ANALYTICAL DETERMINATION OF FREE EDGE STRESSES IN COMPOSITE LAMINATES WITH PIEZOELECTRIC MATERIAL PROPERTIES

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Abstract. *An analytical solution is presented to determine the interlaminar stresses at the free edges of long symmetric cross-ply laminates with piezoelectric material properties subjected to uniform extension in the length direction. The displacement field is assumed within the framework of a second-order shear deformation theory. Also the electromechanical coupling is considered and it is assumed that the electric potential varies linearly through the thickness of the layers. Using the principle of minimum total potential energy, equilibrium equations are obtained in terms of stress, moment, and electric flux resultants. The coupled three-dimensional constitutive equations of piezoelectricity are considered. Finally, a set of ordinary differential equations with constant coefficients are obtained. The solutions of the equations are the unknown assumed functions for displacement field and electric potential. The numerical results obtained from this theory are compared with those obtained by other investigators and finite element method. Numerical results clearly indicate the singular behavior of interlaminar normal and shear stresses in the boundary-layer region of the laminate.*

1 INTRODUCTION

As a historical problem, free edge effect has been a considerable subject in laminated structures. This effect in composite laminates with piezoelectric material properties, as new smart materials should be considered, because of the improvements in the field of smart structures. These improvements are concerned with the development of composite materials with piezoelectric fibers in a resin matrix. These materials have both mechanical and piezoelectric properties and have the potential of application in a wide field of engineering structures. The analysis of structures with these materials can be performed by the use of adequate structural theories. A review of piezoelectric plate theories is given by Gopinathan et al [1].

The state of stress in the vicinity of free edges of laminates is three-dimensional, with nonzero through-thickness stresses. Stress-free boundaries lead to large gradients of interlaminar stresses in order to conserve equilibrium. These stresses exhibit a mathematical singularity at free edges and may cause interlaminar failures such as delamination. This effect can be more severe in the case of greater mismatch of layer properties [2]. A detailed review of analytical and numerical approaches for investigating the free edge effect is given by Kant and Swaminathan [3].

The coupled piezoelectric analysis of the free edge effect has been performed by Davi and Milazzo [4], who investigated a $[+45°/–45°]_s$ laminate under several loading conditions using boundary element formulation. In that work, no significant influence of piezoelectric coupling on the interlaminar stresses could be detected. Artel and Becker [5] analyzed the influence of piezoelectric coupling on interlaminar stresses and electric field near the free edge, using the finite element method. They showed that for the cross-ply laminates under consideration, some interlaminar stress components near the free edge become singular and in the coupled analysis case are usually of higher magnitude than in the uncoupled analysis. Erturk and Tekinalp and also Zhen and Wanji introduced new types of elements, for finite element approach, and analyzed the interlaminar stresses of piezoelectric composite laminated beams and plates [6,7].

As it is seen, yet there are no analytical investigations including the free edge effect in composite laminates with piezoelectric material properties. In this paper an analytic solution is obtained to determine the interlaminar stresses in the vicinity of free edges of a long symmetric composite laminate which has layers with piezoelectric material properties.

2 MATHEMATICAL FORMULATION

It is intended here to determine the interlaminar stresses in a long symmetric cross-ply laminate with piezoelectric properties subjected to unidirectional tension. The geometry of the laminate is shown in Figure 1. The formulation is restricted to linear piezo-elastic material behavior and small strain and displacements. In modeling the boundary-layer effect, the inclusion of the transverse normal strain is important because it is usually a significant stress in this region. The displacement and electric potential fields using a second-order shear deformation plate theory may be represented as:

$$\begin{aligned} &u_1(x,y,z)=\varepsilon_0 x,\\ &u_2(x,y,z)=v(y)+z^2\varphi_y(y),\\ &u_3(x,y,z)=z\psi_z(y),\\ &F(x,y,z)=zf(y). \end{aligned} \tag{1}$$

where u_1, u_2, and u_3 are displacements in x, y, and z directions, respectively, and F is the electric potential of a material point initially located at (x, y, z) in the undeformed laminate. Here u_2, u_3, and F are assumed to be independent of x, because the laminate is long in this direction. Also terms including odd orders of z for u_2 and even orders for u_3 and F are zero due to symmetry.

