Indag. Mathem., N.S., 17 (2), 297–309 June 19, 2006

A product of varieties in relation to Baer invariants [☆]

by Mohsen Parvizi [a] and Saeed Kayvanfar [b,1]

[a] *Department of Mathematics, Ferdowsi University of Mashhad, P.O.Box 1159-91775, Mashhad, Iran*
[b] *Institute for Studies in Theoretical Physics and Mathematics, P.O.Box 5746-19395, Tehran, Iran*

Communicated by Prof. T.A. Springer at the meeting of September 26, 2005

ABSTRACT

The concept of the Baer invariant is useful in classifying groups into isologism classes. In this paper two sequences of varieties $\{^r\mathcal{V}_n\}_{n\in N_0}$ and $\{^\ell\mathcal{V}_n\}_{n\in N_0}$, are considered from a given variety $\mathcal{V}$. The structure of Baer invariants of some groups with respect to these varieties, are determined for some specific $\mathcal{V}$.

1. INTRODUCTION

In 1945 R. Baer [1], generalized the well known notion of Schur multiplier to the more general notion of Baer invariant, it gave rise to a vast amount of research in group theory. In particular lots of facts and claims about the Schur multiplier can be considered in connection to different varieties of groups.

One of the most interesting problems in this area is to compute the Baer invariants of groups with respect to different varieties of groups, for example see [5,11,15]. Although determining the explicit formula for the Baer invariants is so valuable, it is relatively hard, special cases excepted. Therefore it is clearly preferable to compute the Baer invariant via structural properties.

In this article, we intend to reach a similar goal, namely computing the Baer invariants of some groups. To do this we use a product of varieties. Using this

MSC: 20E10

Key words and phrases: Variety, Product of varieties, Baer invariant

☆ This research was in part supported by a grant from IPM (No. 82200028).
E-mails: parvizi@math.um.ac.ir (M. Parvizi), skayvanf@math.um.ac.ir (S. Kayvanfar).

1 Corresponding author.

product, from a given variety, we construct two sequences of varieties. It should be recalled that Hekster [6] had introduced the notion of sequences, but he did used them in a different manner. More precisely, Hekster applied these sequences from point of view of isologism theory, for details see [6, Section 6]. Further information about the mentioned product may be found in [9, Chapter 1].

In this paper we concentrate on Baer invariants of some groups with respect to the terms of these sequences. To avoid unwanted complications, we impose some conditions on groups, where wanted.

2. NOTATIONS AND PRELIMINARIES

Let F_∞ be the free group on the set $X = \{x_1, x_2, \ldots\}$ and $V \subseteq F_\infty$ be a set of words. We assume that the reader is familiar with the notion of the *variety of groups* defined by a set of words, verbal and marginal subgroups. Throughout this article we assume that $\mathcal{V}$ is the variety of groups defined by a set of words V and $V(G)$ and that $V^*(G)$ is the *verbal* and *marginal subgroup* of a group G with respect to $\mathcal{V}$, respectively. For a group G with a normal subgroup N, $[NV^*G]$ is defined to be the subgroup generated by the set:

$$\big\{v(g_1, g_2, \ldots, g_i n, \ldots, g_r)\big(v(g_1, g_2, \ldots, g_r)\big)^{-1} \mid 1 \leqslant i \leqslant r,\ v \in V,\\ g_i \in G,\ n \in N\big\}.$$

One may easily show that $[NV^*G]$ is the least normal subgroup T, say, of G contained in N, satisfying $N/T \subseteq V^*(G/T)$.

Now let G be any group with a free presentation

$$E : 1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1,$$

then the (two) Baer invariants of G with respect to the variety $\mathcal{V}$ are defined as follows:

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]}, \qquad \mathcal{V}P(G) = \frac{V(F)}{[RV^*F]}.$$

It is a known fact that these are invariants of G, that is independent of the choice of free presentation for G, see [9, Lemma 1.8], and that $\mathcal{V}M(G)$ is always abelian.

The following proposition contains some classic facts about varieties. We mention it here, as it is of use later.

Proposition 2.1. *Let G be any group with a normal subgroup N and let $\mathcal{V}$ be any variety. Then*

(i) $V(G/N) = V(G)N/N$, $V^*(G/N) \supseteq V^*(G)N/N$;
(ii) $V(N) \subseteq [NV^*G] \subseteq N \cap V(G)$, *in particular* $V(G) = [GV^*G]$;
(iii) *If* $N \cap V(G) = 1$ *then* $N \subseteq V^*(G)$ *and* $V^*(G/N) = V^*(G)N/N$.

Proof. See [6, Proposition 2.3]. □

3. PRODUCT VARIETIES

In this section we introduce the mentioned product of varieties and state some lemmas to be used in the next sections.

