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Solving fuzzy linear programming problems with piecewise linear membership functions by the determination of a crisp maximizing decision

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Abstract: In this paper, we concentrate on linear programming problems in which both the right-hand side and the technological coefficients are fuzzy numbers.We consider here only the case of fuzzy numbers with linear membership function. The determination of a crisp maximizing decision [2] is used for a defuzzification of these problems. The crisp problems obtained after the defuzzification are non-linear and non-convex in general. We propose here the ”augmented lagrangian penalty function method” and
use it for solving these problems. We also compare the new proposed method with well known ”fuzzy decisive set method”. Finally, we give illustrative example and
this solve by the new proposed method and compare the numerical solution with the solution obtained from fuzzy decisive set method.

Keywords: Fuzzy linear programming, fuzzy number, augmented lagrangian penalty function method, fuzzy decisive set method.

1 Introduction

A model in which the objective function is crisp– that is, has to be maximized or minimized– and in which the constraints are or partially fuzzy is no longer symmetrical. Fuzzy linear programming problem with fuzzy coefficients was formulated by Negoita [3] and called robust programming. Dubois and Prade [4] investigated linear fuzzy constraints. Tanaka and Asai [5] also proposed a formulation of fuzzy linear programming with fuzzy constraints and gave a method for its solution which bases on inequality relation between fuzzy numbers.

We consider linear programming problems in which both technological coefficients and right-hand-side numbers are fuzzy numbers. Each problem is first converted into an equivalent crisp problem. This is a problem of finding a point which satisfies the constraints and the goal with the maximum degree. The crisp problems, obtained by such a manner, can be non-linear (even non-convex), where the non-linearity arises in constraints. For solving these problems we use and compare two methods. One of them called the fuzzy decisive set method, as introduced by Sakawa and Yana [7]. In this methoda combination with the bisection method and phase one of the simplex method of linear programming is used to obtained a feasible solution. The second method we use, is the ”augmented lagrangian penalty method”. In this method a combines the algorithmic aspects of both Lagrangian duality methods and penalty function methods. For this kind of problems we consider, this method is applied to solve concrete examples.

The paper is outlined as follows. In section 2, we study the linear programming problem in which both technological coefficients and right-hand-side are fuzzy numbers. The general principles of the augmented lagrangian penalty method is presented in section 3. In section 4, we examine the application of this method and fuzzy decisive set method to concrete example.

2. Linear Programming Problems with fuzzy technological coefficients and fuzzy right-hand-side numbers

We consider a linear programming problem with fuzzy technological coefficients and fuzzy right-hand-side numbers:

$$\text{Maximize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} \tilde{a}_{ij} x_j \le \tilde{b}_i, \quad 1 \le i \le m \qquad (1)$$

$$x_j \ge 0, \quad 1 \le j \le n$$

where at least one $x_j > 0$ and $\tilde{a}_{ij}$ and $\tilde{b}_i$ are fuzzy numbers with the following linear membership functions:

$$\mu_{\tilde{a}_{ij}}(x) = \begin{cases} 1 & , \quad x < a_{ij} \\ \dfrac{a_{ij} + d_{ij} - x}{d_{ij}} & , \quad a_{ij} \le x \le a_{ij} + d_{ij} \\ 0 & , \quad x \ge a_{ij} + d_{ij} \end{cases}$$

Where $x \in \Re$ and $d_{ij} > 0$ for all $i = 1, \ldots, m$, $j = 1, \ldots, n.$ and

$$\mu_{\tilde{b}_i}(x) = \begin{cases} 1 & , \quad x < b_i \\ \dfrac{b_i + p_i - x}{p_i} & , \quad b_i \le x \le b_i + p_i \\ 0 & , \quad x \ge b_i + p_i \end{cases}$$

Where $p_i > 0$, for $i = 1, \ldots, m$.

The fuzzy set of the ith constraint, C_i, which is a subset of $\Re^n$ is defined by

$$\mu_{C_i}(x) = \begin{cases} 0 & , \quad b_i < \sum_{j=1}^{n} a_{ij} x_j \\ \dfrac{b_i - \sum_{j=1}^{n} a_{ij} x_j}{\sum_{j=1}^{n} d_{ij} x_j + p_i} & , \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i < \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j + p_i \\ 1 & , \quad b_i \ge \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j + p_i \end{cases} \qquad (2)$$

The problem we fact is the determination of an crisp function over a fuzzy domain. The approaches are conceivable:

1. the determination of the fuzzy set "decision".
2. the determination of a crisp "maximizing decision" by aggregating the objective function after appropriate with the constraints.