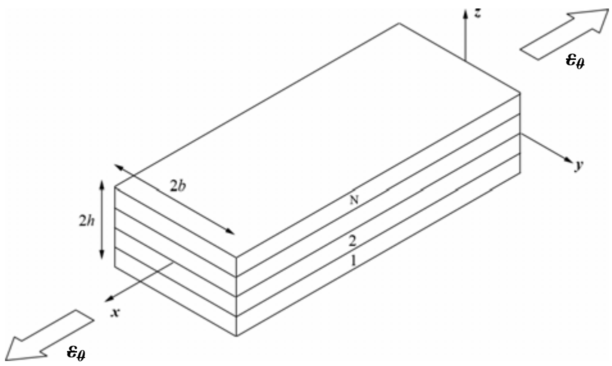

Figure 1. A symmetric long laminate

Substituting Eqs. (1) into the linear strain-displacement relations, the following results will be obtained [8]:

$$\begin{aligned} &\varepsilon_x=\varepsilon_0,\quad \varepsilon_y=v'(y)+z^2\varphi_y'(y),\quad \varepsilon_z=\psi_z(y),\\ &\gamma_{yz}=z[2\varphi_y(y)+\psi_z'(y)],\quad \gamma_{xz}=\gamma_{xy}=0. \end{aligned} \tag{2}$$

Also the components of the electric field vector are:

$$E_x=0,\quad E_y=-zf'(y),\quad E_z=-f(y). \tag{3}$$

where E_i's are the electric field components for the case of electric linearity. A prime in these equations indicates an ordinary derivative with respect to y. Using the principle of minimum total potential energy, equilibrium equations can be obtained in terms of stress, moment, and electric flux resultants as [8]:

$$\begin{aligned} &\frac{dN_y}{dy}=0,\qquad \frac{dP_y}{dy}-2R_y=0,\\ &\frac{dR}{dy}-N_z=0,\quad \frac{dQ_y^P}{dy}-N_z^P=0. \end{aligned} \tag{4}$$

In these equations

$$(N_y, P_y) = \int_{-h}^{h} \sigma_y (1, z^2) dz, \quad N_z = \int_{-h}^{h} \sigma_z dz, \quad R = \int_{-h}^{h} z\sigma_{yz} dz,$$
$$Q_y^P = \int_{-h}^{h} zD_y dz, \quad N_z^P = \int_{-h}^{h} D_z dz. \tag{5}$$

are the stress, moment and electric flux resultants. Here we study the free edge effect of a long composite laminate, for which the boundary conditions are:

$$N_y = P_y = R = Q_y^P = 0 \quad \text{at} \quad y = \pm b. \tag{6}$$

The full coupled three-dimensional piezo-elastic constitutive law of the kth orthotropic piezoelectric lamina with fiber orientations of 0º or 90º only, with respect to the laminate coordinate axes, is utilized as:

$$\{\sigma\}^{(k)} = [\bar{C}]^{(k)}\{\varepsilon\} - ([\bar{e}]^{(k)})^T\{E\},$$
$$\{D\}^{(k)} = [\bar{e}]^{(k)}\{\varepsilon\} + [\in]^{(k)}\{E\}. \tag{7}$$

where $\{\sigma\}$ and $\{\varepsilon\}$ are stress and strain contracted vectors, $\{D\}$ and $\{E\}$ are electric flux and field vectors, and $[\bar{C}]^{(k)}$, $[\bar{e}]^{(k)}$, and $[\in]^{(k)}$, are mechanical stiffness, piezoelectric coupling, and dielectric matrices of the kth layer. Upon substitution of Eqs. (2) and (3) into Eqs. (7) and the subsequent results into Eqs. (5), the stress and moment resultants are obtained which can be presented as follows:

$$N_y = A_{12}^0 \varepsilon_0 + A_{22}^0 v' + A_{22}^2 \varphi_y' + A_{23}^0 \psi_z + D_{32}^0 f,$$
$$P_y = A_{12}^2 \varepsilon_0 + A_{22}^2 v' + A_{22}^4 \varphi_y' + A_{23}^2 \psi_z + D_{32}^2 f,$$
$$N_z = A_{13}^0 \varepsilon_0 + A_{23}^0 v' + A_{23}^2 \varphi_y' + A_{33}^0 \psi_z + D_{33}^0 f,$$
$$R = A_{44}^2 (2\varphi_y + \psi_z') + D_{24}^2 f. \tag{8}$$