Definition 3.1. Let $\mathcal{V}$ and $\mathcal{U}$ be varieties of groups defined by the set of words V and U, respectively. The product $\mathcal{U} * \mathcal{V}$ is the variety of all groups G, such that $U(G) \subseteq V^*(G)$.

See also Hekster's paper [6, pp. 28 (bottom) and 29 (top)] and [9, p. 104].

The following proposition gives the verbal subgroup of this product variety.

Proposition 3.2. *Using the above notations and definitions, the verbal subgroup of a group G with respect to variety $\mathcal{U} * \mathcal{V}$ is $[U(G)V^*G]$.*

Proof. See [9, Proposition 1.5]. □

The next proposition gives a set of defining words for this product variety. It can be used in computations with the words and is used to obtain Example 4.6.

Proposition 3.3. *Using the above notations and definitions, assume $u(x_1, \dots, x_r)$ and $v(x_1, \dots, x_s)$ are words, and for each i $(1 \leqslant i \leqslant s)$*

$$v_{(i)}u = v\big(x_1, \dots, x_{i-1}, x_i u(x_{s+1}, \dots, x_{s+r}), x_{i+1}, \dots, x_s\big)\big(v(x_1, \dots, x_s)\big)^{-1} \quad \textit{and}$$
$$v^{(i)}u = v\big(x_1, \dots, x_{i-1}, u(x_{s+1}, \dots, x_{s+r}), x_{i+1}, \dots, x_s\big) \times \big(v(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_s)\big)^{-1}.$$

Then $\mathcal{U} * \mathcal{V}$ is defined by each of the following sets:

$$W = \big\{v_{(i)}u \mid v(x_1, \dots, x_s) \in V,\ u \in U,\ 1 \leqslant i \leqslant s < \infty\big\};$$
$$W' = \big\{v^{(i)}u \mid v(x_1, \dots, x_s) \in V,\ u \in U,\ 1 \leqslant i \leqslant s < \infty\big\}.$$

Proof. See [9, Proposition 1.5]. □

This product $*$ featuring in $\mathcal{U} * \mathcal{V}$ is not commutative and it is not associative as well (see [9, p. 106, Example 2] and [6, Example 3.7]). Non-commutativity of this product makes it somewhat difficult to work with. On the other hand, the interesting fact about this product is its non-associativity and this is to some extent a lucky occurrence, it presents us to two kinds of powers for a variety, to be called the left and right sequences.

Some relations between the verbal and marginal subgroups of this product variety as stated in the following lemma may be found in [6, Theorem 3.2 and Lemma 3.3].

Lemma 3.4. *Let $\mathcal{U}$ and $\mathcal{V}$ be two varieties, $\mathcal{W} = \mathcal{U} * \mathcal{V}$ and let G be any group with a normal subgroup N. Then*:

(i) $V^*(G) \subseteq W^*(G)$;
(ii) $W^*(G)/V^*(G) \subseteq U^*(G/V^*(G)) \subseteq W^*(G/V^*(G))$;
(iii) $[[NU^*G]V^*G] \subseteq [NW^*G]$.

Some interesting inclusion relations exist between the multiple product varieties as follows:

Lemma 3.5. *Let $\mathcal{U}$, $\mathcal{U}_1$, $\mathcal{V}$, $\mathcal{V}_1$ and $\mathcal{W}$ be varieties for which $\mathcal{U} \subseteq \mathcal{U}_1$ and $\mathcal{V} \subseteq \mathcal{V}_1$. Then*:

(i) $\mathcal{U} * (\mathcal{V} * \mathcal{W}) \subseteq (\mathcal{U} * \mathcal{V}) * \mathcal{W}$;
(ii) $\mathcal{U} * \mathcal{V} \subseteq \mathcal{U}_1 * \mathcal{V}_1$.

Proof. See [6, Propositions 3.5 and 3.6]. □

4. SEQUENCES OF VARIETIES

In the previous section we recalled and presented some information about the product variety and now we are ready to introduce the sequences of varieties which is the main subject to be discussed in the rest of this article.

Given a variety $\mathcal{V}$, two sequences ${}^{\ell}\mathcal{V}_n$ and ${}^{r}\mathcal{V}_n$ may be defined as follows:

(1) ${}^{\ell}\mathcal{V}_0 = \mathcal{T}$ and ${}^{\ell}\mathcal{V}_{n+1} = \mathcal{V} * {}^{\ell}\mathcal{V}_n$,
(2) ${}^{r}\mathcal{V}_0 = \mathcal{T}$ and ${}^{r}\mathcal{V}_{n+1} = {}^{r}\mathcal{V}_n * \mathcal{V}$

in which $\mathcal{T}$ is the variety of trivial groups. Let us call $\{{}^{r}\mathcal{V}_n\}_{n\in N_0}$ and $\{{}^{\ell}\mathcal{V}_n\}_{n\in N_0}$ the *right* and the *left sequence*, respectively.

