Analysis of Elasto-Plastic Plane Cracked Problems using Element-Free Galerkin Method

M.H.Kargarnovin*[1], H.E.Toussi[2], S.J.Fariborz[3]
*Center of Excellence in Design, Robotic and Automation,
School of Mech. Eng., Sharif University of Technology,
P.O. Box 11365-9567, Tehran - Iran.
E-mail: mhkargar@sharif.edu

Abstract

The Element Free Galerkin Method (EFGM) has been extended to be used in the Elastoplastic stress analysis. Stress fields in two different plates with and without a crack have been calculated and the results have been compared with other similar works in the literature. The value of J-integral has been used as a base for the assesment of the open-edge cracked plate problem.

Keywords: Element Free Galerkin Method; Elastoplastic; Stress analysis; J-integral

Introduction

In the last decade Belytschko et. al. (1994) [1] introduced the EFG method to reduce some of the shortcomings of Finite Element Method. The paper of Nayroles et. al. (1992) [2] namely "Generalizing the FEM" was a close work prior to the former one and this work by itself seems to be inspired by another work which is in the area of Moving Least Square interpolants (MLS) [3]. After introducing of the EFGM, this method has been used in a wide range of different subjects such as dynamic fracture [4, 5], crack growth [6 and 7], elastic plates and shells [8], 3-D problems [9 and 10] and non-elastic stress analysis [11]. Even the theoretical foundations of this method have been reconsidered, developed or revised for several times [12 to 16].

Two main characteristics of the EFG Method seem to be a unique approximate function for the whole field and relatively easier kind of crack modeling. In this paper, the application of EFGM in solving field equations for the incremental plastic behavior of material has been examined.

Related to the historical background of the numerical stress analysis in the Elastoplastic range, it should be mentioned that the proposed techniques in this range are as old as those in the elastic range. For example in [17] a method for Elastoplastic stress analysis has been devised.

In this paper primarily EFGM and incremental plasticity is overviewed. Then, a new method of Elastic-Plastic Element Free Galerkin Method (EP-EFGM) is constructed. The composed method is intrinsically based on nonlinear relations; therefore its solution technique should be iterative too. A section is devoted to present the solution technique. The requirements of solution and solved examples and their results in comparison with other similar techniques are proposed in another section.

EFG Method

As like as FEM, the first step in EFGM modeling is assuming an appropriate description for the approximate function. Once an approximate function type is assumed the shape functions can be calculated. It should be pointed out that the basic difference between the EFGM and FEM is that in EFGM, the approximate function is an

---

[1] Associate Professor
[2] Ph.D. candidate, School of Mech. Eng., Sharif University of Technology, Tehran, Iran
[3] Associate Professor, Amirkabir University of Technology, Tehran, Iran

Analysis of Elasto-Plastic Plane Cracked Problems Using Element Free Galerkin Method

M.H.Kargarnovin*[١], H.E.Toussi[٢], S.J.Fariborz[٣]

* School of Mech. Eng., Sharif University of Technology, P.O. Box ١١٣٦٥-٩٥٦٧, Tehran - Iran.

E-mail: mhkargar@sharif.edu

Abstract

The Element Free Galerkin Method (EFGM) has been extended to be used in the Elastoplastic stress analysis. Stress fields in two different plates with and without a crack have been calculated and the results have been compared with other similar works in the literature. The value of J-integral has been used as a base for the assesment of the open-edge cracked plate problem.

Keywords: Element Free Galerkin Method; Elastoplastic; Stress analysis; J-integral

Introduction

In the last decade Belytschko et. al. (١٩٩٤) [١] introduced the EFG method to reduce some of the shortcomings of Finite Element Method. The paper of Nayroles et. al. (١٩٩٢) [٢] namely "Generalizing the FEM" was a close work prior to the former one and this work by itself seems to be inspired by another work which is in the area of Moving Least Square interpolants (MLS) [٣]. After introducing of the EFGM, this method has been used in a wide range of different subjects such as dynamic fracture [٤, ٥], crack growth [٦ and ٧], elastic plates and shells [٨], ٣-D problems [٩ and ١٠] and non-elastic stress analysis [١١]. Even the theoretical foundations of this method have been reconsidered, developed or revised for several times [١٢ to ١٦].

Two main characteristics of the EFG Method seem to be a unique approximate function for the whole field and relatively easier kind of crack modeling. In this paper, the application of EFGM in solving field equations for the incremental plastic behavior of material has been examined.

Related to the historical background of the numerical stress analysis in the Elastoplastic range,

it should be mentioned that the proposed techniques in this range are as old as those in the elastic range. For example in [١٧] a method for Elastoplastic stress analysis has been devised.

In this paper primarily EFGM and incremental plasticity is overviewed. Then, a new method of Elastic-Plastic Element Free Galerkin Method (EP-EFGM) is constructed. The composed method is intrinsically based on nonlinear relations; therefore its solution technique should be iterative too. A section is devoted to present the solution technique. The requirements of solution and solved examples and their results in comparison with other similar techniques are proposed in another section.