Also the electric flux resultants are:

$$Q_y^P = D_{24}^2 (2\varphi_y' + \psi_z') + H_{22}^2 f, \quad N_z^P = D_{31}^0 \varepsilon_0 + D_{32}^0 v' + D_{32}^2 \varphi_y' + D_{33}^0 \psi_z + H_{33}^0 f \tag{9}$$

where

$$A_{ij}^l = \int_{-h}^{+h} \bar{C}_{ij}^{(k)} z^l dz, \quad D_{ij}^l = \int_{-h}^{+h} -\bar{e}_{ij}^{(k)} z^l dz, \quad H_{ij}^l = \int_{-h}^{+h} \in_{ij}^{(k)} z^l dz. \tag{10}$$

Substitution of Eqs. (8) and (9) into equilibrium equations in (4) leads to the governing equations of equilibrium as:

$$A_{22}^0 v'' + A_{22}^2 \varphi_y'' + A_{23}^0 \psi_z' + D_{32}^0 f' = 0, \tag{11a}$$

$$A_{22}^2 v'' + A_{22}^4 \varphi_y'' - 4A_{44}^2 \varphi_y + (A_{23}^2 - 2A_{44}^2)\psi_z' + (D_{32}^2 - 2D_{24}^2) f' = 0, \tag{11b}$$

$$-A_{23}^0 v' + (2A_{44}^2 - A_{32}^2)\varphi_y' + A_{44}^2 \psi_z'' + A_{33}^0 \psi_z + D_{24}^2 f'' - D_{33}^0 f = A_{13}^0 \varepsilon_0, \tag{11c}$$

$$-D_{32}^0 v' + (2D_{24}^2 - D_{32}^2)\varphi_y' + D_{24}^2 \psi_z'' - D_{33}^0 \psi_z + H_{22}^2 f'' - H_{33}^0 f = D_{31}^0 \varepsilon_0. \tag{11d}$$

These equations are a set of ordinary differential equations with constant coefficients. Note that the first equilibrium equation in (4) or (11) tells that N_y is constant and from the first boundary condition in (6), we conclude that $N_y = 0$. Hence:

$$f = -\frac{1}{D_{32}^0}\left(A_{12}^0 \varepsilon_0 + A_{22}^0 v' + A_{22}^2 \varphi_y' + A_{23}^0 \psi_z\right). \tag{12}$$

Substituting f from Eq. (12) into Eqs. (11b), (11c), and (11d) yields:

$$[a]\begin{Bmatrix} v''' \\ v'' \\ v' \end{Bmatrix} + [b]\begin{Bmatrix} \varphi_y''' \\ \varphi_y'' \\ \varphi_y' \\ \varphi_y \end{Bmatrix} + [c]\begin{Bmatrix} \psi_z'' \\ \psi_z' \\ \psi_z \end{Bmatrix} = \varepsilon_0 \{d\}. \tag{13}$$

The coefficient matrices [a], [b], and [c], and the vector {d} appearing in Eqs. (13) are presented in Appendix. The particular solution of Eqs. (13) is given by:

$$v_p(y) = \gamma_1 \varepsilon_0 y, \quad \varphi_{yp}(y) = 0, \quad \psi_{zp}(y) = \gamma_2 \varepsilon_0. \tag{14}$$

where the constants γ_i's are given in Appendix. The complementary solution of Eqs. (13) may be found by solving the following homogeneous form:

$$[a]\begin{Bmatrix} v''' \\ v'' \\ v' \end{Bmatrix} + [b]\begin{Bmatrix} \varphi_y''' \\ \varphi_y'' \\ \varphi_y' \\ \varphi_y \end{Bmatrix} + [c]\begin{Bmatrix} \psi_z'' \\ \psi_z' \\ \psi_z \end{Bmatrix} = \{0\}. \tag{15}$$

To this end, let:

$$v_c(y) = Ae^{\lambda y}, \quad \varphi_{yc}(y) = Be^{\lambda y}, \quad \psi_{zc}(y) = Ce^{\lambda y}. \tag{16}$$

