

ENTROPY GENERATION ANALYSIS IN ERROR ESTIMATION OF AN APPROXIMATE SOLUTION: A CONSTANT SURFACE TEMPERATURE SEMI-INFINITE CONDUCTIVE PROBLEM

by

Javad Abolfazli ESFAHANI and Seama KOOHI-FAYEGH

Original scientific paper

UDC: 536.24:517.957

BIBLID: 0354-9836, 13 (2009), 2, 133-140

DOI: 10.2298/TSCI0902133E

The integral solution of one-dimensional heat conduction in a semi-infinite wall with constant temperature at its surface has been reviewed and compared with the exact solution for three temperature profiles. Then, an entropy generation analysis has been carried out for all solutions. Introducing an average normalized entropy generation, the error of the integral solution is found to show values in the same order as the values calculated for the normalized entropy generation. Therefore, it can be concluded that when no exact solution is available for a similar problem, one can verify the error of the available approximate solutions simply by applying an entropy generation analysis on the problem.

Key words: *heat conduction, semi-infinite, constant surface temperature, approximate solution, entropy generation, estimation error*

Introduction

Thermodynamic irreversibility which is an inevitable phenomenon in all types of thermal processes is associated with entropy generation or in other words the one-way destruction of available work. The most common source of irreversibility in conductive systems is thermal irreversibility due to the finite temperature differences in thermal contacts. The growing need for higher quality performance in thermal engineering systems have proposed the second law analysis as a great tool in improving such systems.

Bejan [1] pioneered the method of entropy generation minimization (EGM) in various configurations and flow regimes. In his book [2], he conducted the second law analysis of thermodynamics via the minimization of entropy generation for the single phase convection heat transfer. Fowler *et al.* [3] used thermo-economic analysis to study the optimal sizes of geometries with specified external forced in convection heat transfer. Shuja *et al.* [4] presented a thermo-economic design and optimization of fins with constant cross-sectional area. This thermo-economic design considered capital costs and irreversibility penalty costs. Sahin [5] investigated entropy generation of a laminar flow in a tube with constant wall temperature. Also Esfahani *et al.* [6] investigated the relation among entropy generation, geometry and fluid properties in the heat transfer process in circular pipes with constant wall temperature. They presented an optimum design based on minimizing thermal and frictional entropy generation. Walsh *et al.* [7] developed a quick, simple, and relatively accurate method for the prediction of entropy generation in steady, two-dimensional, incompressible, adiabatic boundary layer flows

of turbo machines, which gives both the distribution and magnitude of the entropy generation rate. Griffin *et al.* [8] investigated the effect of Reynolds number, compressibility and free stream turbulence on profile of entropy generation rate; in their results, increased free stream turbulence had a greater effect on the generated entropy. It was observed that the amount of entropy generated in the turbulent boundary layer was approximately equivalent for two turbulence levels at comparable Reynolds numbers.

Simple heat conduction problems may have analytical solutions in simple geometries, but the solutions are always in closed series form and do not give a rapid insight of temperature field behavior. On the other hand, this solution may not be available in case of complex geometries, mixed boundary conditions or non-linear source terms. Thus, approximate analytical methods of solution to heat conduction problems have received considerable attention for many decades. One approximate solution applied in a number of conduction problems is the heat-balance integral method first employed by Goodman [9-11] to investigate a unidirectional melting problem. Since then, this method has been used extensively in many problems of non-linear heat input and in problems involving temperature dependant material properties [12, 13]. One point worth discussing for all temperature profiles gained by the integral method in a problem is that among several assumed temperature profiles, which one is more similar to the natural shape of the temperature profile? In problems for which the exact solution is available, the exactness of the approximate temperature profiles can be examined simply by comparing them to the exact one. But when no exact solution is available for a problem, a parameter is needed to represent the error of various integral solutions available for the problem.