---

[١] Associate Professor

[٢] Ph.D. candidate, School of Mech. Eng., Sharif University of Technology, Tehran, Iran

[٣] Associate Professor, Amirkabir University of Technology, Tehran, Iran

EFG Method

As like as FEM, the first step in EFGM modeling is assuming an appropriate description for the approximate function. Once an approximate function type is assumed the shape functions can be calculated. It should be pointed out that the basic difference between the EFGM and FEM is that in EFGM, the approximate function is an approximation through the nodal values while in FEM it is an interpolation between nodal values, (See Figure ١). Where, the approximation and interpolation have the same explanation as, for example, in [١٨].

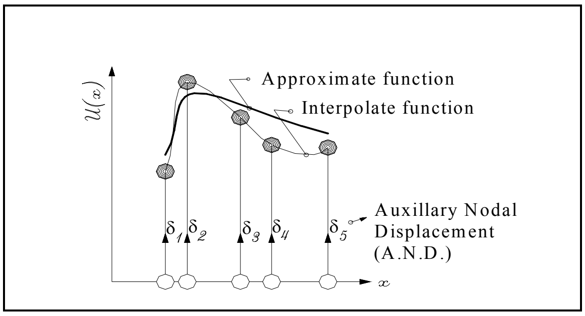

Figure ١. A comparison between the concepts of the interpolation as used in FEM and the approximation as used in EFG.

Calculated shape functions should be inserted in weak form of the integral governing equations of the system and convert them to a new system of nonlinear equations. Solving this system of equation results in the primary unknowns, which in solid mechanics, are generally, displacement field components.

The following notational rules have been applied throughout this section,

١- As most variables are function of position, for the sake of brevity, the argument (**x**) has been omitted. Moreover, the italic symbols have been used for variables that have no explicit dependence on (**x**).

٢- Subscripts in capital letters are used to count nodal variables.

In the following section, the construction of EFGM has been described.

Moving Least Square interpolants(MLS)

To construct a matrix form used in variational principle, an approximate function must be chosen such as,

$$u = \boldsymbol{\varphi}^T \delta \qquad (١)$$

Where, δ is the vector of nodal unknown parameters and $\boldsymbol{\varphi}$ is the vector of shape functions for n different number of nodes.

To obtain the best approximation, a criterion should be chosen and then minimized. In MLS this criteria is as follows [١];

$$R = \sum_{I=1}^{n} w_I [\, u(\mathbf{x}_I) - \delta_I \,]^2 \qquad (٢)$$

in which, $w_I = w(\mathbf{x} - \mathbf{x}_I)$ is the weight function and $u(\mathbf{x}_I)$ is the magnitude of approximate scalar function in point $\mathbf{x}_I$. The approximate function is chosen as,

$$u = \mathbf{p}^T \mathbf{c} \qquad (٣)$$

where **c** is a vector of m unknown variable coefficients which are to be calculated, and **p** is called the base vector. In this paper the base vector is chosen as, $\mathbf{p}^T = \lfloor 1, x, y \rfloor$. However in [١٢] some other kinds of base vectors are introduced.

Inserting (٣) in (٢), the stationary of R with respect to **c** leads to the following linear relation between **c** and δ.

$$\mathbf{A}\ \mathbf{c} = \mathbf{B}\ \delta \qquad (٤)$$

Where, **A** and **B** are matrices as follows,

$$\mathbf{A} = \sum_{I=1}^{n} w_I\, \boldsymbol{p}_I\, \boldsymbol{p}_I^T \qquad (٥a)$$

$$\mathbf{B} = [\mathbf{b}_1, ..., \mathbf{b}_I, ..., \mathbf{b}_n] \qquad (٥b)$$

in which

$$\mathbf{b}_I = w_I\, \boldsymbol{p}_I \qquad (٦)$$

B and **A** are matrices similar to the first and second moment matrices, respectively. Especially, **A** is called moment matrix [١٩]. After solving (٤) for **c** and substitute it in (٣) and combining the result with (١) we obtain,

$$\varphi_I = \mathbf{p}^T \mathbf{A}^{-1}\, \mathbf{b}_I \qquad (٧)$$

As it is shown in [١ and ٤] weight function has a dominant effect on the final solution. Usually a bell type function is chosen as weight function. Hence, the resulting shape function is also nearly a bell type function. As shown in [٤ and ٥] exponential type of weight function causes more accurate solution. Therefore, in this paper, this type of weight function has been used. i.e.,

$$w(\rho) = \frac{e^{-(\beta.\rho)^2} - e^{-\beta^2}}{1 - e^{-\beta^2}} \qquad (٨)$$

where, in this relation $\rho = r / r_m$ is the dimensionless radius, r is the radial distance, r_m is the radius of the support domain for the weight function and β is a parameter that controls the bell-type shape of weight function and has the value of ٤.

Because of existence of the $\mathbf{A}^{-1}$ in the expression for φ_I, evaluation of φ_I and its derivatives in all quadrature points can take up a great amount of computing time. By application of the following technique the mathematical manipulations and therefore the execution time will be reduced.

