dimensional case for some of these examples in Section 3 to solve the Wave equations.

Example 2.5 Let $f(x) = J_0(2x^{1/2})$. The

$$\phi(s) = \frac{1}{s}\exp\left(-\frac{1}{s}\right), \quad \mathrm{Re}\, s > 0,$$

$$\zeta(s) = \frac{\pi^{1/2}}{(s+1)^{1/2}}, \quad \mathrm{Re}\, s > 0,$$

so that

$$\eta(s) = \frac{\pi^{1/2}(vs^2 + v - 1)}{s^v(1+s^2)^{3/2}}, \quad \mathrm{Re}\, s > 0.$$

$$F_j(s) = \frac{\Gamma(j-1/2)}{s^{j-1/2}}\exp\left(-\frac{1}{s}\right){}_1F_1\left[\begin{matrix} j-\frac{1}{2}; \\ 1; \end{matrix}\ 1s\right] \quad \text{for } j = 0, 1, \ \ \mathrm{Re}\, s > 0.$$

Hence,

$$\mathbb{L}_n\left\{\frac{\begin{matrix}\pi^{1/2}v\exp[-1/p_1(\overline{x^{-1}})]{}_1F_1\left[\begin{matrix}1/2; \\ 1;\end{matrix}\ 1/p_1(\overline{x^{-1}})\right] \\ -2\pi^{1/2}/2\exp[-1/p_1(\overline{x^{-1}})]{}_1F_1\left[\begin{matrix}-\frac{1}{2}; \\ 1;\end{matrix}\ 1/p_1(\overline{x^{-1}})\right]\end{matrix}}{[p_1(\overline{\overline{x^{-1}}})]^{1/2}p_n(\overline{x^{1/2}})}\right\}$$

$$= \frac{\pi^{\frac{n-1}{2}} \cdot \pi^{1/2}[v[p_1(\overline{s^{1/2}})]^{-2} + v - 1][p_1(\overline{s^{1/2}})]^v}{p_n(\overline{s^{1/2}})[p_1(\overline{s^{1/2}})]^{v+1}[1 + (p_1(\overline{s^{1/2}}))^{-2}]^{3/2}}.$$

Thus,

$$\mathbb{L}_n \left\{ \frac{\exp[-1/p_1(\overline{x^{-1}})] \left\{ v_1 \; F_1 \left[\begin{matrix} 1/2; \\ 1; \end{matrix} \; 1 p_1(\overline{x^{-1}}) \right] - {}_1F_1 \left[\begin{matrix} -1/2; \\ 1; \end{matrix} \; 1 p_1(\overline{x^{-1}}) \right] \right.}{p_n(\overline{x^{1/2}})[p_1(\overline{x^{-1}})]^{1/2}} ; \bar{s} \right\}$$

$$= \frac{\pi^{(n-1)/2}[v + (v-1)p_1^2(\overline{s^{1/2}})]}{p_n(\overline{s^{1/2}})[1 + p_1^2(\overline{s^{1/2}})]^{3/2}}, \tag{33}$$

where Re $p_n(\overline{s^{1/2}}) > 0$.

Remark 2.1 Equation (33) in two-dimensional case for $v = 0$ or $v = 1$ reduces to the following new results, respectively.

(i)

$$\mathbb{L}_2 \left\{ \frac{\exp[-xy/(x+y)]_1F_1 \left[\begin{matrix} -1/2; \\ 1; \end{matrix} \; xy/(x+y) \right]}{(x+y)^{1/2}} ; s_1, s_2 \right\}$$

$$= \frac{\pi^{1/2}(s_1^{1/2} + s_2^{1/2})^2}{(s_1 s_2)^{1/2}[1 + (s_1^{1/2} + s_2^{1/2})^2]^{3/2}}, \tag{34}$$

where Re $[s_1^{1/2} + s_2^{1/2}] > 0$.

$$\mathbb{L}_2 \left\{ \frac{\exp[-xy/(x+y)] \left\{ {}_1F_1 \left[\begin{matrix} 1/2; \\ 1; \end{matrix} \; xy/(x+y) \right] - {}_1F_1 \left[\begin{matrix} -1/2; \\ 1; \end{matrix} \; xy/(x+y) \right] \right\}}{(x+y)^{1/2}} ; s_1, s_2 \right\}$$

$$= \frac{\pi^{1/2}}{(s_1 s_2)^{1/2}[1 + (s_1^{1/2} + s_2^{1/2})^2]^{3/2}}$$

using the following result

$${}_1F_1 \left[\begin{matrix} 1/2; \\ 1; \end{matrix} \; xyx + y \right] - {}_1F_1 \left[\begin{matrix} -1/2; \\ 1; \end{matrix} \; xyx + y \right] = \frac{xy}{x+y} {}_1F_1 \left[\begin{matrix} 1/2; \\ 2; \end{matrix} \; xyx + y \right].$$