The introduction of these sequences causes arising several questions and problems about them. Some of them (such as relations between these sequences and their verbal and marginal subgroup) are easily established from the facts stated earlier about the product, but the others (like Baer invariants and some further ones) are not so routine. The main purpose of this work concerned with some works on Bear invariants.

As one can see in [6, Lemma 6.11], it holds that ${}^{\ell}\mathcal{V}_n \neq {}^{r}\mathcal{V}_n$ in general. So it is preferable to state lemmas and theorems for each sequence, separately.

Some useful properties of the left sequence are listed in the following.

Lemma 4.1. *Let $\mathcal{V}$ be any variety and let G be any group with a normal subgroup N. Then*:

(i) ${}^{\ell}V_{n+1}(G) = [V(G){}^{\ell}V_n^*G]$;
(ii) ${}^{\ell}V_{n+1}^*(G)/{}^{\ell}V_n^*(G) \subseteq V^*(G/{}^{\ell}V_n^*(G))$;
(iii) ${}^{\ell}V_n(G)/{}^{\ell}V_{n+1}(G) \subseteq V^*(G/{}^{\ell}V_{n+1}(G))$;
(iv) $[[NV^*G]{}^{\ell}V_n^*G] \subseteq [N{}^{\ell}V_{n+1}^*G]$.

Proof. (i) Can be easily obtained by considering the definition of ${}^{\ell}\mathcal{V}_n$ and Proposition 3.2(ii).

For (ii) and (iii) use the definition of ${}^{\ell}\mathcal{V}_n$.

For (iv) see (iii) of Lemma 3.4. □

Parts (ii) and (iii) of the above lemma allow us to spot two sequences of verbal and marginal subgroups of an arbitrary group G, as follows:

$$V(G) = {}^{\ell}V_1(G) \supseteq {}^{\ell}V_2(G) \supseteq \cdots \supseteq {}^{\ell}V_n(G) \supseteq {}^{\ell}V_{n+1}(G) \supseteq \cdots,$$
$$V^*(G) = {}^{\ell}V_1^*(G) \subseteq {}^{\ell}V_2^*(G) \subseteq \cdots \subseteq {}^{\ell}V_n^*(G) \subseteq {}^{\ell}V_{n+1}^*(G) \subseteq \cdots.$$

The following lemma, which is stated for the right sequence, shows the influence of non-associativity of the product. Although it resembles Lemma 4.1, there are some basic differences to absence between the Lemmas 4.1 and 4.2.

Lemma 4.2. *Let $\mathcal{V}$ be any variety and let G be any group with a normal subgroup N. Then*:

(i) ${}^{r}V_{n+1}(G) = [{}^{r}V_n(G)V^*G]$;
(ii) ${}^{r}V_{n+1}^*(G)/{}^{r}V^*(G) \subseteq {}^{r}V_n^*(G/{}^{r}V^*(G))$;
(iii) ${}^{r}V_n(G)/{}^{r}V_{n+1}(G) \subseteq V^*(G/{}^{r}V_{n+1}(G))$;
(iv) ${}^{r}V_n^*(G/V^*(G)) \subseteq {}^{r}V_{n+1}^*(G/V^*(G))$;
(v) $[[N\,{}^{r}V_n^*G]V^*G] \subseteq [N\,{}^{r}V_{n+1}^*G]$.

Proof. It can be done similar to Lemma 4.1. □

The above lemma provides a sequence of verbal subgroups for the right sequence as we had it for the left one:

$$V(G) = {}^{r}V_1(G) \supseteq {}^{r}V_2(G) \supseteq \cdots \supseteq {}^{r}V_n(G) \supseteq {}^{r}V_{n+1}(G) \supseteq \cdots.$$

It is of interest to see some examples of these sequences in order to show applications of the sequences. But before this, showing some relations between the terms of the sequences may be helpful.

Lemma 4.3. *For any variety $\mathcal{V}$ we have ${}^{\ell}\mathcal{V}_n \subseteq {}^{r}\mathcal{V}_n$, for all $n \in \mathbf{N}_0$.*

Proof. For each $n \geqslant 0$ we have

$$\mathcal{V} * {}^{r}\mathcal{V}_{n+1} = \mathcal{V} * \left({}^{r}\mathcal{V}_n * \mathcal{V}\right) \subseteq \left(\mathcal{V} * {}^{r}\mathcal{V}_n\right) * \mathcal{V} \subseteq {}^{r}\mathcal{V}_{n+1} * \mathcal{V} = {}^{r}\mathcal{V}_{n+2},$$

which implies $\mathcal{V} * {}^{r}\mathcal{V}_n \subseteq {}^{r}\mathcal{V}_{n+1}$ ($n \geqslant 0$). Therefore induction on n provides

$${}^{\ell}\mathcal{V}_{n+1} = \mathcal{V} * {}^{\ell}\mathcal{V}_n \subseteq \mathcal{V} * {}^{r}\mathcal{V}_n \subseteq {}^{r}\mathcal{V}_{n+1}. \qquad □$$

An immediate consequence of this lemma is the following corollary.