Entropy generation analysis of thermal problems has been shown to be a good parameter in the area of error estimation. Bejan [14] showed that the natural shape of the velocity and temperature profiles of a two-dimensional turbulent jet is the one that minimizes the total entropy generation rate. Esfahani *et al.* [15] examined different integral solutions in a flat plate boundary layer and introduced entropy generation analysis as a valid representative of exactness of an approximate solution. Later, Esfahani *et al.* [16] performed an entropy generation analysis of a two dimensional steady conduction problem and discussed the correlation between the entropy generation and the average error. Hristov [17] performed an entropy generation of the heat-balance integral of Goodman for two classical problems with known exact solutions to exemplify the second law approach in defining the appropriate exponent n assumed in the temperature profiles.

Since the application of entropy generation in error estimation of problems has not been studied for a wide range of problems yet, in the present work, the objective is to focus on errors of the solutions gained by applying an integral method on a well known conduction problem of constant surface temperature for three temperature profiles. This classical problem is selected as the exact solution is available and the comparison between the approximate and the exact results can be made. Then, an entropy generation analysis of the problem is made which is followed by introducing a normalized entropy generation. It is found that the normalized rate of entropy generation behaves in the same manner as average error calculated for each approximate solution.

Governing equations

A thick wall, initially at uniform temperature T_i , is suddenly brought into contact with a warm steam of temperature T_0 . Perfect thermal contact is assumed to exist between the wall and the steam. As time progresses, the part of the wall adjacent to the side in contact with the fluid warms up. The mathematical model for the problem is

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \quad (1)$$

with the following boundary conditions

$$T(0, t) = T_0 \quad (2)$$

$$T(\delta, t) = T_i \quad (3)$$

$$T(x, 0) = T_i \quad (4)$$

Due to the prescribed boundary conditions, two solution methods are applied. The first is the similarity method which will result in an exact solution, and the second is the integral method which is an approximate solution.

Similarity solution

The exact solution of the problem via the similarity solution method can be found in [18] as below:

$$\theta = 1 - \operatorname{erf} \frac{\eta}{2} \quad (5)$$

where θ and η are the dimensionless temperature and the dimensionless length, respectively, and are introduced as:

$$\theta = \frac{T - T_i}{T_0 - T_i}, \quad \eta = \frac{x}{\sqrt{\alpha t}} \quad (6)$$

Integral solution

The simplest temperature profile can be a first degree polynomial assumed as:

$$\frac{T - T_i}{T_0 - T_i} = A + B \frac{x}{\delta} \quad (7)$$

Applying the two boundary conditions presented in eqs. (2) and (3) on eq. (7) will result in the following expression for the temperature profile:

$$\frac{T - T_i}{T_0 - T_i} = 1 - \frac{x}{\delta} \quad (8)$$

The length of the temperature domain (δ) in eq. (8) is defined writing the integral form of the energy equation. However, the assumption of a linear temperature profile will result in a constant δ which is not physically acceptable as it varies with time.

Therefore, putting one step forward, a second degree polynomial expression for the temperature profile can be assumed as:

$$\frac{T - T_i}{T_0 - T_i} = A + B \frac{x}{\delta} + C \frac{x^2}{\delta^2} \quad (9)$$

Three boundary conditions are required for the evaluation of the unknown constants in the temperature profile. Boundary conditions (2) and (3) yield two of the constants while the third one is obtained realizing that:

$$\frac{\partial T}{\partial x} \Big|_{x=\delta} = 0 \quad (10)$$

Thus, the solution for the temperature profile is defined as:

$$\theta = 1 - \frac{\eta^2}{\sqrt{12}} \quad (11)$$

Assuming a third degree polynomial expression for the temperature profile as:

$$\frac{T - T_i}{T_0 - T_i} = A + B \frac{x}{\delta} + C \frac{x^2}{\delta^2} + D \frac{x^3}{\delta^3} \quad (12)$$