2-1. The determination of a fuzzy set "decision"

Orlovski [1977] suggests computing, for all α-level sets of the solution space, the corresponding optimal values of the objective function and considering as the fuzzy set "decision" the optimal values of the objective functions, with the degree of membership equal to the corresponding α-level of the solution space.

Definition 1. [Werners 1984]

Let $R_\alpha = \{x \mid x \in \Re^n, \mu_{C_i}(x) \ge \alpha\}$ be the α-level sets of the solution space and $N(\alpha) = \{x \mid x \in R_\alpha, f(x) = \sup_{y \in R_\alpha} f(y)\}$ the set of optimal solutions for each the set of optimal solutions for each α-level set, where f is an objective function.

The fuzzy set "decision" is then defined by the membership function

$$\mu_{opt}(x) = \begin{cases} \sup_{x \in N(\alpha)} \alpha & , \quad x \in \cup_{\alpha > 0} N(\alpha) \\ 0 & , \quad otherwise \end{cases}$$

For the case of linear programming, the determination of the $\mu_{opt}(x)$ can be obtained by parametric programming [Chanas 1983].

For each α, an LP of the following kind would have to be solved:

$$\text{Maximize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \quad \alpha \le \mu_{C_i}(x), \quad i = 1, \ldots, m \qquad (3)$$

$$x \in X.$$

In the following, we shall consider an approach that suggests a crisp solution depen-
dent on the solution space.

2-2. The determination of a crisp maximizing decision

We shall present a model that is particularly suitable for the type of linear programming model we are considering here. Werners [1984] suggests the following definition.

Definition 2. let $f :\to \Re^1$ be the objective function, $\tilde{R}$ a fuzzy region (solution space), and $S(\tilde{R})$ the support of this region, i.e. $S(\tilde{R}) = \{x \in \Re^n | \mu_{C_i}(x) > 0\}$. The maximizing set over the fuzzy region, $M\tilde{R}(f)$, is then defined by its membership function

$$\mu_{M\tilde{R}(f)}(x) = \begin{cases} 0 & , \quad f(x) \le \inf_{x\in S(\tilde{R})} f(x) \\ \dfrac{f(x) - \inf_{x\in S(\tilde{R})} f(x)}{\sup_{x\in S(\tilde{R})} f(x) - \inf_{x\in S(\tilde{R})} f(x)} & , \quad \inf_{x\in S(\tilde{R})} f(x) < f(x) < \sup_{x\in S(\tilde{R})} f(x) \\ 1 & , \quad \sup_{x\in S(\tilde{R})} f(x) \le f(x) \end{cases}$$

For defuzzification of the problem (1), we first calculate $f_1 = \inf_{x\in S(\tilde{R})} c^t x$ and $f_2 = \sup_{x\in S(\tilde{R})} c^t x$, for which $c^t x = \sum_{j=1}^n c_j x_j$. In fact f_1 and f_2 are the lower and upper bounds of the optimal values, respectively. The optimal values f_1 and f_2 can be defined by solving the following standard linear programming problems, for which we assume that all they have the finite optimal values.

$$f_1 = \text{Maximize} \quad \sum_{j=1}^n c_j x_j$$
$$\text{Subject to} \quad \sum_{j=1}^n (a_{ij} + d_{ij}) x_j \le b_i, \; 1 \le i \le m \qquad (5)$$
$$x_j \ge 0, \; 1 \le j \le n$$

$$f_2 = \text{Maximize} \quad \sum_{j=1}^n c_j x_j$$
$$\text{Subject to} \quad \sum_{j=1}^n (a_{ij} + d_{ij}) x_j \le b_i + p_i, \; 1 \le i \le m$$
$$x_j \ge 0, \; 1 \le j \le n \qquad (6)$$

The objective function takes values between f_1 and f_2 while technological coefficients take values between a_{ij} and $a_{ij} + d_{ij}$ and the right-hand side numbers take values between b_i and $b_i + p_i$.