Corollary 4.4. *Let G be any group and let $\mathcal{V}$ be any variety. Then, for each $n \geqslant 0$ we have ${}^{r}V_n(G) \leqslant {}^{\ell}V_n(G)$.*

Of course occurrences under which equality holds may be of interest. One of these is to be found in the following lemma which is not straightforward enough to work with, in some sense.

Lemma 4.5. *Let $\mathcal{V}$ be any variety. Then, ${}^{\ell}\mathcal{V}_{n+1} = {}^{\ell}\mathcal{V}_n * \mathcal{V}$, for all $n \geqslant 0$, if and only if ${}^{\ell}\mathcal{V}_{m+n} = {}^{\ell}\mathcal{V}_m * {}^{\ell}\mathcal{V}_n$, for all $m, n \geqslant 0$.*

To illustrate applications of these sequences, we state some examples in the following.

Example 4.6. (1) If $\mathcal{A}$ is the variety of abelian groups, then the left and right sequences coincide. In fact in this case ${}^{\ell}\mathcal{A}_n = {}^{r}\mathcal{A}_n = \mathcal{N}_n$, the variety of nilpotent groups of class at most n.

Many papers are devoted to the variety of nilpotent groups, so according to the above example, each conclusion in this work can be regarded as a generalization of work on nilpotent varieties.

Another usage is generalizing some theorems stated for $\mathcal{N}_c$, in two directions, for the left and right sequence separately.

(2) As a slight generalization, starting with $\mathcal{N}_c$ one may easily show that the left and right sequence are the same and the nth term is $\mathcal{N}_{cn}$, so considering the previous example, it provides no more information.

(3) Suppose $\mathcal{S}_\ell$ is the variety of solvable groups of length at most ℓ. Then an straightforward computation shows that for any group G,

$$
{}^{r}(S_\ell)_n(G) =
$$
$$
\big[G^{(n)}, \overbrace{\underbrace{G, G', G'', \ldots, G^{(n-1)}}, \underbrace{G, G', G'', \ldots, G^{(n-1)}}, \ldots, \underbrace{G, G', G'', \ldots, G^{(n-1)}}}^{(n-1)\text{-}times}\big];
$$

here $G^{(i)}$ denotes the ith term of the derived series of G.

(4) Assume that $\mathcal{A}_m$ is the variety of abelian groups of exponent m. It is known that the defining set of words for this variety is $\{[x_1, x_2], x_3^m\}$. By induction we see that the defining set of laws for the nth term of the left sequences is

$$
\big\{x_1^{1^{m^n}}, \big[x_1^2, x_2^2\big]^{m^{n-1}}, \big[x_1^3, x_2^3, x_3^3\big]^{m^{n-2}}, \ldots, \big[x_1^n, x_2^n, \ldots, x_n^n\big]^m, \\ \big[x_1^{n+1}, x_2^{n+1}, \ldots, x_{n+1}^{n+1}\big]\big\}.
$$

It is of interest to know which outer commutator varieties can be the terms of the left and right sequences.

(5) Let $\mathcal{N}_{1,2}$ be the polynilpotent variety of class row $(1, 2)$, that is the variety defined by the set $\{[[x_1, x_2], [x_3, x_4], [x_5, x_6]]\}$. Then for any group G we have $(N_{1,2})_2(G) = [G', G', G', G, G', G']$.

(6) Let $\mathcal{N}_{2,1}$ be the polynilpotent variety of class row (2,1), that is the variety defined by the set $\{[[x_1,x_2,x_3],[x_4,x_5,x_6]]\}$. Then for any group G we have $(N_{2,1})_2(G)=[\gamma_3(G),\gamma_3(G),G,G,\gamma_3(G)]$.

(7) Let $\mathcal{V}$ be the variety defined by the set of words $\{[[x_1,x_2,x_3],[x_4,x_5]]\}$. Then for any group G we have

$$V_2(G)=\big[\gamma_3(G),\gamma_2(G),G,G,\gamma_2(G)\big]\big[\gamma_3(G),\gamma_2(G),G,\big[G,\gamma_2(G)\big]\big].$$

5. APPLICATION IN BAER INVARIANTS

This section contains computations of some Baer invariants using the sequences of varieties as an instrument. We restrict our attention to $\mathcal{V}$-perfect groups. A group G is called $\mathcal{V}$-perfect if $V(G)=G$. One may find some information on perfect groups in [7], and on $\mathcal{V}$-perfect groups, in [8,12]. $\mathcal{V}$-perfect groups are so well-behaved in computing Baer invariants. The following theorem of Ellis [3, Proposition 2.11] justifies it.