four boundary conditions are required for the evaluation of the unknown constants in the temperature profile. Boundary conditions (2), (3), and (10) yield three of the constants while the last one can be obtained writing an energy equation either at $x = 0$ or $x = \delta$, respectively as:

$$\frac{\partial^2 T}{\partial x^2} \Big|_{x=0} = 0 \quad (13-1)$$

$$\frac{\partial^2 T}{\partial x^2} \Big|_{x=\delta} = 0 \quad (13-2)$$

Therefore, the solution for the temperature profile is further divided into two categories regarding eqs. (13-1) and (13-2) as :

$$\theta = 1 - \frac{3}{2} \frac{\eta}{\sqrt{8}} + \frac{1}{2} \frac{\eta^3}{\sqrt{8}} \quad (14-1)$$

$$\theta = 1 - \frac{\eta^3}{\sqrt{24}} \quad (14-2)$$

To compare the three integral profiles with the exact solution, an average error is introduced as:

$$\overline{\text{Error}} = \frac{1}{\eta_\delta} \int_0^{\eta_\delta} \left| \frac{\theta_{\text{exact}} - \theta_{\text{approximate}}}{\theta_{\text{exact}}} \right| d\eta \quad (15)$$

The present study tries to introduce a parameter which can evaluate the error behavior of the approximate solutions. Esfahani *et al.* [15] showed the validity of entropy generation analysis in determination of the most accurate temperature and velocity profiles in the flat plate boundary layer. Also, Esfahani *et al.* [16] performed an entropy generation analysis of a two-dimensional steady conduction and discussed the correlation between the entropy generation and the average error. Here, the entropy generation of the approximate solutions related to three temperature profiles is compared with the average error.

Entropy generation

It is easy to show that the rate of one-way destruction of useful work in an engineering system, W_{lost} , is directly proportional to the rate of entropy generation:

$$W_{\text{lost}} = T_0 S_{\text{gen}} \quad (16)$$

where T_0 is the absolute temperature of the ambient reservoir ($T_0 = \text{constant}$). Assuming a finite-size control volume at an arbitrary point in a two dimensional convective field and applying the second law of thermodynamics, the entropy generation per unit time and per unit volume, S_{gen} , is defined as:

$$S_{\text{gen}} = \frac{k}{T^2} \left(\frac{\partial T}{\partial x} \right)^2 + \frac{\mu}{T} \left(2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right) \quad (17)$$

where k and μ are the conductivity and viscosity of the fluid, T represents the absolute temperature of the point where S_{gen} is being evaluated [2]. In the current study, attention is focused only on diffusion of energy in the x-direction. Therefore, the above relation reduces to:

$$S_{\text{gen}} = \frac{k}{T^2} \left(\frac{\partial T}{\partial x} \right)^2 \quad (18)$$

This source of entropy generation is called thermal entropy generation. As it is seen, the entropy generation depends on determination of the temperature profile. On the other hand, the temperature profile depends on the method of solution, which was reviewed in the previous section.

Furthermore, a non-dimensional entropy generation is introduced as:

$$S_{\text{gen}}^m = \frac{S_{\text{gen}}}{\frac{k}{\alpha t} \frac{T_0 - T_i}{T_i}} \left(\frac{\partial \theta}{\partial \eta} \right)^2 \quad (19)$$

and the average normalized entropy generation is defined as:

$$\overline{S_{\text{gen}}^m} = \int_0^{\eta_s} S_{\text{gen}}^m d\eta \quad (20)$$

In the following section, attention is focused on entropy generation behavior of the approximate solutions as well as the exact one and a correlation is found between entropy generation and the average error of the approximate solutions.

Results and discussion

In the current study, entropy generation is examined in two different ways; firstly we concentrate on the local behavior of non-dimensional entropy generation, and secondly the correlation between the average normalized entropy generation and the average error is discussed.