The last formula can be simplified as

$$\mathbb{L}_2\left\{\exp\left[-\frac{xy}{x+y}\right]xy(x+y)^{1/2}{}_1F_1\left[\begin{matrix}1/2;\\ 2;\end{matrix}\ xyx+y\right];s_1,s_2\right\}$$
$$=\frac{\pi^{1/2}}{(s_1s_2)^{1/2}[1+(s_1^{1/2}+s_2^{1/2})^2]^{3/2}}, \tag{35}$$

where Re $[s_1^{1/2}+s_2^{1/2}]>0$.

Using the following version of F_j

$$F_j(s)=\frac{\Gamma(j-1/2)}{s^{j-1/2}}{}_1F_1\left[\begin{matrix}j-1/2;\\ 1;\end{matrix}\ -1s\right]\quad \text{for } j=0,1.$$

The Formula (33) turns out to be

$$\mathbb{L}_n\left\{\frac{v\ {}_1F_1\left[\begin{matrix}1/2;\\ 1;\end{matrix}\ -1p_1\right]-{}_1F_1\left[\begin{matrix}3/2;\\ 1;\end{matrix}\ -1p_1(\overline{x^{-1}})\right]}{p_1(\overline{x^{1/2}})[p_1(\overline{x^{-1}})]^{1/2}};\bar{s}\right\}$$
$$=\frac{\pi^{(n-1)/2}[v+(v-1)p_1^2(\overline{s^{1/2}})]}{p_n(\overline{s^{1/2}})[1+p_1^2(\overline{s^{1/2}})]^{3/2}}, \tag{36}$$

where Re $[p_1(\overline{s^{1/2}})]>0$.

Special cases of Eq. (36) for $n=2$, $v=0$ or $v=1$ turn out to be

$$\mathbb{L}_2\left\{\frac{1}{(x+y)^{1/2}}{}_1F_1\left[\begin{matrix}\frac{3}{2};\\ 1;\end{matrix}\ -xyx+y\right];s_1,s_2\right\}$$
$$=\frac{\pi^{1/2}}{(s_1s_2)^{1/2}[1+(s_1^{1/2}+s_2^{1/2})^2]^{3/2}},\quad \text{Re}\,[s_1^{1/2}+s_2^{1/2}]>0. \tag{37}$$

$$\mathbb{L}_2\left\{xy(x+y)^{1/2}{}_1F_1\left[\begin{matrix}\frac{1}{2};\\ 2;\end{matrix}\ -xyx+y\right];s_1,s_2\right\}$$
$$=\frac{\pi^{1/2}}{(s_1s_2)^{1/2}[1+(s_1^{1/2}+s_2^{1/2})^2]^{3/2}},\quad \text{Re}\,[s_1^{1/2}+s_2^{1/2}]>0. \tag{38}$$

Example 2.6 If we let $f(x)$ to be ${}_0F_1\left[\begin{matrix}\cdot;\\ 1;\end{matrix}\ x\right]$, $x^{1/4}J_{1/2}(2x^{1/2})$ or $\cos 2x^{1/2}$, then using Theorem 2.2 we derive the following results

(i)

$$\mathbb{L}_n\left\{\frac{1}{p_1(\overline{x^{-1}})p_n(\overline{x^{1/2}})}\left\{v\ {}_1F_1\left[\begin{matrix}\frac{1}{2};\\ 1;\end{matrix}\ 1p_1(\overline{x^{-1}})\right]\right.\right.$$

$$
\begin{aligned}
&\quad - {}_1F_1\left[\begin{matrix}\frac{3}{2};\\ 1;\end{matrix}\ 1p_1(\overline{x^{-1}})\right]\Bigg\};\bar{s}\Bigg\}\\
&= \frac{\pi^{(n-1)/2}}{p_1(\overline{s^{1/2}})p_n(\overline{s^{1/2}})}\left\{(v-1){}_2F_1\left[\begin{matrix}1,\ \frac{1}{2};\\ 1;\end{matrix}\ 1p_1^2(\overline{s^{12}})\right]\right.\\
&\quad \times \frac{1}{p_1(\overline{s^{1/2}})}{}_2F_1\left[\begin{matrix}2,\ \frac{3}{2};\\ 2;\end{matrix}\ 1p_1^2(\overline{s^{1/2}})\right]\Bigg\},
\end{aligned}
\tag{39}
$$

where Re $[p_1(\overline{s^{1/2}})] > 1$ and $n = 2, 3, \ldots, N$.