If one rewrites Equation (٧) in the form of :

$$\varphi_I = \gamma^T \mathbf{b}_I$$ (٩)

Where, $\gamma = \mathbf{A}^{-1}\mathbf{p}$, then one can write

$$\mathbf{p} = \mathbf{A}\gamma$$ (١٠)

Now we can decompose **A** into the upper and lower triangular parts and with fewer operations, calculate γ and φ_I . To find the derivatives of the shape function, after taking derivatives from Equation (١٠), we have

$$\mathbf{A}\gamma_{,i} = \mathbf{p}_{,i} - \mathbf{A}_{,i}\gamma$$ (١١)

The terms in right hand side of Equation (١١) can easily be estimated and $\gamma_{,i}$ can be calculated similar to γ calculation. Later on for some other purposes, we need to calculate $\varphi_{I,i}$,namely the derivative of φ_I with respect to some variable. From Equation (٩) we have

$$\varphi_{I,i} = \gamma^T_{,i}\mathbf{b}_I + \gamma^T \mathbf{b}_{I,i}$$ (١٢)

in witch all terms at right hand side are known quantities.

Discritized variational formulation

In the field of solid mechanics the equilibrium equation for a continuous media under small displacements is given as;

$$\nabla . \sigma + \mathbf{b} = 0 \quad \text{in } \Omega$$ (١٣)

with essential and natural boundary conditions as follows.

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u$$ (١٤a)

$$\boldsymbol{\sigma}.\mathbf{n} = \bar{\mathbf{t}} \quad \text{on } \Gamma_t$$ (١٤b)

In these relations, σ is the stress tensor, **b** is the body force vector, **u** is the displacement vector, ∇ is the gradient operator, **t** is the traction force and **n** is the unit normal vector to the boundary. Also the superscripted bar denotes a prescribed boundary value.

As in [٥], the weak form of equilibrium equation is given as follows,

$$\int \delta(\nabla_s u) : \sigma \, d\Omega - \int \delta u . b \, d\Omega - \int \delta u . \bar{\mathbf{t}} \, d\Gamma_t - \int \delta l . (\mathbf{u} - \bar{\mathbf{u}}) \, d\Gamma_u - \int \delta u . l \, d\Gamma_u = 0$$ (١٥)

In this relation $\nabla_s \mathbf{u}$ is the symmetric part of $\nabla\mathbf{u}$ term. Double dot product (**:**) represents the dyadic scalar product and **l** is the vector of Lagrange multipliers. **l** and δ**l** belong to Sobolov space of degree ٠ whereas **u** and δ**u** belong to Sobolov space of degree ١.

Note that in this method in the absence of Lagrange multipliers, it will be impossible to obtain a solution, which can satisfy essential boundary conditions. To impose essential boundary conditions, apart from the aforementioned method, some other methods can be found in the EFGM literature. For example in [١١] by using singular weight functions and with the cost of some alteration in the class of continuity of shape function or in [١٣] by combining FEM near boundary zone with EFG the problem of boundary conditions implication has been simplified. In some other works such as [٦ and ١٣] physical meaning of Lagrange multipliers has been used. They have used boundary traction force in place of Lagrange multiplier, **l** in Equation (١٥).

To discritize the final variational formulation we substitute $u_i = \varphi^T \delta$ and $l_i = \mathbf{N}^T \lambda$ in Equation (١٥) where u_i and l_i are components of **u** and **l** , respectively and **N** is the vector of local shape functions. Having in mind that the relation between strain and small displacement is $\boldsymbol{\varepsilon} = \nabla_s \mathbf{u}$ and constitutive relation is $\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\varepsilon}$ we obtain,

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{G} \\ \boldsymbol{G}^T & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \delta \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{q} \end{Bmatrix}$$ (١٦)

in which

$$K_{IJ} = \int \mathbf{B}_I^T \mathbf{D} \mathbf{B}_J \, d\Omega$$ (١٧a)

$$G_{IK} = -\int \varphi_I N_K \, d\Gamma_u$$ (١٧b)

$$f_I = \int \varphi_I \bar{t} \, d\Gamma_t + \int \varphi_I b \, d\Omega$$ (١٧c)

$$q_K = -\int N_K \bar{u} \, d\Gamma_u$$ (١٧d)

also in this relations we have

$$\mathbf{B}_I = \begin{bmatrix} \phi_{I,x} & 0 \\ 0 & \phi_{I,y} \\ \phi_{I,y} & \phi_{I,x} \end{bmatrix}$$ (١٨a)

$$\mathbf{N}_K = \begin{bmatrix} N_K & 0 \\ 0 & N_K \end{bmatrix} \qquad (۱۸b)$$

and **D** is property matrix, which relates different components of stress and strain to each other.

Elastoplastic constotutive equation

At least in the last couple of decades, different approaches for the modeling of incremental behavior of plastic deformation especially in the isotropic and homogeneous materials have been introduced. Following an Elastoplastic derivation for an isotropic material including some modifications made related to this special usage has been proposed.

When material is loaded in Elastoplastic range, property matrix **D** in Equation (۱۷a), must be revised to comply with this complex behavior.

From now on, the stress and strain tensors are expressed in the form of single column array matrices, e.g., $\boldsymbol{\varepsilon}^T = \lfloor \varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz} \rfloor$.