Upon substitution of Eqs. (16) into (15) results in:

$$\begin{bmatrix} a_{12}\lambda^2 & b_{12}\lambda^2 + b_{14} & c_{12}\lambda^2 \\ a_{21}\lambda^3 + a_{23}\lambda & b_{21}\lambda^3 + b_{23}\lambda & c_{21}\lambda^3 + c_{23}\lambda \\ a_{31}\lambda^3 + a_{33}\lambda & b_{31}\lambda^3 + b_{33}\lambda & c_{31}\lambda^3 + c_{33}\lambda \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \{0\} \tag{17}$$

For nontrivial solution, the determinant of the coefficient matrix should be equal to zero. This gives us seven values for λ, such a way that we can write:

$$\lambda_1 = \bar{\lambda}_1, \quad \lambda_2 = -\bar{\lambda}_1, \quad \lambda_3 = \bar{\lambda}_2, \quad \lambda_4 = -\bar{\lambda}_2, \quad \lambda_5 = \bar{\lambda}_3, \quad \lambda_6 = -\bar{\lambda}_3, \quad \lambda_7 = 0. \tag{18}$$

The boundary conditions at $y = -b$ and $y = +b$ appearing in (6) are identical. To be able to impose the boundary conditions only at one edge, say, at $y = +b$ and, this way, save some computational time, instead of writing the unknown functions as those in (16), we may write them as:

$$\begin{aligned} v_c(y) &= \sum_{i=1}^{3} A_i \sinh(\bar{\lambda}_i y) + \sum_{i=1}^{3} A_{i+3} \cosh(\bar{\lambda}_i y) + A_7, \\ \varphi_{yc}(y) &= \sum_{i=1}^{3} B_i \sinh(\bar{\lambda}_i y) + \sum_{i=1}^{3} B_{i+3} \cosh(\bar{\lambda}_i y) + B_7, \\ \psi_{zc}(y) &= \sum_{i=1}^{3} C_{i+3} \sinh(\bar{\lambda}_i y) + \sum_{i=1}^{3} C_i \cosh(\bar{\lambda}_i y) + C_7. \end{aligned} \tag{19}$$

Finally, the complete solution of Eqs. (11) are summation of the complementary and particular solutions. Since the variables $v(y)$ and $\varphi_y(y)$ should be odd functions of y and, on the other hand, the variable $\psi_z(y)$ must be even function of y, the complete solution can be represented as:

$$v(y) = \gamma_1 \varepsilon_0 y + \sum_{i=1}^{3} A_i \sinh(\bar{\lambda}_i y),$$

$$\varphi_y(y) = \sum_{i=1}^{3} B_i \sinh(\bar{\lambda}_i y),$$
$$\psi_z(y) = \gamma_2 \varepsilon_0 + \sum_{i=1}^{3} C_i \cosh(\bar{\lambda}_i y). \tag{20}$$

The constants B_i's and C_i's may be found in terms of A_i's by substituting Eqs. (20) into any two of the Eqs. (13), as:

$$B_i = \bar{B}_i A_i, \quad C_i = \bar{C}_i A_i. \tag{21}$$

The constant coefficients $\bar{B}_i$'s and $\bar{C}_i$'s are presented in Appendix. Finally, from Eq. (12) we obtain:

$$f(y) = \gamma_3 \varepsilon_0 + \sum_{i=1}^{3} \bar{D}_i A_i \sinh(\bar{\lambda}_i y). \tag{22}$$

The constants coefficients $\bar{D}_i$'s are also given in Appendix. The constants of integration A_i's may be found from the boundary conditions only at one edge, say, at $y = +b$. The remaining boundary conditions in (6) at $y = +b$ (or $-b$) are stated as:

$$\begin{aligned} &A_{22}^2 v' + A_{22}^4 \varphi_y' + A_{23}^2 \psi_z + D_{32}^2 f = -A_{12}^2 \varepsilon_0, \\ &A_{44}^2 (2\varphi_y + \psi_z') + D_{24}^2 f = 0, \\ &D_{24}^2 (2\varphi_y' + \psi_z') + H_{22}^2 f = 0. \end{aligned} \tag{23}$$