Theorem 5.1. *Let G be any perfect group. Then there are two isomorphisms as follows*

$$\mathcal{N}_c P(G)\cong P(G),\qquad \mathcal{N}_c M(G)\cong M(G);$$

in which $M(G)$ and $P(G)$ are the introduced Baer invariants of G with respect to the variety of abelian groups.

Now since the variety of nilpotent groups of class at most c, $\mathcal{N}_c$, is both the cth term of the left and right sequences of $\mathcal{A}$, so the above theorem can be generalized (Theorem 5.2). On the other hand, bearing in mind Example 4.6, one sees that computing Baer invariants with respect to some outer commutator varieties are very complicated, and so, obtaining a method to compute these Baer invariants, is valuable. The following theorem provides the above purpose.

Theorem 5.2. *Let $\mathcal{V}$ be any variety with the property $Z(H)\subseteq V^*(H)$ for any $\mathcal{V}$-perfect group H. Then the following isomorphisms hold for any $\mathcal{V}$-perfect group G;*

$${}^{\ell}\mathcal{V}_n M(G)\cong \mathcal{V}M(G)\cong {}^{r}\mathcal{V}_n M(G),$$

$${}^{\ell}\mathcal{V}_n P(G)\cong \mathcal{V}P(G)\cong {}^{r}\mathcal{V}_n P(G)\quad \forall n\in\mathbf{N}.$$

In order to prove Theorem 5.2, we need some notations from [8] and the following easily deduced fact.

Lemma 5.3. *Let $\mathcal{V}$ be any variety and let G be any group. Then the following are equivalent*:

(i) *G is ${}^{r}\mathcal{V}_i$-perfect (${}^{\ell}\mathcal{V}_i$-perfect) for some i;*
(ii) *G is ${}^{r}\mathcal{V}_i$-perfect (${}^{\ell}\mathcal{V}_i$-perfect) for each i.*

The following theorems and lemmas are all taken from [8] and play an important role in the proof of Theorem 5.2.

Theorem 5.4. *Let $\mathcal{V}$ be any variety and let G be any group. If G is $\mathcal{V}$-perfect then so is $\mathcal{V}P(G)$.*

The second author in his paper [8], using a universal property of some extensions, showed the uniqueness of a kind of exact sequence which he called $\mathcal{V}$-perfect-marginal extension. The following is a brief description of what he stated.

Definition 5.5. Let $\mathcal{V}$ be any variety defined by a set of laws V and let G be any group. An exact sequence $E: 1 \to A \to H \to G \to 1$ is called a $\mathcal{V}$-marginal extension by G if $A \subseteq V^*(H)$. It is said that a $\mathcal{V}$-marginal extension E covers (uniquely covers) another $\mathcal{V}$-marginal extension $E_1: 1 \to A_1 \to H_1 \to G \to 1$, if there exists a homomorphism (resp. a unique homomorphism) $\theta: H \to H_1$ such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & A_1 & \longrightarrow & H_1 & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

commutes.

Definition 5.6. An extension $E': 1 \to A \to B \to G \to 1$ by a group G is called $\mathcal{V}$-perfect if B is $\mathcal{V}$-perfect. If in addition E' is a $\mathcal{V}$-marginal extension, then it is said to be a $\mathcal{V}$-perfect-marginal extension. A $\mathcal{V}$-marginal extension by a group G is called universal if it uniquely covers any $\mathcal{V}$-perfect-marginal extension by the group G.

Remark 5.7. Let $\mathcal{V}$ be any variety and let G be any $\mathcal{V}$-perfect group. Then the sequence $1 \to \mathcal{V}M(G) \to \mathcal{V}P(G) \to G \to 1$ is both $\mathcal{V}$-marginal and $\mathcal{A}$-marginal extension.

The following theorem is essential for the proof of Theorem 5.2.