To find $\mathbf{D}_{ep}$ matrix, following relations are assumed to be known:

a- Total strain increment is the summation of elastic and plastic parts i.e.,

$$d\varepsilon = d\varepsilon_e + d\varepsilon_p \qquad (۱۹)$$

b- Elastic stress-strain relation in the incremental form is similar to the relation in its total form, e.g.,

$$d\boldsymbol{\sigma} = \mathbf{D}_e \, d\varepsilon_e \qquad (۲۰)$$

c- Failure criteria is,

$$F(\sigma) = f(\bar{\sigma}) \qquad (۲۱)$$

in which F and f are two different forms of failure functions, $\boldsymbol{\sigma}$ is the stress tensor and $\bar{\sigma}$ is the equivalent stress.

d- Flow rule that relates strain increment to other quantities, is the gradient of a function called plastic potential. If one assumes that the plastic potential function is the same as the failure function, then one can get the following relation known as normality rule,

$$d\varepsilon_p = \nabla F . \, d\lambda \qquad (۲۲)$$

e- Definition of plastic modulus is as follows.

$$H' = \frac{d\bar{\sigma}}{d\bar{\varepsilon}_p} \qquad (۲۳)$$

f- For a given strain energy, δw, and according to the definition of $d\bar{\varepsilon}_p$ we must have,

$$\delta w = \bar{\sigma} . d\bar{\varepsilon}_p \qquad (۲٤)$$

g- According to the Von Mises criteria F = $J_۲$, [۱۲], where $J_۲$ is the second invariant of deviatoric stress tensor. So we must have $f(\bar{\sigma}) = \bar{\sigma}^۲/۳$.

Now, combining (۱۹) and (۲۰) results in

$$d\boldsymbol{\sigma} = \mathbf{D}_e \{ d\varepsilon - d\varepsilon_p \} \qquad (۲٥)$$

After taking the derivatives from both sides of Equation (۲۱) we obtain,

$$\left(\frac{\partial F}{\partial \boldsymbol{\sigma}} \bullet d\boldsymbol{\sigma}\right) = \frac{\partial f}{\partial \bar{\sigma}} \cdot \frac{\partial \bar{\sigma}}{\partial \bar{\varepsilon}_p} \cdot \frac{\partial \bar{\varepsilon}_p}{\partial w} \cdot \left(\frac{\partial w}{\partial \boldsymbol{\varepsilon}_p} \bullet d\varepsilon_p\right) \qquad (۲٦)$$

In Equation (۲٦) vector quantities are enclosed in parentheses and symbol (•) stands for the so-called dot vector product. For simplicity we take $\frac{\partial F}{\partial \boldsymbol{\sigma}} = \mathbf{a}$, and $\frac{\partial f}{\partial \bar{\sigma}} = \bar{a}$. Also by means of equations (۲۳) and (۲٤) we can rewrite (۲٦) in the form of,

$$\mathbf{a} \bullet d\boldsymbol{\sigma} = \bar{a} \; . \; H' \; . \; \left(\frac{1}{\bar{\sigma}}\right) \; . \; \boldsymbol{\sigma} \bullet d\varepsilon_p \qquad (۲۷)$$

dλ is calculated by omitting $d\boldsymbol{\sigma}$ between Equation (۲۵) and (۲۷) and substituting $d\varepsilon_p$ from Equation (۲۲). If we back substitute dλ in Equation (۲۲), the final form of material matrix of Elastoplastic equation will be obtained. i.e.,

$$\mathbf{D}_{ep} = \mathbf{D}_e - \mathbf{D}_p \qquad (۲۸)$$

Where,

$$\mathbf{D_p} = \frac{\mathbf{D a a^T D}}{\frac{\bar{a}}{\bar{\sigma}} \, H' \boldsymbol{\sigma}^\mathbf{T} \mathbf{a} + \mathbf{a^T D a}} \qquad (۲۹)$$

In order to obtain $\mathbf{D}_p$ or the standard rank one correction to the elastic modulus, similar methods are outlined in [۲۰ to ۲۲] and some nearly similar relations are represented in those works.

In terms of Von Mises criteria, we have $\bar{a} = 2\bar{\sigma}/3$. In this case using a single column matrix representation for the flow vector **a** in Equation (۲۷) we have, $\mathbf{a}^T = \lfloor s_x, s_y, s_z, ۲s_{xy}, ۲s_{yz}, ۲s_{zx} \rfloor$ in which, s, stands for a deviatoric stress components. Now, if we use the general stress-strain matrix of property for an isotropic material in place of **D**, we will obtain **D a** = E′ **s**, in which, E′ = E / (١+ν) and **s** is a column of deviatoric stresses, i.e., $\mathbf{s}^T = \lfloor s_x, s_y, s_z, s_{xy}, s_{yz}, s_{zx} \rfloor$. After substituting these relations into the Equation (۲۹) and some simplification, we obtain:

$$\mathbf{D}_p = \mathbf{s}^T \mathbf{s} / c \qquad (۳۰)$$

in which, c = (H′+ ١,٥ E′) (۲ $\bar{\sigma}$ / ۳E′)^۲.