(ii)

$$
\begin{aligned}
&\mathbb{L}_n\left\{\frac{1}{p_1(\overline{x^{-1}})p_n(\overline{x^{1/2}})}\left\{v\ {}_1F_1\left[\begin{matrix}1;\\ \frac{3}{2};\end{matrix}\ -1p_1(\overline{x^{-1}})\right]-2{}_1F_1\left[\begin{matrix}2;\\ \frac{3}{2};\end{matrix}\ -1p_1(\overline{x^{-1}})\right]\right\};\bar{s}\right\}\\
&= \frac{\pi^{n/2}[v+(v-2)p_1^2(\overline{s^{1/2}})]}{2p_n(\overline{s^{1/2}})[1+p_1^2(\overline{s^{1/2}})]^2}, \quad \text{Re}\,[p_1(\overline{s^{1/2}})] > 0, \quad n = 2, 3, \ldots, N.
\end{aligned}
\tag{40}
$$

(iii)

$$
\begin{aligned}
&\mathbb{L}_n\left\{\frac{1}{p_1^{3/2}(\overline{x^{-1}})p_n(\overline{x^{1/2}})}\left[2+(v-1)p_1(\overline{x^{-1}})\exp\left[-1p_1(\overline{x^{-1}})\right]\right];\bar{s}\right\}\\
&= \frac{\pi^{(n-1)/2}p_1(\overline{s^{1/2}})}{p_n(\overline{s^{1/2}})}\left[\frac{(v+1)+(v-1)p_1^2(\overline{s^{1/2}})}{(1+p_1^2(\overline{s^{1/2}}))^2}\right],\\
&\text{Re}\,[p_1(\overline{s^{1/2}})] > 0, \quad n = 2, 3, \ldots, N.
\end{aligned}
\tag{41}
$$

Example 2.6′ (Two-dimensions)

(i) Upon substituting $n = 2$ and $v = 1$ in Eq. (39) we arrive at a new result as follows

$$
\begin{aligned}
&\mathbb{L}_2\left\{\frac{(xy)}{(x+y)^{3/2}}{}_1F_1\left[\begin{matrix}\frac{5}{2};\\ 2;\end{matrix}\ xyx+y\right];s_1,s_2\right\}\\
&= \frac{\pi^{1/2}}{(s_1s_2)^{1/2}(s_1^{1/2}+s_2^{1/2})^3}{}_2F_1\left[\begin{matrix}2,\ \frac{3}{2};\\ 2;\end{matrix}\ 1(s_1^{12}+s_2^{1/2})^2\right],
\end{aligned}
\tag{42}
$$

where Re $[s_1^{1/2}+s_2^{1/2}] > 1$.

(ii) On substitution of $n = 2$ and $v = 2$ or $v = 0$ in Eq. (40), we obtain the following results, respectively

$$
\mathbb{L}_2\left\{\frac{(xy)^{3/2}}{(x+y)^2}{}_1F_1\left[\begin{matrix}2;\\ \frac{5}{2};\end{matrix}\ -xy\ x+y\right];s_1,s_2\right\}
$$

$$= \frac{3\pi}{4(s_1 s_2)^{1/2}[1 + (s_1^{1/2} + s_2^{1/2})]^2}, \quad \text{Re}\,[s_1^{1/2} + s_2^{1/2}] > 0. \tag{43}$$

$$\mathbb{L}_2 \left\{ \frac{(xy)^{1/2}}{(x+y)} {}_1F_1 \left[\begin{matrix} 2; \\ \frac{3}{2}; \end{matrix} -xyx + y \right]; s_1, s_2 \right\}$$

$$= \frac{\pi(s_1^{1/2} + s_2^{1/2})^2}{2(s_1 s_2)^{1/2}[1 + (s_1^{1/2} + s_2^{1/2})]^2}, \quad \text{Re}[s_1^{1/2} + s_2^{1/2}] > 0. \tag{44}$$