Theorem 5.8. *Let G be any $\mathcal{V}$-perfect group and let $1 \to A \to H \to G \to 1$ be a universal $\mathcal{V}$-perfect-marginal extension by G. Then $A \cong \mathcal{V}M(G)$ and $H \cong \mathcal{V}P(G)$.*

Proof. See [8, Corollary 3.11]. $\square$

Proof of Theorem 5.2. We state the proof only for the right sequence. The proof of the left one being similar. It suffices to show that for an arbitrary natural number n, the exact sequence

$$E : 1 \longrightarrow {}^{r}\mathcal{V}_n M(G) \longrightarrow {}^{r}\mathcal{V}_n P(G) \longrightarrow G \longrightarrow 1$$

is a universal $\mathcal{V}$-perfect-marginal extension by G. Hence the result follows by Theorem 5.8. As before mentioned the above sequence is central so ${}^{r}\mathcal{V}_n M(G) \subseteq Z({}^{r}\mathcal{V}_n P(G))$. But by the assumption we have $Z({}^{r}\mathcal{V}_n P(G)) \subseteq V^*({}^{r}\mathcal{V}_n P(G))$ and so E is a $\mathcal{V}$-perfect-marginal extension. On the other hand, each $\mathcal{V}$-perfect-marginal extension is a ${}^{r}\mathcal{V}_n$-perfect-marginal extension too. So it is uniquely covered by E and the result holds. □

The next theorem contains some conditions under which $\mathcal{V}M(G)$ is trivial. To express the importance of triviality the Baer invariants, it is necessary to state the notion of $\mathcal{V}$-covering groups.

Definition 5.9. Let G be a group, and let E be the following exact sequence

$$E : 1 \longrightarrow A \longrightarrow G^* \longrightarrow G \longrightarrow 1,$$

in which $A \subseteq V^*(G^*) \cap V(G^*)$ and $A \cong \mathcal{V}M(G)$. Then G^* is said to be a $\mathcal{V}$-covering group of G.

In general, the problem of existence and uniqueness of $\mathcal{V}$-covering group is unsolved. Computing covering groups is also interesting but certainly not straightforward in many cases (for more information see [10]).

In the special case $\mathcal{V}M(G) = 1$, the mentioned problem is completely solved.

The following result is worthy enough to be considered.

Theorem 5.10. *Let G be any group with G^{ab} free. If $M(G) = 0$ then so are $\mathcal{N}_c M(G)$ for all $c \geqslant 1$.*

Proof. See [3, Theorem 3.2]. □

Considering $\mathcal{N}_c$ as the right sequence of $\mathcal{A}$, we can state the following generalization of Theorem 5.10.

Theorem 5.11. *Let G be an arbitrary group for which the group $G/V(G)$ is a free $\mathcal{V}$-group and $\mathcal{V}M(G) = 0$. Then for all $n \geqslant 1$ we have ${}^{r}\mathcal{V}_n M(G) = 0$.*

Before preparing to prove the theorem, it is necessary to show that the condition of being V-free for $G/V(G)$ is essential which the following remark shows.

Remark 5.12. It is a well-known fact that $M(S_3) = 1$ [7, Corollary 2.1.12], and the second author in his joint paper [14, Theorem 5.6], showed that $\mathcal{N}_2 M(S_3) \neq 1$.

To prove the latest theorem we need some further information which are listed below.

It was Fröhlich [4] who proved that for any variety $\mathcal{V}$ and any group G with a normal subgroup N, there exists an exact sequence as follows:

$$\mathcal{V}M(G) \longrightarrow \mathcal{V}M(G/N) \longrightarrow N/[NV^*G] \longrightarrow G/V(G) \longrightarrow G/NV(G) \longrightarrow 1.$$

The next lemma is a key to prove Theorem 5.10.

Lemma 5.13. *Let $f: A \to B$ be a group homomorphism such that induces an isomorphism $A/V(A) \cong B/V(B)$ and an epimorphism $\mathcal{V}M(A) \to \mathcal{V}M(B)$. Then f induces isomorphisms ${}^rV_n(A)/{}^rV_{n+1}(A) \cong {}^rV_n(B)/{}^rV_{n+1}(B)$ and $A/{}^rV_n(A) \cong B/{}^rV_n(B)$ for each $n \geqslant 1$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccccc}
\mathcal{V}M(A) & \longrightarrow & \mathcal{V}M\left(\frac{A}{{}^rV_n(A)}\right) & \longrightarrow & \frac{{}^rV_n(A)}{{}^rV_{n+1}(A)} & \longrightarrow & \frac{A}{V(A)} & \longrightarrow & \frac{A}{{}^rV_n(A)V(A)} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{V}M(B) & \longrightarrow & \mathcal{V}M\left(\frac{B}{{}^rV_n(B)}\right) & \longrightarrow & \frac{{}^rV_n(B)}{{}^rV_{n+1}(B)} & \longrightarrow & \frac{B}{V(B)} & \longrightarrow & \frac{B}{{}^rV_n(B)V(B)} & \longrightarrow & 1
\end{array}$$

By the assumption and induction hypothesis the first vertical map is an epimorphism and the second, forth and fifth ones are isomorphisms, so the middle one is an isomorphism according to the five lemma. Now the following diagram is commutative:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & {}^rV_n(A)/{}^rV_{n+1}(A) & \longrightarrow & A/{}^rV_{n+1}(A) & \longrightarrow & A/{}^rV_n(A) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & {}^rV_n(B)/{}^rV_{n+1}(B) & \longrightarrow & B/{}^rV_{n+1}(B) & \longrightarrow & B/{}^rV_n(B) & \longrightarrow & 1
\end{array}$$

Now the result holds by induction. □

We are ready to prove Theorem 5.11.