Figure 1 demonstrates the temperature profiles of the integral solution as well as the exact solution. It is seen that the approximate temperature profiles differ from the exact one in the whole domain. This is expected as the

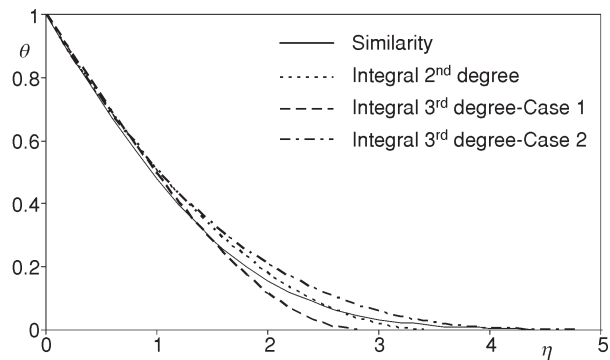


Figure 1. Temperature distribution of various solutions in a semi-infinite slab

simplification of exact solution to a polynomial function ignored some physical aspects of the real behavior of temperature field. It is seen that the temperature profile for the second degree polynomial temperature profile – eq. (11), almost overlap the exact solution at distances near the edge of the slab while it differs from the exact solution at longer distances. Also it is seen that the third degree polynomial temperature profile for the second case – eq. (14-2), has a more similar profile to the exact solution regarding its length of the temperature domain δ and its shape but shows slight temperature differences in the whole domain. The third degree polynomial temperature profile of the first case – eq. (14-1), is the least similar profile to the exact one regarding its length of temperature domain and its values at longer distances from the edge of the slab. Therefore, one can conclude that higher degree polynomials do not necessarily result in more accurate temperature profiles.

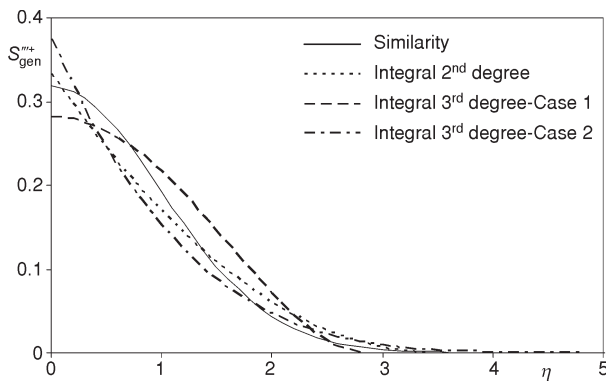


Figure 2. Non-dimensional entropy distribution of various solutions

entropy generation profile of the first case is more similar in shape to the exact profile than the second case which is due to the extra boundary condition applied at this region – eq. (13-1). Similarly, the dimensionless entropy generation values for the second case where the extra boundary condition is applied at δ – eq. (13-2), is more similar to the exact values at longer distances from the edge of the slab.

The local error of the dimensionless temperature profiles is shown in fig. 3. It is seen that all solutions have more accurate results at the beginning and the end of the temperature domain which is due to the existing boundary conditions at these two regions for all solutions.