Substituting Equation (۳۰) in (۲۸) and expanding (۲۸) for general state of stress results in the finale form of incremental Elastoplastic constitutive relation. That is,

$$\begin{Bmatrix} d\sigma_x \\ d\sigma_y \\ d\sigma_z \\ d\tau_{xy} \\ d\tau_{yz} \\ d\tau_{zx} \end{Bmatrix} = E'\left[\mathbf{C}_1 + \frac{1}{(H' + \frac{3}{2}E')(\frac{2\bar{\sigma}}{3E'})^2}\mathbf{C}_2\right]\begin{Bmatrix} d\varepsilon_x \\ d\varepsilon_y \\ d\varepsilon_z \\ d\gamma_{xy} \\ d\gamma_{yz} \\ d\gamma_{zx} \end{Bmatrix} \qquad (۳۱)$$

in which

$$\mathbf{C}_1 = \begin{pmatrix} C_1 & C_2 & C_2 & 0 & 0 & 0 \\ C_2 & C_1 & C_2 & 0 & 0 & 0 \\ C_2 & C_2 & C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix}, \quad \mathbf{C}_2 = \begin{pmatrix} S_x^2 & S_xS_y & S_xS_z & S_xS_{xy} & S_xS_{yz} & S_xS_{zx} \\ S_yS_x & S_y^2 & S_yS_z & S_yS_{xy} & S_yS_{yz} & S_yS_{zx} \\ S_zS_x & S_zS_y & S_z^2 & S_zS_{xy} & S_zS_{yz} & S_zS_{zx} \\ S_{xy}S_x & S_{xy}S_y & S_{xy}S_z & S_{xy}^2 & S_{xy}S_{yz} & S_{xy}S_{zx} \\ S_{yz}S_x & S_{yz}S_y & S_{yz}S_z & S_{yz}S_{xy} & S_{yz}^2 & S_{yz}S_{zx} \\ S_{zx}S_x & S_{zx}S_y & S_{zx}S_z & S_{zx}S_{xy} & S_{zx}S_{yz} & S_{zx}^2 \end{pmatrix},$$

$$E' = \frac{E}{1+\nu}, \; C_1 = \frac{1-\nu}{1-2\nu} \text{ and } C_2 = \frac{\nu}{1-2\nu}.$$

In plane stress conditions the out of plane components of stress are zero. Using this fact and the equation (۳١), the planar incremental stress-strain relationship is easily derived.

EP-EFGM solution algorithm

As indicated before if incremental solution is sought, the derived Elastoplastic constitutive relation is needed. Now if in Equations (١٦) and (١٧a), total auxiliary nodal displacement vector is replaced by incremental auxiliary nodal displacement and property matrix is replaced by Elastoplastic property matrix, then a new set of equations will be obtained which describes incremental Elastoplastic behavior. The incremental form of Equation (١٦), is

$$\begin{bmatrix} \boldsymbol{K}_{ep} & \boldsymbol{G} \\ \boldsymbol{G}^T & \boldsymbol{0} \end{bmatrix}\begin{Bmatrix} \Delta\delta \\ \Delta\lambda \end{Bmatrix} = \begin{Bmatrix} \Delta\boldsymbol{f} \\ \Delta\boldsymbol{q} \end{Bmatrix} \qquad (۳۲)$$

which can be shown in more compact form with,

$$\boldsymbol{S}_{ep}\,\delta = \boldsymbol{f} \qquad (۳۳)$$

for the sake of brevity, $\boldsymbol{S}_{ep}$ will be called as the stiffness matrix, δ as the Auxiliary Nodal Displacement (AND) vector and $\boldsymbol{f}$ as the force vector. In this manner, to obtain the total displacement, we should change boundary conditions gradually and solve related incremental equations and finally imply a summation over the resulted field quantities.

Apart from incremental behavior, there is still another difference between the form of Equation (١٦) and (۳۲). That is in this model the behavior of Elastoplastic EFG stiffness matrix in the incremental form is nonlinear. In fact stiffness matrix $\boldsymbol{S}_{ep}$, which has to be used to obtain displacement field, by itself depends on material properties. In other words it can easily be verified that the Elastoplastic material property matrix $\mathbf{D}_{ep}$ indirectly depends on displacement field. So, in order to obtain unknown $\Delta\delta$s in Equation (۳۲) a nonlinear solution technique has been chosen.

Owen and Hinton (١٩٨٠) [۲١] have introduced different methods for solving nonlinear Elastoplastic FEM equations. In this paper we chose one of those methods, which is called initial stiffness method and tailor it to EFGM.

Numerical solution

To examine the validity of this method, the solution of two well known problems are compared with their respective bench marking analytical solutions in the literature. Both examples devoted to the stress analysis in a thin rectangular plate. The simply supported plate is under uniform tensile traction in each side. In the first example it is assumed that the plate has no crack and should carry a uniform tensile Elastoplastic state of stress in entire field. In the second example it is assumed that the plate has a middle edge crack with the length of ١٠٠mm and carries the uniform tensile traction over end boundaries.

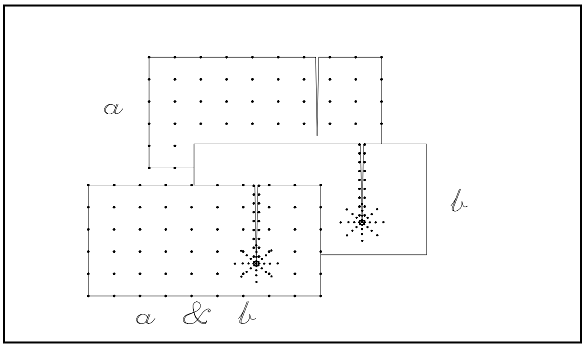

Figure ۲. Node distribution in an overlay made of field nodes (a) and crack tip nodes (b).