Proof of Theorem 5.11. Suppose $G/V(G)$ is $\mathcal{V}$-free on X and put F the free group on X, so there exists a homomorphism $\phi: F \to G$ which induces an isomorphism $G/V(G) \cong F/V(F)$. Hence by Lemma 4.11 it induces isomorphisms $\phi^n: F/{}^rV_n(F) \to G/{}^rV_n(G)$. Put $R = \ker\phi$, then $R \subseteq {}^rV_n(F)$ for all $n \geqslant 1$, so we have ${}^r\mathcal{V}_nM(G) = R/[RV^*F]$. But the triviality of $\mathcal{V}M(G)$ implies $R = [RV^*F]$ so by induction $R = [R\,{}^rV_n^*F]$ and hence the result holds. □

6. APPLICATION IN VARIETIES OF EXPONENT m

In this section using the notions of the left and right sequences and in a different manner some results on Baer invariants of $\mathcal{V}$-perfect groups is obtained. To do this, the following theorem about varieties is needed.

Theorem 6.1. *Let $\mathcal{V}$ be any variety. The following are equivalent*:

(i) *$\mathcal{V}$ contains all abelian groups*;
(ii) *$\mathcal{V}$ contains the infinite cyclic group* $\mathbf{Z}$;
(iii) *$\mathcal{V}$ is defined by outer commutator words.*

Proof. See [2, Remark 5.6]. □

Definition 6.2. A variety which satisfies one of the conditions of the above theorem is said to be of exponent zero. If $\mathcal{V}$ contains $\mathbf{Z}_m$ but no larger cyclic group, then $\mathcal{V}$ is of exponent m.

It can be found in [16, Theorem 12.12] that a variety of exponent m is defined by a set of outer commutator words in conjunction with the single word x^m, so each group in this variety is of exponent dividing m.

The following theorem is about $\mathcal{V}$-perfect groups and has an important role in the rest of this article.

Theorem 6.3. *Let $\mathcal{V}$ be any variety of exponent m, and let G be any $\mathcal{V}$-perfect group. Then $\mathcal{V}M(G) \cong M(G) \otimes \mathbf{Z}_m$.*

Proof. See [9, p. 142]. □

An outer commutator variety is exactly a variety of exponent zero. Theorem 6.3 completely determines the structure of the Baer invariants of a varietal perfect group with respect to this varieties in terms of its Schur multiplier, that is the Baer invariant with respect to the variety of abelian groups. It is easy to see that for an outer commutator variety $\mathcal{V}$, each term of the left and right sequence of $\mathcal{V}$ is an outer commutator too. Considering Theorem 5.2, it is easy to see that outer commutator varieties satisfy the condition of that theorem. So Theorem 5.2 can be considered as a slighter version of Theorem 6.3, but it is interesting to solve a problem in part, which have been solved, in a quiet different manner.

The following lemma almost does everything we want.

Lemma 6.4. *Let $\mathcal{V}$ be any variety of exponent m. Then ${}^{\ell}\mathcal{V}_n$ and ${}^{r}\mathcal{V}_n$ are varieties of exponent m^n.*

Proof. Straightforward. □

The immediate consequence of Theorem 6.3 and Lemma 6.5 is

Corollary 6.5. *Let $\mathcal{V}$ be any variety of exponent m, and let G be any $\mathcal{V}$-perfect group. Then ${}^{r}\mathcal{V}_n M(G) \cong M(G) \otimes \mathbf{Z}_{m^n} \cong {}^{\ell}\mathcal{V}_n M(G)$.*

In the special cases where $M(G)$ is finite or even finitely generated, we have the following corollaries.

Corollary 6.6. *Let $\mathcal{V}$ be any variety of exponent m and let G be any $\mathcal{V}$-perfect group with finitely generated multiplier. Then ${}^r\mathcal{V}_n M(G)$ and ${}^\ell\mathcal{V}_n M(G)$ are epimorphic images of ${}^r\mathcal{V}_{n+1} M(G)$ and ${}^\ell\mathcal{V}_{n+1} M(G)$, respectively.*

Corollary 6.7. *Under the above assumption when $M(G)$ is finite, it holds that*

(i) *${}^r\mathcal{V}_{n+1} M(G)$ and ${}^\ell\mathcal{V}_{n+1} M(G)$ are subgroups of ${}^r\mathcal{V}_n M(G)$ and ${}^\ell\mathcal{V}_n M(G)$, respectively;*
(ii) *there exists a natural number n_0 such that ${}^r\mathcal{V}_{n_0} M(G) \cong {}^r\mathcal{V}_{n_0+i} M(G)$ and ${}^\ell\mathcal{V}_{n_0} M(G) \cong {}^\ell\mathcal{V}_{n_0+i} M(G)$ for each $i \in \mathbf{N}$.*

The number n_0 (as occurring in Corollary 6.8) for any group with finite multiplier can be easily computed as follows.