(iii) If we let $n = 2$, $v = 1$ in Eq. (41), we obtain

$$\mathbb{L}_2 \left\{ \frac{xy}{(x+y)^{3/2}} \exp\left(-\frac{xy}{x+y}\right); s_1, s_2 \right\}$$

$$= \frac{\pi^{1/2}(s_1^{1/2} + s_2^{1/2})}{(s_1 s_2)^{1/2}[1 + (s_1^{1/2} + s_2^{1/2})^2]^2}, \quad \text{Re}[s_1^{1/2} + s_2^{1/2}] > 0. \tag{45}$$

Furthermore, if we substitute $v = 3$ and with the help of Eq. (45), we arrive at

$$\mathbb{L}_2 \left\{ \frac{1}{x+y} \exp\left(-\frac{xy}{x+y}\right); s_1, s_2 \right\}$$

$$= \frac{\pi^{1/2}(s_1^{1/2} + s_2^{1/2})}{(s_1 s_2)^{1/2}[1 + (s_1^{1/2} + s_2^{1/2})^2]}, \quad \text{Re}[s_1^{1/2} + s_2^{1/2}] > 0. \tag{46}$$

Both the operational relations (45) and (46) are well-known results.

Similarly, many more double Laplace transforms can be derived by taking different values of v in Eqs. (39)–(41).

Example 2.7 Suppose $f(x) = x^\alpha \exp(-bx)$. Then

$$\phi(s) = \frac{\Gamma(\alpha+1)}{(s+b)^{\alpha+1}}, \quad \text{Re}\,\alpha > -1, \ \ \text{Re}\,s > -\text{Re}\,b,$$

$$\xi(s) = \left(\frac{2}{b}\right)^{\alpha+1} \Gamma(\alpha+1)\Gamma\left(\alpha + \frac{1}{2}\right) \exp\left(\frac{s}{2b}\right) D_{-2\alpha-1}\left[\left(\frac{2s}{b}\right)\right], \quad \text{Re}\,\alpha > -\frac{1}{2}.$$

For $v = 0$

$$\eta(s) = -(\frac{2}{b})^{a+3/2}\Gamma(\alpha+1)\Gamma\left(\alpha + \frac{1}{2}\right)(2\alpha+1) \cdot \frac{\exp(1/(2bs^2))}{s^2} \, D_{-2\alpha-2}\left[\left(\frac{2}{b}\right)^{1/2} \frac{1}{s}\right],$$

where $\text{Re}\,\alpha > -1/2$, $\text{Re}\,s > 0$, and $|\arg 1/b| < \pi$.

Next,

$$F_1(s) = \frac{\Gamma(\alpha + 3/2)}{(s+b)^{\alpha+3/2}}, \quad \text{Re}\,s > -\text{Re}\,b, \ \ \text{Re}\,\alpha > -\frac{3}{2}.$$

Therefore, from Eq. (26) for $v = 0$, we obtain

$$\begin{aligned}
&\mathbb{L}_n\left\{\frac{p_1(\overline{x^{-1}})}{p_n(\overline{x^{1/2}})[b+p_1(\overline{x^{-1}})]^{a+3/2}};\ \bar{s}\right\}\\
&\quad=\pi^{(n-1)/2}\left(\frac{2}{b}\right)^{\alpha+3/2}\frac{\Gamma(\alpha+1)p_1(\overline{s^{1/2}})}{p_n(\overline{s^{1/2}})}\\
&\qquad\times\exp\left[\frac{1}{2b}p_1^2(\overline{s^{1/2}})\right]D_{-2\alpha-2}\left[\left(\frac{2}{b}\right)^{1/2}p_1(\overline{s^{1/2}})\right],
\end{aligned}\tag{47}$$

where $\operatorname{Re}\alpha > -1/2$, $\operatorname{Re}[p_1(\overline{s^{1/2}})] > 0$ and $n = 2, 3, \ldots, N$.

Example 2.7′ If we let $n = 2$ in Eq. (47) we obtain the following new result

$$\begin{aligned}
&\mathbb{L}_2\left\{\frac{(x+y)(xy)^\alpha}{(x+y+bxy)^{\alpha+3/2}};s_1,s_2\right\}\\
&\quad=\frac{(2/b)^{\alpha+3/2}\pi^{1/2}\Gamma(\alpha+1)(s_1^{1/2}+s_2^{1/2})}{(s_1s_2)^{1/2}}\\
&\qquad\times\exp\left[\frac{1}{2b}(s_1^{1/2}+s_2^{1/2})^2\right]D_{-2\alpha-2}\left[\left(\frac{2}{b}\right)^{1/2}(s_1^{1/2}+s_2^{1/2})\right].
\end{aligned}\tag{48}$$

One can develop many other operational relations from given function f in Example 2.7 via using Eq. (26) for different values of v.