Now the displacement and electric field components are determined. Next, by using relations (2) and (7), the components of mechanical stress and electric flux are determined as functions of y and z. To improve the accuracy of the interlaminar stresses, we integrate the local equilibrium equations. For example σ_z and σ_{yz} at $z=t$ may be obtained as:

$$\begin{aligned} \sigma_z(y, z=t) &= \int_t^{2t} \frac{\partial \sigma_{yz}}{\partial y} dz = \frac{3t^2}{2} \sum_{i=1}^{3} \left[\bar{C}_{44}^{(1)} (2\bar{B}_i \bar{\lambda}_i + \bar{C}_i \bar{\lambda}_i^2) + \bar{e}_{24}^{(1)} \bar{D}_i \bar{\lambda}_i^2 \right] A_i \cosh(\bar{\lambda}_i y) \\ \sigma_{yz}(y, z=t) &= \int_t^{2t} \frac{\partial \sigma_y}{\partial y} dz = \sum_{i=1}^{3} \left[\bar{C}_{22}^{(1)} (t + \frac{7t^3}{3} \bar{B}_i) \bar{\lambda}_i^2 + \bar{C}_{23}^{(1)} t \bar{C}_i \bar{\lambda}_i + \bar{e}_{32}^{(1)} t \bar{D}_i \bar{\lambda}_i \right] A_i \sinh(\bar{\lambda}_i y). \end{aligned} \tag{24}$$

3 NUMERICAL RESULTS

Here, the numerical results of the free edge effect for a symmetric cross-ply laminate is investigated. The plate is under constant axial strain ε=0.1%. The laminate with the ply configuration $[0°/90°]_s$ is assumed to have the width $2b$ and thickness $2h$. Also the length of the laminate is large, so the displacement components are independent of x, as in Equ. (1). The width $2b$ is chosen ten times larger than the thickness $2h$ and each of the material layers are of equal thickness $t=h/2$ (see Figure 1).

The employed material is supposed to be idealized as a homogeneous piezoelectric orthotropic composite with the mechanical properties of T300/Epoxy and piezoelectric and electrical properties of PZT-5A [5]. These properties are shown in Table 1.

Elastic Stiffnesses	Piezoelectric coefficients
C_{11} = 137 GPa	$e_{31} = e_{31} = -5.4$ C/m^2
$C_{22} = C_{33}$ = 10.9 GPa	e_{33} = 15.8 C/m^2
$C_{12} = C_{13}$ = 3.75 GPa	$e_{24} = e_{15}$ = 12.3 C/m^2
C_{23} = 3 GPa	**Dielectric Properties**
C_{44} = 3.97 GPa	$\epsilon_{11} = \epsilon_{22} = 1730\,\epsilon_0$
$C_{55} = C_{66}$ = 5 GPa	$\epsilon_{33} = 1700\,\epsilon_0$
	$\epsilon_0 = 8.895 \times 10^{-12}$ C^2/Jm

Table 1: Mechanical and electric properties of the composite lamina

To check the correctness and accuracy of the present method, the results achieved from this theory will be compared with those obtained by Artel and Becker [5] and also by utilizing the commercial finite element package of ANSYS [10]. In the latter method, the mesh is refined till no significant change in stress distributions is obtained.

Numerical results are presented in the form of the interfacial and through the thickness stress distributions. For all results, the interlaminar stresses are computed by integrating the local equations of equilibrium.

Figure 2 shows the distribution of the normal interlaminar stress σ_z at the 0°/90° interface ($z=t$) of $[0°/90°]_s$ laminate. The boundary layer of edge effect and singular behavior of the stress is obviously seen in the figure. Generally, decent agreements between the present result and those of Artel and Becker [5] and finite element method are observed. The distribution of transverse shear stress σ_{yz} along the 0°/90° interface in the $[0°/90°]_s$ laminate is shown in Figure 3. The numerical value of σ_{yz} may not become zero at the free edge, contrary to what is expected. This is attributed to the facts that firstly the generalized stress resultants *R*, rather than σ_{yz}, is forced to vanish at the free edge, and secondly, because of the possible singularity that may exist at the 0°/90° interface [11-13].