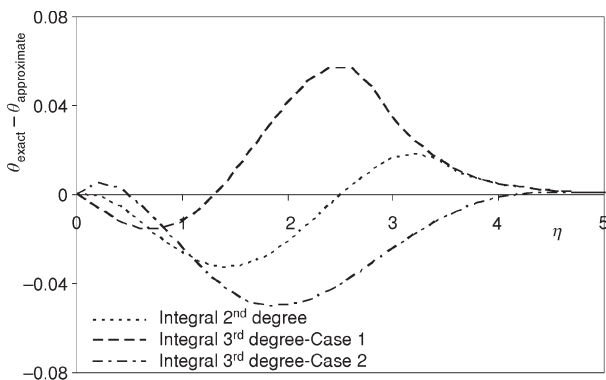


Figure 3. Temperature errors of the approximate solutions

Figure 2 shows the local non-dimensional entropy generation distribution of the three integral solutions as well as the exact solution. It is seen that the entropy generation shows a similar trend for all solutions. In other words, the entropy generation is in very low level inside the slab and increases consistently near the edge of the slab where higher temperature gradients exist. This can be due to the higher heat fluxes at distances near the edge of the slab. Comparing the two third degree polynomial temperature profiles, one can notice that at distances near the surface, the dimensionless entropy generation profile of the first case is more similar in shape to the exact profile than the second case which is due to the extra boundary condition applied at this region – eq. (13-1). Similarly, the dimensionless entropy generation values for the second case where the extra boundary condition is applied at δ – eq. (13-2), is more similar to the exact values at longer distances from the edge of the slab.

Also, one can recognize that the third degree polynomial temperature profile for the first case overpredicts the temperature values in most of the temperature domain while the one for the second case underpredicts the temperature. As for the second degree polynomial temperature profile, it is seen that the values of maximum errors are less than the other two integral solutions which can be counted as a good characteristic of this profile.

Table 1 shows the values of the average error for the integral solutions as well as the values of normalized entropy generation. Considering the whole domain of solution, it is seen that the third degree polynomial integral solution for the second case generates the minimum error among the three solutions while the one for the first case generates the maximum error. The same trend is observed for the values of average normalized entropy generation. Therefore, the similarity between entropy generation and exactness of the approximate solutions can be used as a tool to predict the error of the approximate solutions when there is no exact solution available.

Table 1. Average error and the normalized entropy generation

Integral solution	Average error	Average normalized entropy generation
2 nd degree polynomial (eq. 11)	0.68	0.38
3 rd degree polynomial – 1 st (eq. 14-1)	0.76	0.42
3 rd degree polynomial – 2 nd (eq. 14-2)	0.66	0.36

Concluding remarks

The entropy generation analysis of a one-dimensional steady conduction in a semi-infinite wall with constant temperature at its surface has been carried out with an exact solution method as well as an integral method resulting in three approximate solutions. Based on this study, it can be concluded that the normalized entropy generation rate behaves similarly to the average error. So, in more complicated cases when no exact solution is available, the normalized entropy generation rate can be used to examine the exactness of a given solution.

In addition to other applications of entropy generation analysis in thermo-fluid problems, from this work, one can recommend the analysis of entropy generation as a procedure for evaluating the solution methods and estimating the error of approximate solutions in the field of thermal problems.

Nomenclature

k	– thermal conductivity, [$\text{Wm}^{-1}\text{K}^{-1}$]
S_{gen}	– entropy generation, [WK^{-1}]
$S_{\text{gen}}^{\text{local}}$	– local entropy generation, [$\text{Wm}^{-3}\text{K}^{-1}$]
$\overline{S_{\text{gen}}}$	– dimensionless entropy generation
$\overline{S_{\text{gen}}}$	– average normalized entropy generation, [-]
T	– temperature, [K]
T_0	– temperature at the edge, [K]
t	– time, [s]
u	– velocity in the x-direction, [ms^{-2}]
v	– velocity in the y-direction, [ms^{-2}]
W_{lost}	– lost work, [W]
x	– x-direction, [m]

Greek letters

α	– thermal diffusivity, [m^2s^{-1}]
δ	– temperature domain, [m]
η	– dimensionless length
μ	– viscosity, [Nsm^{-2}]
θ	– dimensionless temperature

Subscripts

i	– initial
δ	– at the end of temperature domain

References

- [1] Bejan, A., A Study of Entropy Generation in Fundamental Convective Heat Transfer, *Trans. ASME*, 101 (1979), 4, pp.718-725
- [2] Bejan, A., Entropy Generation Minimization, CRC Press, Boca Raton, Fla., USA, 1996