An overlay of the two different networks of nodes (a) and (b) in Figure ۲ is used for nodal modeling in our solution domain. Amongst them the arrangement (a) is individually used in example ۱ and the arrangement (a&b) is used for the solution of example ۲.

In order to conduct the field integrations we need another network of cells and a web of points to be used in a quaderative rule of integration. At these points, the stiffness matrix components have to be calculated and integrated.

In the following the solution for two benchmarking problems using our methods is described. In both examples, assumptions are:

- Mechanical behavior of material is supposed to be isotropic and homogeneous with an elastic modulus equal to ۲۰۰ GPa, a Poisson's ratio equal to ۰٫۳.
- In the elastic range a linear relation for stress and strain is assumed.
- In Elastoplastic region, material yields according to Von-Misses criteria and also normality rule prevails.
- Imposed boundary traction or extensions are planar. Hence induced deformation remains planar. On the other hand, no off-plane deformation or buckling is allowed to occur.

Example ۱: Uniform Elastoplastic Tensile Stretch

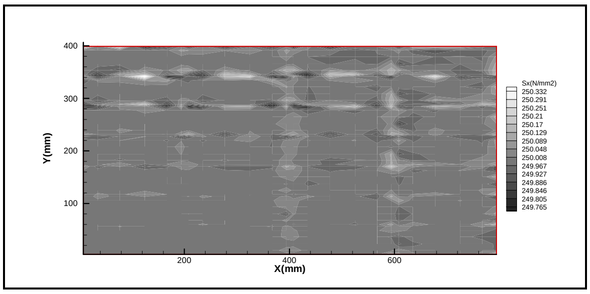

Figure ۳. Distribution of σ_x in the Elastoplastic uniaxial stretch of a rectangular thin plate.

Consider a ٤۰۰×۸۰۰mm rectangular simply supported plate under a uniform in-plane tensile traction, which is made of isotropic material with elastic modulus ۲۰۰GPa and yield stress equal to ۲۰۰Mpa. We assume an elastic linearly plastic strain-hardening behavior for the material with slope of stress vs. plastic strain curve equal to ۲۰GPa. Then if we stretch the plate uniformly in both opposite sides up to ۳mm, a simple stress-strain analysis shows that the uniform tensile stress should be ۲٥۰Mpa throughout the plate. Figure ۳ shows the result obtained by using a network of ۸×٥ nodes to model the plate surface. As can be seen in the TECPLOT contour plot of the results, the deviation of stress distribution from true values is very slim.

Example ۲:Capturing the mode (I) of Elastoplastic crack tip plane stress singularity

To show the utilization of this method in crack stress analysis, numerical solution for plane stress distribution around the tip of a crack is sought. It is assumed that the ۱۰۰mm open crack is situated in the middle of a rectangular plate of ٤۰۰×۸۰۰mm size. See Figure ۲. The plate is loaded uniformly and gradually up to the traction of ۱۰۰ MPa in both sides.

In EFG method a crack can rather be model more easily than other methods. Here the rule is to omit that part of the shape function of any node that situated in other side a crack line. According to this rule the support of any point is limited to the visible region through that point. In this method any kind of boundary such as crack edges are visualized to be barriers against vision.

Generally in Elastoplastic situations J-integral is used as a representative to show the magnitude of stress singularity in crack tip. We have also used J-integral to represent a numerical value for stress singularity and to compare our results with others.

J-integral is an integral over a special function of stress, which is defined as follows [۲۳].

$$J = \int \left(w\mathbf{n}_1 - \mathbf{t}_i \frac{\partial \mathbf{u}_i}{\partial x} \right) d\Omega \qquad (۳٤)$$

In this relation w stands for elastic energy density, $\mathbf{t}_i$ is vector of traction applied to the integration contour, x is the coordinate axes which places along the crack wall in direction of crack growth, $\mathbf{n}_1$ is x component of the unit normal vector to the contour and finally $\mathbf{u}_i$ is the displacement vector. Using Green's theorem, the path integral may be transformed to a domain form [۲٤ and ۲٥].

Though it is easy to pinpoint a numerous number of literatures in Elastoplastic stress analysis for crack tip, using J-integral approach, e.g. [۲٦ to ۲۹], nonetheless there are limited number of works that analytically relate different geometry and material to the value of J-integral. One of the well-known work in this field seems to be the work done in Electric power research Institute [۳۰]. In this work according to different geometry of crack and loading, based on the power law behavior of material's hardening the value of J-integral has been proposed. For power law hardening we have:

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n \qquad (۳۵)$$

In this relation σ stands for stress, ε for strain and σ_o, ε_o and α are some adjusting parameters.