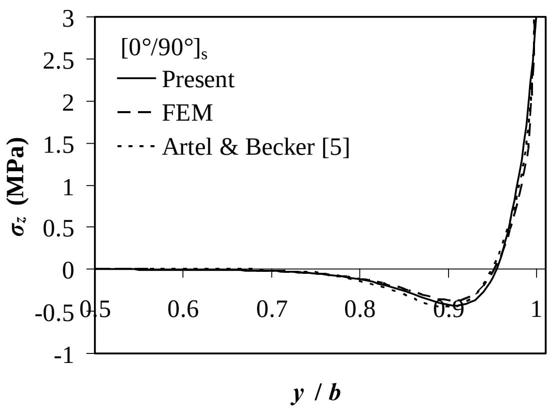

Figure 2. Distribution of interlaminar normal stress σ_z along the 0°/90° interface in $[0°/90°]_s$ laminate under uniform axial strain

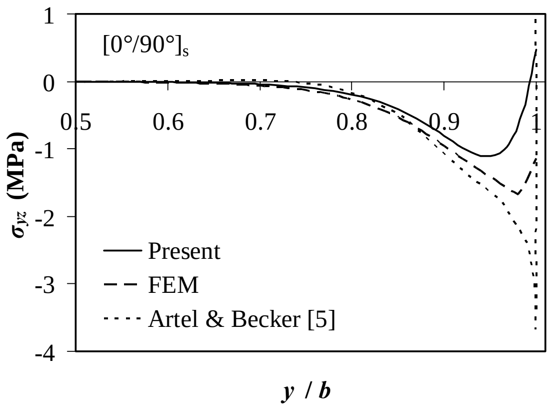

Figure 3. Distribution of interlaminar shear stress σ_{yz} along the 0°/90° interface in $[0°/90°]_s$ laminate under uniform axial strain

Figure 4 shows the distribution of interlaminar normal stress σ_z at the middle plane of the $[0°/90°]_s$ laminate. It is seen that there is good agreement between the present solutions and those of finite element method. Finally, Figure 5 shows through the thickness distribution of interlaminar normal stress σ_z in the upper two layers at the free edge $y=b$ of the $[0°/90°]_s$ laminate.

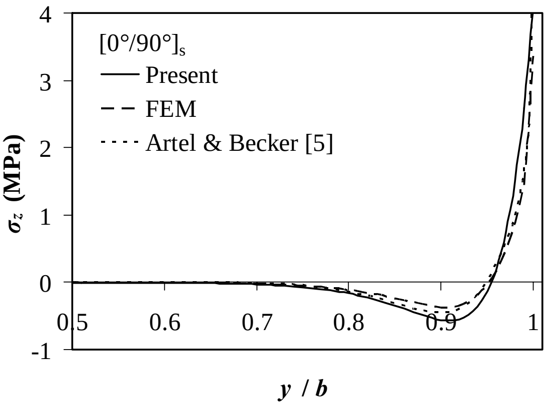

Figure 4. Distribution of interlaminar normal stress σ_z along the middle plane in $[0°/90°]_s$ laminate under uniform axial strain

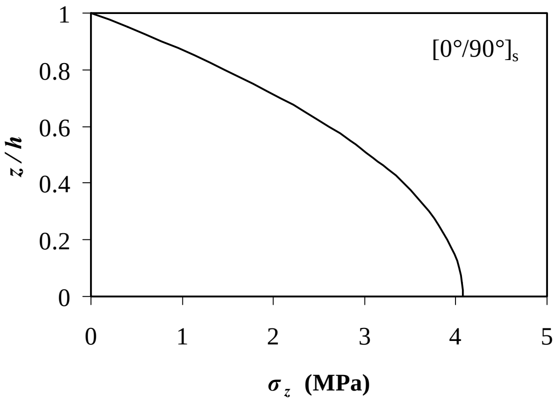

Figure 5. Through the thickness distribution of interlaminar normal stress σ_z in the upper two layers at the free edge $y = b$ in $[0°/90°]_s$ laminate under uniform axial strain

4 CONCLUSIONS

An analytical solution has been performed to investigate the free edge interlaminar stresses in long symmetric cross-ply composite laminates with piezoelectric material properties. Equilibrium equations have been obtained within the framework of a second-order shear deformation theory. Also the electromechanical coupling is considered and it is assumed that the electric potential varies linearly through the thickness of the piezoelectric layers. The singular behavior of the interlaminar stresses in the boundary-layer region is obviously seen in the numerical results. Furthermore, it should be noted that this approach considers the electromechanical coupling, which makes it an acceptable approach.