3 THE WAVE EQUATIONS

To illustrate the use of some of our results which are derived in Section 2 in *wave mechanics*, we shall consider the one-dimensional wave equation in a normalized form

$$u_{xx} - u_{yy} = f(x, y), \quad 0 < x < \infty,\ \ 0 < y < \infty. \tag{49}$$

Under the initial and boundary conditions (necessary in the first view of the application of the Laplace transformation)

$$\begin{aligned}
u(x,0) &= \alpha(x), \quad u(0,y) = \beta(y)\\
u_y(x,0) &= \theta(x), \quad u_x(0,y) = \delta(y),\\
\alpha(0) &= \beta(0),
\end{aligned}\tag{50}$$

subject to the following condition of compatibility:

$$\int_0^x f(x-\tau,\tau)\,\mathrm{d}\tau + \delta(x) - \theta(x) + \frac{\partial}{\partial x}[\beta(x)-\alpha(x)] = 0. \tag{51}$$

Example 3.1 Determination of a solution $u = u(x, y)$ of Eqs. (49) and (50) for

(a)

$$\begin{aligned} f(x, y) &= 0, \\ \alpha(x) &= x^{1/2}, \quad \beta(y) = y^{1/2}, \\ \theta(x) &= \frac{1}{2}x^{-1/2} \text{ and } \quad \delta(y) = \frac{1}{2}y^{-1/2}. \end{aligned}$$

(b)

$$\begin{aligned} f(x, y) &= y^n - x^n, \\ \alpha(x) &= x^{1/2}, \quad \beta(y) = y^{1/2}, \\ \theta(x) &= \frac{1}{2}x^{-1/2} \text{ and } \quad \delta(y) = \frac{1}{2}y^{-1/2}. \end{aligned}$$

(c)

$$\begin{aligned} f(x, y) &= x^{v-2}y^v - x^v y^{v-2}, \quad \text{Re } v > 1, \\ \alpha(x) &= x^{1/2}, \quad \beta(y) = y^{1/2}, \\ \theta(x) &= \frac{1}{2}x^{-1/2} \text{ and } \quad \delta(y) = \frac{1}{2}y^{-1/2}. \end{aligned}$$

We provide the solution for part (c). By an argument similar to that employed in part (c) the solution for the two other parts are established in brief.

(c) Taking the two-dimensional Laplace transformation from each term of Examples 3.1(a) and 3.2(a) with the aid of Eqs. (6) and (7), yield the following transform equation

$$\begin{aligned} (s_1^2 - s_2^2)U(s_1, s_2) &- \frac{\Gamma(3/2)s_1}{s_2^{3/2}} - \frac{\Gamma(3/2)}{s_2^{1/2}} + \frac{\Gamma(3/2)s_2}{s_1^{3/2}} + \frac{\Gamma(3/2)}{s_1^{1/2}} \\ &= \frac{\Gamma(v-1)\Gamma(v+1)}{s_1^{v-1}s_2^{v+1}} - \frac{\Gamma(v-1)\Gamma(v+1)}{s_1^{v+1}s_2^{v-1}} \end{aligned}$$

or, equivalently

$$\begin{aligned} U(s_1, s_2) = \Gamma\left(\frac{3}{2}\right) &\left[\frac{s_1^{1/2} + s_2^{1/2}}{(s_1 s_2)^{3/2}} - \frac{1}{s_1 s_2 (s_1^{1/2} + s_2^{1/2})}\right] \\ &+ \frac{\Gamma(v-1)\Gamma(v+1)}{s_1^{v+1}s_2^{v+1}}, \quad \text{Re}[s_1^{1/2} + s_2^{1/2}] > 0. \end{aligned} \tag{52}$$

For the inversion of Eqs. (52), we combine Example 2.1′ with Formula 87 in Voelker and Doetsch [21, p. 236]. Finally, we obtain the solution in the form

$$u(x, y) = (x + y)^{1/2} + \frac{1}{v(v-1)}x^v y^v, \quad \text{Re } v > 1. \tag{53}$$