In order to comply further with the type of loading assumed in [۳۰], instead of applying stepwise traction, a single step loading is employed. Finally, in our stress analysis, as it is the case in [۳۰], the contribution of elastic part of Equation (۳۵) is neglected. For the aforementioned geometrical specifications, J-integral in different paths has been calculated and compared with those which can be obtained by using the relation in [۳۰]. In our analysis, we have taken n=۱۰, σ_o=۲۰۰ Mpa, ε_o=.۰۰۱ for and α = ۱. According to [۳۰] and based on the geometrical and material parameters, J_p should be ۲,۸٤۲N/mm.

To test the results of the program, the following entries are taken to be the value of the node distribution parameters:

The number of nodes in x direction is ۱۰, the number of nodes in y direction is ٦, the number of circular and radial arrays of nodes both equals ٤ and finally the number of nodes used for modeling of each side of the crack wall is ٤, too.

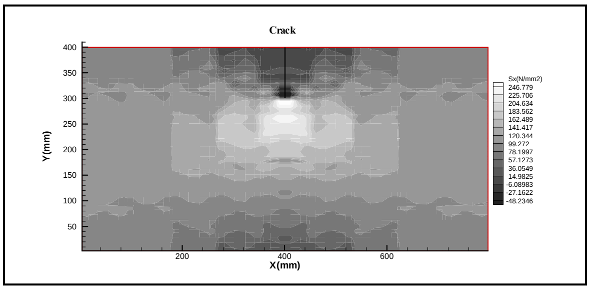

Figure ٤. Equivalent stress ($\overline{\sigma}$) developed in a plate having an open crack and loaded in both sides

For such an arrangement of nodes we have found the values of J۱ to J٤ on the square paths with side lengths ٤۰, ٦۰, ۸۰ and ۱۰۰ to be ۸,٦۹۹, ٥,۰۰۰, ۳,۹۲۰, ۲,۸٤۷ N/mm respectively. Figure ٤ shows the typical distribution of the equivalent stress in such a plate.

As one can imagine the shape of the stress contours are very much similar to the well-known plastic zone proposed in the literature such as [۳۱ to ۳۳]. In this typical example it is seen that the value of J-integral closely approaches that obtained by using the relations in [۳۰].

Conclusion

In this paper by combining EFG and incremental plasticity, a new solution method has been proposed. Using two benchmarking examples the capabilities of the method has been examined. It is shown that the extension of EFGM to Elasto-Plastic stress analysis including the stress analysis in crack problems is feasible and that its results are reasonable and have close agreement with other near works in the literature. Especially in the simple geometeries such as the first example the results are nearly perfect.

Compared to FEM, there are some drawbacks such as insertion of essential boundary conditions, complexity of solution algorithms, sensitivity of the method to adjusting some indices like weight function and the need for base function enrichment and shape function repair around crack tip. Our further experience shows that in complicated and singular conditions such as the crack problems results have some kind of complex and unknown dependence on the number and distribution of nodes. Such kind of disorder has also been anticipated by a recently published paper [۳٤].

With all these drawbacks the method seems to be nearly expensive. Nevertheless in Elastoplastic problems with complex geometeries, especially where there are two or three-dimensional cracks, because of the benefits of meshless techniques the method sounds to be useful.

References

۱. Belytschko T, Lu YY, Gu L. Element-free Galerkin methods, International Journal for Numerical Methods in Engineering, Vol. ۳۷, pp ۲۲۹-۲٥٦, ۱۹۹٤.

۲. Nayroles B, Touzot G, Villon P. Generalizing the finite element method: diffuse approximation and diffuse element. Computational Mechanics, Vol. ۱۰, pp ۳۰۷-۳۱۸, ۱۹۹۲.

۳. Lancaster P, Salkauskas K. Surfaces generated by moving least squares methods. Mathematics of Computations, Vol. ۳۷, pp ۱٤۱-۱٥۸, ۱۹۸۱.

٤. Belytschko T, Lu YY, Gu L, Tabbara M. Element-free Galerkin method for static and dynamic fracture. International Journal of Solids and Structures, Vol. ۳۲, pp ۲٥٤۷-۲٥۷۰, ۱۹۹٥.

٥. Belytschko T, Tabarra M. Dynamic fracture using element-free Galerkin methods.

International Journal for Numerical Methods in Engineering, Vol. ٣٩, pp ٩٢٣-٩٣٨, ١٩٩٦.

٦. Belytschko T, Lu YY, Gu L. Crack propagation by element-free Galerkin methods. Engineering Fracture Mechanics, Vol. ٩٥, pp ٢٩٥-٣١٥, ١٩٩٥.

٧. Xu Y, Saigal S. An element-free Galerkin formulation for stable crack growth in an elastic solid. International Journal of Solids and Structures, Vol. ١٥٤, pp ٣٣١-٣٤٣, ١٩٩٨.

٨. Krysl P, Belytschko T. Analysis of thin plates by element-free Galerkin method. Computational Mechanics, Vol. ١٧, pp ٢٦-٣٥, ١٩٩٦.

٩. Sukumar N, Moran B, Black T, Belytschko T. An element-free Galerkin method for three dimensional fracture mechanics. Computational Mechanics, Vol. ٢٠, pp ١٧٠-١٧٥, ١٩٩٧.