Let $\pi(m) = \{p_1, p_2, \dots, p_r\}$ and $\pi(|M(G)|) = \{q_1, q_2, \dots, q_s\}$, where $\pi(X)$ is the set of all primes dividing the natural number X. Consider two cases:

(i) $\pi(m) \cap \pi(|M(G)|) = \emptyset$,
(ii) $\pi(m) \cap \pi(|M(G)|) = \{p_1, p_2, \dots, p_k\}$.

In case (i) we have ${}^r\mathcal{V}_n M(G) \cong {}^\ell\mathcal{V}_n M(G) \cong 0$, for all natural number n. In case (ii) let $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ and $|M(G)| = q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}$, then $n_0 = \max\{[\frac{\alpha_i}{\beta_i}] \mid 1 \leqslant i \leqslant k\}$, in which $[x]$ is the greatest integer not exceeding x.

In such a case the following sequences with finitely many distinct elements exist:

$$ {}^r\mathcal{V}_1 M(G) \supseteq {}^r\mathcal{V}_2 M(G) \supseteq \cdots \supseteq {}^r\mathcal{V}_n M(G) \supseteq {}^r\mathcal{V}_{n+1} M(G) \supseteq \cdots, $$

$$ {}^\ell\mathcal{V}_1 M(G) \supseteq {}^\ell\mathcal{V}_2 M(G) \supseteq \cdots \supseteq {}^\ell\mathcal{V}_n M(G) \supseteq {}^\ell\mathcal{V}_{n+1} M(G) \supseteq \cdots. $$

REFERENCES

[1] Baer R. – Representations of groups as quotient groups, I, II, III, Trans. Amer. Math. Soc. **58** (1945) 295–419.
[2] Beyl F.R., Tappe J. – Group Extension, Representations, and the Schur Multiplicator, Lecture Notes in Math., vol. 958, Springer, Berlin, 1982.
[3] Burns J., Ellis G. – On the nilpotent multipliers of a group, Math. Z. **226** (1997) 405–428.
[4] Fröhlich A. – Baer invariants of algebras, Trans. Amer. Soc. **109** (1963) 221–244.
[5] Gupta N.D., Moghaddam M.R.R. – Higher Schur multiplicators of nilpotent dihedral groups, C. R. Math. Rep. Acad. Sci. Canada **XIV** (5) (1992) 225–230.
[6] Hekster N.S. – Varieties of groups and isologism, J. Austral. Math. Soc. Ser. A **46** (1989) 22–60.
[7] Karpilovsky G. – The Schur multiplier, London Math. Soc. Monographs (N.S.) **2** (1987).
[8] Kayvanfar S. – The category $\mathcal{PME}(G, \mathcal{V})$ and generalized covering groups, J. Algebra **238** (2001) 126–138.
[9] Leedham-Green C.R., McKay S. – Baer-invariant, isologism, varietal laws and homology, Acta Math. **137** (1976) 99–150.
[10] Mashayekhy, B. – The Baer-invariant and generalized covering groups, Ph.D. Thesis, Ferdowsi University of Mashhad, 1996.
[11] Mashayekhy B., Moghaddam M.R.R. – Higher Schur multiplicator of a finite Abelian group, Algebra Colloquium **4** (3) (1997) 317–322.

[12] Moghaddam M.R.R., Kayvanfar S. – A new notion derived from varieties of groups, Algebra Colloquium **4** (1) (1997) 1–11.
[13] Moghaddam M.R.R., Kayvanfar S. – $\mathcal{V}$-perfect groups, Indag. Math. (N.S.) **8** (4) (1997) 537–542.
[14] Moghaddam M.R.R., Mashayekhy B., Kayvanfar S. – Subgroup theorems for the Baer invariant of groups, J. Algebra **206** (1998) 17–32.
[15] Moghaddam M.R.R., Mashayekhy B., Kayvanfar S. – The higher Schur-multiplier of certain classes of groups, Southeast Asian Bull. Math. **27** (2003) 1–8.
[16] Neumann H. – Varieties of Groups, Springer, Berlin, 1967.
[17] Stallings J. – Homology and central series of groups, J. Algebra **2** (1965) 170–181.

(Received September 2004)