Proceeding in the same way as in the establishment of solution (53), we can then show that the transform equations for parts (a) and (b) are as follows, respectively

$$U(s_1, s_2) = \frac{\Gamma(3/2)(s_1^{1/2} + s_2^{1/2})}{(s_1 s_2)^{3/2}} - \frac{1}{s_1 s_2 (s_1^{1/2} + s_2^{1/2})},$$

$$U(s_1, s_2) = \frac{\Gamma(3/2)(s_1^{1/2} + s_2^{1/2})}{(s_1 s_2)^{3/2}} - \frac{1}{s_1 s_2 (s_1^{1/2} + s_2^{1/2})} + \frac{\sum_{\substack{i,j=1 \\ i<j}}^{n-1} s_i s_j}{(s_1 s_2)^{n+1}(s_1 + s_2)}.$$

Therefore,

$$u(x, y) = (x + y)^{1/2},$$

$$u(x, y) = (x + y)^{1/2} + \begin{cases} \dfrac{xy^{n+1}}{\Gamma(1)\Gamma(n)\,{}_2F_1\left[\begin{matrix} 1, -1; \\ n+2; \end{matrix}\ yx\right]} + \dfrac{x^2 y^n}{\Gamma(2)\Gamma(n-1)\,{}_2F_1\left[\begin{matrix} 1, -2; \\ n+1; \end{matrix}\ yx\right]} \\ \quad + \cdots + \dfrac{x^{n+1} y}{\Gamma(n)\Gamma(1)\,{}_2F_1\left[\begin{matrix} 1, -n; \\ 3; \end{matrix}\ yx\right]} \quad \text{if } y > x \\ \dfrac{x^{n+1} y}{\Gamma(n)\Gamma(1)\,{}_2F_1\left[\begin{matrix} 1, -n; \\ 3; \end{matrix}\ xy\right]} + \dfrac{x^n y^2}{\Gamma(n-1)\Gamma(2)\,{}_2F_1\left[\begin{matrix} 1, 1-n; \\ 4; \end{matrix}\ xy\right]} \\ \quad + \cdots + \dfrac{xy^{n+1}}{\Gamma(n)\Gamma(1)\,{}_2F_1\left[\begin{matrix} 1, -1; \\ n+2; \end{matrix}\ xy\right]} \quad \text{if } y < x, \end{cases}$$

by virtue of Example 2.1 and the following inversion formula given in the unpublished monograph by Dahiya is

$$\mathbb{L}_2^{-1}\{s_2^{k-\mu-1} s_2^{k-v-1}(s_1 + s_2)^{-k}; x, y\} = \begin{cases} \dfrac{x^{\mu-k} y^v}{\Gamma(\mu-k)\Gamma(v+1)\,{}_2F_1\left[\begin{matrix} k, k-\mu; \\ v+1; \end{matrix}\ yx\right]} & \text{if } y > x \quad \mu, v > -1. \\ \dfrac{x^{\mu} y^{v-k}}{\Gamma(\mu+1)\Gamma(v-k)\,{}_2F_1\left[\begin{matrix} k, k-v; \\ \mu+1; \end{matrix}\ xy\right]} & \text{if } y < x. \end{cases}$$

Example 3.2 Determination of a solution $u = u(x, y)$ of Eqs. (49) and (50) for

(a)

$$f(x, y) = \frac{x^{v-1} y^v - x^v y^{v-1}}{(x + y)^{v+1/2}} \text{ and}$$

$$\alpha(x) = 0 = \beta(y),$$

$$\theta(x) = x^{-1/2} \text{ and } \quad \delta(y) = y^{-1/2}.$$

(b)

$$f(x, y) = \frac{x^{v-1}y^v - x^v y^{v-1}}{(x+y)^{v+(3/2)}} \text{ and}$$
$$\alpha(x) = 0 = \beta(y),$$
$$\theta(x) = x^{-(1/2)} \text{ and } \quad \delta(y) = y^{-1/2}.$$

We only present the solution of part (b) completely, the solution of part (a) is similar to that of part (b), hence the details of part (a) have been omitted.