١٠. Krysl P, Belytschko T. The element free Galerkin method for Dynamic propagation of arbitrary ٣-D cracks. International Journal for Numerical Methods in Engineering, Vol. ٤٤, pp ٧٦٧-٨٠٠, ١٩٩٩.

١١. Barry S, Saigal S. A three-dimensional element-free Galerkin elastic and Elastoplastic formulation. International Journal for Numerical Methods in Engineering, Vol. ٤٦, pp ٦٧١-٦٩٣, ١٩٩٩.

١٢. Fleming M, Chu YA, Moran B, Belytschko T. Enriched element-free Galerkin methods for crack tip fields. International Journal for Numerical Methods in Engineering, Vol. ٤٠, pp ١٤٨٣-١٥٠٤, ١٩٩٧.

١٣. Krongauz T, Belytschko T. Enforcement of essential boundary conditions in mesh-less approximations using finite elements. Computer Methods in Applied Mechanics and Engineering, Vol. ١٣١, pp ١٣٣-١٤٥, ١٩٩٦.

١٤. Dolbow J, Belytschko T. Volumetric locking in the element free Galerkin method. International Journal for Numerical Methods in Engineering, Vol. ٤٦, pp ٩٢٥-٩٤٢, ١٩٩٩.

١٥. Belytschko T, Krongauz Y, Dolbow J, Gerlach C. On the completeness of meshfree particle methods. International Journal for Numerical Methods in Engineering, Vol. ٤٣, pp ٧٨٥-٨١٩, ١٩٩٩.

١٦. Krysl P, Belytschko T. Element-free Galerkin method: Convergence of the continuous and discontinuous shape functions. Computer Methods in Applied Mechanics and Engineering, Vol. ١٤٨, pp ٢٥٧-٢٧٧, ١٩٩٧.

١٧. Zienkiewicz OC. Elastoplastic solutions of engineering problems initial stress finite element approach. International Journal for Numerical Methods in Engineering, Vol. ١, pp ٧٥-١٠٠, ١٩٦٩.

١٨. Carnahan B. Applied numerical methods. John wiley & son Inc. :١٩٦٩.

١٩. Belytschko T, Krongauz Y, Organ D, Feleming M, Krysl P. Meshless methods: an overview and recent developments. Computer Methods in Applied Mechanics and Engineering, Vol. ١٣٩, pp ٣-٤٧, ١٩٩٦.

٢٠. Chakrabarty J. Theory of plasticity. Mc Graw Hill:١٩٨٧.

٢١. Owen DR, Hinton E. Finite element in plasticity. Pineridge press: U.K., ١٩٨٠.

٢٢. Hsu TR. The finite element in thermomechanics. Prantice-Hall Inc.: Canada, ١٩٨٧.

٢٣. Rice JR. A path independent integral and the approximate analysis of strain concentration by notches and cracks. Journal of Applied Mechanics, Vol. ٣٥, pp ٣٧٩-٣٨٦, ١٩٦٨.

٢٤. Moran B, Shih CF. Crack tip and assciated domain integrals from momentum and energy balance. Engineering Fracture Mechanics, Vol. ٢٧, pp ٦١٥-٦٤١, ١٩٨٧.

٢٥. Nishioka T. Theoretical and computational aspects of the dynamic J-integral in dynamic fracture mechanics for homogeneous and nonhomogeneous materials. Computer Modeling and Simulation in Engineering, Vol. ٤, pp ٢٩-٣٦, ١٩٩٩.

٢٦. Rice JR, Rosengren GF. Plane Strain Deformation near a Crack Tip in a power-law Hardening Material. Journal of the Mechanics and Physics of Solids, Vol. ١٦, pp ١-١٢, ١٩٦٨.

٢٧. Hutchinson JW. Singular Behavior at the End of a Tensile Crack Tip in a Hardening Material. Journal of the Mechanics and Physics of Solids, Vol. ١٦, pp ١٣-٣١, ١٩٦٨.

٢٨. Shih CF, Hutchinson JW. Fully Plastic Solution and Large Scale Yielding Estimates for Plane Stress Crack Problems. Journal of Engineering Material and Technology, pp ٢٨٩-٢٩٥, ١٩٧٦.

٢٩. Shih CF, Needleman A. Fully Plastic Crack Problems, Part ١: Solution by a Penalty Method. Journal of Applied Mechanics, Vol. ٥١, pp ٤٨-٥٦, ١٩٨٤.

۳۰. Kumar V, German MD, Shih CF. An Engineering Approach for Elastic Plastic Fracture Analysis. EPRI Report NP-۱۹۳۱, Electric Power Research Institute, Palo Alto, CA, ۱۹۸۱.

۳۱. Tada H, Paris PC, Irwin GR. The Stress Analysis of Cracks Handbook. Paris Production, Inc.: St. Louis, ۱۹۸۵.

۳۲. Atkins AG, Mai YW. Elastic and plastic fracture. Ellis Harwood limited: England, ۱۹۸۸.

۳۳. Anderson TL. FRACTURE MECHANICS Fundamentals and applications. CRC Press, Inc., ۱۹۹۵.

۳٤. Belytschko T, Mish K. Computability in non-linear solid mechanics. International Journal for Numerical Methods in Engineering, Vol. ۵۲, pp ۳-۲۱, ۲۰۰۱.