- [3] Fowler, A. J., Bejan, A., Correlation of Optimal Sizes of Bodies with External Forced Convection Heat Transfer, *Int. Commun. in Heat and Mass Transfer*, 21(1994), 1, pp. 17-27
- [4] Shuja, S. Z., Zubair, S. M., Khan, M. S., Thermo Economic Design and Analysis of Constant Cross-Sectional Area Fins, *J. Heat Mass Transfer*, 34 (1999), 5, pp. 357-364
- [5] Sahin, A. S., Second Law Analysis of Laminar Viscous Flow Through a Duct Subjected to Constant Wall Temperature, *ASME J. Heat Transfer*, 120 (1998), 1, pp.76-83
- [6] Esfahani, J. A., Alizadeh, A., Thermodynamic Optimization of Geometry in Laminar Convective Heat Transfer through a Pipe with Constant Wall Temperature, *Proceedings*, First International Exergy, Energy and Environmental Symposium, Izmir, Turkey, 2003, pp. 77-83
- [7] Walsh, E., Myose, R., Davies, M., A Prediction Method for the Local Entropy Generation Rate in a Transitional Boundary Layer with a Free Stream Pressure Gradient, Stokes Research Institute, ASME, International Gas Turbine Institute, Turbo Expo (Publication) IGTI, 2002, Vol. 3A, pp. 637-646
- [8] Griffin, P. C., *et al.*, The Effect of Reynolds Number, Compressibility and Free Stream Turbulence on Profile Entropy Generation Rate, ASME, International Gas Turbine Institute, Turbo Expo (Publication) IGTI, 2002, Vol. 5A, pp. 61-69
- [9] Goodman, T. R., The Heat Balance Integral and its Application to Problems Involving a Change of Phase, *Trans. ASME*, 80 (1958), 2, pp.335-342
- [10] Goodman, T.R., Heat Balance Integral Further Considerations and Refinement, *J. Heat Transfer*, 83c (1961), 1, pp. 83-86
- [11] Goodman, T. R., Shea, J., The Melting of Finite Slabs, *J. Appl. Mech.*, 32 (1960), 1, pp. 16-24
- [12] Venkateshan, S. P., Solaiappan, O., Approximate Solution of Nonlinear Transient Heat Conduction in One Dimension, *Heat Mass Transfer*, 26 (1988), 2, pp. 229-233
- [13] Venkateshan, S. P., Rao, V. R., Approximate Solution of Nonlinear Transient Heat Conduction in Cylindrical Geometry, *Heat Mass Transfer*, 26 (1991), 1, pp. 97-102
- [14] Bejan, A., Thermodynamic of an Isothermal Flow: The Two-Dimensional Turbulent Jet, *Int. J. Heat Mass Transfer*, 34 (1991), 2, pp.407-413
- [15] Esfahani, J. A., Jafarian, M., Entropy Generation Analysis of a Flat Plate Boundary Layer with Various Solution Methods, *J. Iranica Scientia*, 12 (2005), 2, pp. 233-240
- [16] Esfahani, J. A., Roohi, E., Entropy Generation Analysis in Error Estimation of Approximate Solution Methods Implemented to Conductive Problem, *Proceedings*, 18th National and 7th ISHMT-ASME Heat and Mass Transfer Conference, IIT Guahati, India, 2006, pp. 832-839
- [17] Hristov, J., Research Note on a Parabolic Heat-Balance Integral Method with Unspecified Exponent: An Entropy Generation Approach in Optimal Profile Determination, *Thermal Science*, 13 (2009), 2, pp. 49-59 (present issue)
- [18] Poulidakos, D., *Conduction Heat Transfer*, Prentice Hall, New Jersey, USA, 1994

Authors' affiliations:

J. A. Esfahani (corresponding author)
Mechanical Engineering Department,
Ferdowsi University of Mashhad,
Mashhad 91755-1111, Iran
E-mail: jaesfahani@gmail.com

S. Koohi
Mechanical Engineering Department,
Ferdowsi University of Mashhad,
Mashhad, Iran

Paper submitted: October 15, 2008
Paper revised: October 25, 2008
Paper accepted: November 1, 2008