(b) Applying the double Laplace transform to the terms of Examples 3.1(b) and 3.2(b) with the aid of Eqs. (6) and (7), Formula 47 in Voelker and Doetsch [21, p. 159] and Formula 181 in Brychkov *et al*. [1, p. 300], finally we arrive at

$$(s_1 - s_2)U(s_1, s_2) - \frac{\pi^{1/2}}{s_2^{1/2}} + \frac{\pi^{1/2}}{s_1^{1/2}}$$
$$= \frac{\pi\Gamma(2v+1)}{2^{2v}\Gamma(v+3/2)}\left\{\int_{s_1}^{\infty} \frac{\mathrm{d}\lambda}{(\lambda^{1/2} + s_2^{1/2})^{2v+1}} - \int_{s_2}^{\infty} \frac{\mathrm{d}\lambda}{(s_1^{1/2} + \lambda^{1/2})^{2v+1}}\right\}.$$

Evaluating the integrals and simplifying, we get

$$U(s_1, s_2) = \frac{\pi^{1/2}}{(s_1 s_2)^{1/2}(s_1^{1/2} + s_2^{1/2})(s_1 + s_2)} + \frac{\pi\Gamma(2v+1)}{v2^{2v}\Gamma(v+3/2)} \cdot \frac{s_1^{1/2} - s_2^{1/2}}{(s_1^{1/2} + s_2^{1/2})(s_1 + s_2)}, \tag{54}$$

where $\mathrm{Re}[s_1^{1/2} + s_2^{1/2}] > 0$.

The inversion of Eq. (54) will be accomplished from Formula 42 in Voelker and Doetsch [21, p. 186], Example 2.1 and Formula 181 in Brychkov *et al*. [1]

$$u(x, y) = \frac{1}{v}\begin{cases} \int_0^x \dfrac{v(x+y-2\xi)^{v+1} + [(x-\xi)(y-\xi)]^v}{(x+y-2\xi)^{v+(3/2)}}\,\mathrm{d}\xi & \text{if } y > x \\ \int_0^y \dfrac{v(x+y-2\xi)^{v+1} + [(x-\xi)(y-\xi)]^v}{(x+y-2\xi)^{v+(3/2)}}\,\mathrm{d}\xi & \text{if } y < x. \end{cases} \tag{55}$$

Substituting $x + y - 2\xi = t$ in Eq. (5), we obtain

$$u(x, y) = \begin{cases} (x+y)^{1/2} - (y-x)^{1/2} \\ \quad - \dfrac{1}{v2^{2v+1}} \int_{x+y}^{y-x} [t^2 - (x-y)^2]^v t^{-v-3/2}\,\mathrm{d}t & \text{if } y > x, \\ (x+y)^{1/2} - (x-y)^{1/2} \\ \quad - \dfrac{1}{v2^{2v+1}} \int_{x+y}^{x-y} [t^2 - (x-y)^2]^v t^{-v-3/2}\,\mathrm{d}t & \text{if } y < x, \quad \mathrm{Re}\, v > 0. \end{cases} \tag{56}$$

(a) Similarly, the transform equation for part (a) is as follows:

$$U(s_1, s_2) = \frac{\pi^{1/2}}{(s_1 s_2)^{1/2}(s_1^{1/2} + s_2^{1/2})(s_1 + s_2)}$$
$$+ \frac{\pi^{1/2}\Gamma(v+1)}{v(s_1 s_2)^{1/2}(s_1^{1/2} + s_2^{1/2})(s_1 + s_2)}, \quad \mathrm{Re}\,[s_1^{1/2} + s_2^{1/2}] > 0.$$

The inversion of $u(s_1, s_2)$ will be accomplished from Formula 42 in Voelker and Doetsch [21, p. 186] and Relation (2.12′) in Example 2.1′

$$u(x, y) = \frac{1}{v}\begin{cases} \int_0^x \dfrac{v(x+y)-2\xi)^v + [(x-\xi)(y-\xi)]^v}{(x+y-2\xi)^{v+(1/2)}}\,\mathrm{d}\xi & \text{if } y > x \\ \int_0^y \dfrac{v(x+y-2\xi)^v + [(x-\xi)(y-\xi)]^v}{(x+y-2\xi)^{v+(1/2)}}\,\mathrm{d}\xi & \text{if } y < x. \end{cases}$$

Remark 3.1 One may easily check that the condition of compatibility (51) holds true for all IBVPs given in Example 3.1.