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APPENDIX

The coefficient matrices [*a*], [*b*], and [*c*] appearing in Eqs. (13) are defined as:

$$[a]=\begin{bmatrix}0 & a_{12} & 0\\ a_{21} & 0 & a_{23}\\ a_{31} & 0 & a_{33}\end{bmatrix},\quad [b]=\begin{bmatrix}0 & b_{12} & 0 & b_{14}\\ b_{21} & 0 & b_{23} & 0\\ b_{31} & 0 & b_{33} & 0\end{bmatrix},\quad [c]=\begin{bmatrix}0 & c_{12} & 0\\ c_{21} & 0 & c_{23}\\ c_{31} & 0 & c_{33}\end{bmatrix}$$

where

$$a_{12}=A_{22}^2+\frac{A_{22}^0}{D_{32}^0}\left(2D_{24}^2-D_{32}^2\right),\quad a_{21}=-\frac{A_{22}^0D_{24}^2}{D_{32}^0},\quad a_{23}=-A_{23}^0+\frac{A_{22}^0D_{33}^0}{D_{32}^0},\quad a_{31}=-\frac{A_{22}^0H_{22}^2}{D_{32}^0},$$

$$a_{33}=-D_{32}^0+\frac{A_{22}^0H_{33}^0}{D_{32}^0},\quad b_{12}=A_{22}^4+\frac{A_{22}^2}{D_{32}^0}\left(2D_{24}^2-D_{32}^2\right),\quad b_{14}=-4A_{44}^2,$$

$$b_{21}=-\frac{A_{22}^2D_{24}^2}{D_{32}^0},\quad b_{23}=2A_{44}^2-A_{23}^2+\frac{A_{22}^2D_{33}^0}{D_{32}^0},\quad b_{31}=-\frac{A_{22}^2H_{22}^2}{D_{32}^0},$$

$$b_{33}=2D_{24}^2-D_{32}^0+\frac{A_{22}^2H_{33}^0}{D_{32}^0},\quad c_{12}=A_{23}^2-2A_{44}^2+\frac{A_{23}^0}{D_{32}^0}\left(2D_{24}^2-D_{32}^2\right),\quad c_{21}=A_{44}^2+\frac{A_{23}^0D_{24}^2}{D_{32}^0},$$

$$c_{23}=-A_{33}^0+\frac{A_{23}^0D_{33}^0}{D_{32}^0},\quad c_{31}=D_{24}^2-\frac{A_{23}^0H_{22}^2}{D_{32}^0},\quad c_{33}=-D_{33}^0+\frac{A_{23}^0H_{33}^0}{D_{32}^0}.$$

Also the vector {*d*} in Eqs. (13) is defined as $\{d\}^T=[0,d_2,d_3]$. where,

$$d_2=A_{13}^0-\frac{A_{12}^0D_{33}^0}{D_{32}^0},\quad d_3=D_{31}^0-\frac{A_{12}^0H_{33}^0}{D_{32}^0}.$$

The constants γ_i 's in Eqs. (14) and (22) are given as:

$$\gamma_1=\frac{c_{33}d_2-c_{23}d_3}{a_{23}c_{33}-a_{33}c_{23}},\quad \gamma_2=\frac{a_{23}d_3-a_{33}d_2}{a_{23}c_{33}-a_{33}c_{23}},\quad \gamma_3=-\frac{\left(A_{12}^0+A_{22}^0\gamma_2+A_{23}^0\gamma_1\right)}{D_{32}^0}.$$

And finally the constants $\overline{B}_i$'s and $\overline{C}_i$'s in Eqs. (21) and $\overline{D}_i$'s in Eq. (22) are:

$$\overline{B}_i=-\overline{\lambda}_i^2\left[\left(-a_{31}c_{12}+a_{12}c_{31}\right)\overline{\lambda}_i^2-a_{33}c_{12}+a_{12}c_{33}\right]/\left[(-b_{31}c_{12}+b_{12}c_{31})\overline{\lambda}_i^4\right.$$
$$\left.+(-b_{33}c_{12}+b_{14}c_{31}+b_{12}c_{33})\overline{\lambda}_i^2+b_{14}c_{33}\right],$$
$$\overline{C}_i=-\frac{(a_{12}+b_{12}\overline{B}_i)\overline{\lambda}_i^2+b_{14}\overline{B}_i}{c_{12}},\quad \overline{D}_i=-\frac{(A_{22}^0+A_{22}^2\overline{B}_i)\overline{\lambda}_i+A_{23}^0\overline{C}_i}{D_{32}^0}.$$