Remark 3.2 If we let $v = 1$, in part (a) and part (b) of Example 3.2, we obtain the following solutions, respectively

$$u(x, y) = \begin{cases} (x+y)^{1/2} - (y-x)^{1/2} - \dfrac{1}{12}[(y-x)^{3/2} - (x+y)^{3/2}] \\ \qquad -\dfrac{1}{4}(x-y)^2[(y-x)^{-1/2} - (x+y)^{-1/2}] & \text{if } y > x, \\ (x+y)^{1/2} - (x-y)^{1/2} - \dfrac{1}{12}[(x-y)^{3/2} - (x+y)^{3/2}] \\ \qquad -\dfrac{1}{4}(x-y)^2[(x-y)^{-1/2} - (x+y)^{-1/2}] & \text{if } y < x. \end{cases}$$

$$u(x, y) = \begin{cases} \dfrac{5}{4}[(x+y)^{1/2} - (y-x)^{1/2}] \\ \qquad -\dfrac{1}{12}(x-y)^2[(y-x)^{-3/2} - (x+y)^{-3/2}] & \text{if } y > x, \\ \dfrac{5}{4}[(x+y)^{1/2} - (x-y)^{1/2}] \\ \qquad -\dfrac{1}{12}(x-y)^2[(x-y)^{-3/2} - (x+y)^{-3/2}] & \text{if } y < x. \end{cases}$$

Example 3.3 Determination of a solution $u = u(x, y)$ of Eqs. (49) and (50) for

$$f(x, y) = \frac{(xy)(x-y)}{(x+y)^{3/2}} \exp\left(-\frac{xy}{x+y}\right) \text{ and}$$
$$\alpha(x) = \exp(-x), \quad \beta(y) = \exp(-y)$$
$$\theta(x) = 0 = \delta(y).$$

We begin to solve the IBVP by taking the two-dimensional Laplace transformation of each term of Eqs. (49) and (50), with the help of Eqs. (6) and (7), and also using the Relation (45′) in Example 2.6′ and with the aid of Formula 47 in Voelker and Doetsch [21, p. 159], we arrive at

$$(s_1 - s_2)U(s_1, s_2) = \frac{(s_1^2 - s_2^2) + (s_1 - s_2)}{(s_1+1)(s_2+1)} + \frac{\pi(s_1^{1/2} - s_2^{1/2})}{(s_1 s_2)^{1/2}} \int_{s_1^{1/2}+s_2^{1/2}}^{\infty} \frac{t}{(1+t^2)^2}\,\mathrm{d}t$$

or, equivalently

$$U(s_1, s_2) = \frac{1}{(s_1+1)(s_2+1)} + \frac{1}{(s_1+1)(s_2+1)(s_1+s_2)} + \frac{\pi}{2(s_1 s_2)^{1/2}(s_1^{1/2} + s_2^{1/2})[1 + (s_1^{1/2} + s_2^{1/2})](s_1+s_2)}.$$

Making use of Relations (1.2) in Ditkin and Prudnikov [10, p. 100] together with Relation (201) in Brychkov *et al.* [1, p. 303] and finally with the aid of Formula 42 in Voelker and Doetsch [21; p. 186], we obtain the following solution

$$u(x, y) = \exp[-(x - y)] + \begin{cases} \int_0^x \exp[-(x + y - 2\xi)]\, d\xi \\ \qquad + \int_0^x \dfrac{1}{(x + y - 2\xi)^{1/2}} \gamma\left(1, \dfrac{(x - \xi)(y - \xi)}{(x + y - 2\xi)}\right) d\xi & \text{if } y > x \\ \int_0^y \exp[-(x + y - 2\xi)]\, d\xi \\ \qquad + \int_0^y \dfrac{1}{(x + y - 2\xi)^{1/2}} \gamma\left(1, \dfrac{(x - \xi)(y - \xi)}{(x + y - 2\xi)}\right) d\xi, & \text{if } y < x. \end{cases}$$

Hence,

$$u(x, y) = \frac{1}{2} \begin{cases} \exp[-(y - x)] + [\exp -(x + y)] \\ \qquad - \int_{(x+y)}^{(y-x)} t^{-1/2} \gamma\left(1, \dfrac{t^2 - (x - y)^2}{4t}\right) dt & \text{if } y > x \\ \exp[-(x - y)] + \exp[-(x + y)] \\ \qquad - \int_{(x+y)}^{(x-y)} t^{-1/2} \gamma\left(1, \dfrac{t^2 - (x - y)^2}{4t}\right) dt & \text{if } y < x. \end{cases}$$

Remark 3.3 It is easy to check that the condition of compatibility (51) holds true for the IBVPs in Example 3.3.

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