The intersection of this maximizing set with the fuzzy set "decision" could then be used to compute a maximizing decision x0 as the solution with the highest degree of membership in this fuzzy set (see [2]). Hence, we have the following lemma:

Lemma 1. x is a crisp optimal solution of the problem (1) if and only if

$$\mu_{opt}(x) = \mu_{M\tilde{R}(f)}(x).$$

If we forgo alternative optimal solutions of the problem (3), according to above lemma we have:

$$\alpha = \frac{c^t x - \inf_{x\in S(\tilde{R})} c^t x}{\sup_{x\in S(\tilde{R})} c^t x - \inf_{x\in S(\tilde{R})} c^t x}.$$

By adding this the constraint to (3) and $\alpha \in [0,1]$, we have:

$$\text{Maximize} \quad \sum_{j=1}^n c_j x_j$$
$$\text{Subject to} \quad \sum_{j=1}^n c_j x_j - \alpha(f_2 - f_1) - f_1 = 0$$
$$\mu_{C_i}(x) \ge \alpha, \quad 1 \le i \le m \qquad (7)$$
$$x \ge 0, \quad 0 \le \alpha \le 1.$$

By using (2), the problem (6) can be written as

$$\text{Maximize} \quad \sum_{j=1}^n c_j x_j$$
$$\text{Subject to} \quad \sum_{j=1}^n c_j x_j - \alpha(f_2 - f_1) - f_1 = 0$$
$$\sum_{j=1}^n (a_{ij} + \alpha d_{ij}) x_j + \alpha p_i - b_i \le 0, \; 1 \le i \le m \qquad (8)$$
$$x \ge 0, \quad 0 \le \alpha \le 1.$$

Notice that, the constraints in problem (8) containing the cross product terms αx_i are not convex. Therefore the solution of this problem requires the special approach adopted for solving general non convex optimization problems.

3. The Augmented Lagrangian Penalty Function Method

The approach used is to convert the problem into an equivalent unconstrained problem. This method is called the penalty or the exterior penalty function method, in whicha penalty term is added to the objective function for any violation of the constraints. This method generates a sequence of infeasible points, hence its name, whose limit is an optimal solution to the original problem. The constraints are placed into the objective function via a penalty parameter in a way that penalizes any violation of the constraints.

In this section, we present and prove an important result that justifies using exterior penalty functions as a means for solving constrained problems.

Consider the following primal and penalty problems:

Primal Problem:

Minimize $-\sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} c_j x_j - \alpha(f_2 - f_1) - f_1 = 0$

$$\sum_{j=1}^{n}(a_{ij} + \alpha d_{ij})x_j + \alpha p_i - b_i \le 0,\ 1 \le i \le m \quad (9)$$

$$\alpha - 1 \le 0,\ -\alpha \le 0,\ -x_j \le 0,\ 1 \le j \le n$$

Penalty problem:

let ρ be a continuous function of the form

$$\begin{aligned}&\rho(x_1,\ldots,x_n,\alpha) = \\ &\sum_{i=1}^{m}\phi\left(\sum_{j=1}^{n}(a_{ij}+\alpha d_{ij})x_j + \alpha p_i - b_i\right) \\ &+ \sum_{j=1}^{n}\phi(-x_j) + \phi(-\alpha) + \phi(\alpha - 1) \\ &+ \psi\left(\sum_{j=1}^{n} c_j x_j - \alpha(f_2 - f_1) - f_1\right)\end{aligned} \quad (10)$$

Where ϕ and ψ are continuous functions satisfying the following:

$$\phi(y) = 0, \text{if } \ y \le 0 \ \text{ and } \ \phi(y) > 0, \text{if } \ y > 0$$
$$\psi(y) = 0, \text{if } \ y = 0 \ \text{ and } \ \psi(y) > 0, \text{if } \ y \ne 0 \quad (11)$$

3.1. Augmented Lagrangian Penalty Functions

Augmented lagrangian penalty functions for the problem (9) is as:

$$\begin{aligned}F_{AL}(x,\alpha,u,v) &= -\sum_{j=1}^{n} c_j x_j \\ &+ \sum_{i=1}^{m+n+2}\mu_i \max\left\{g_i(x,\alpha) + \frac{u_i}{2\mu_i}, 0\right\}^2 \\ &- \sum_{i=1}^{m+n+2}\frac{u_i^2}{2\mu_i} + vh(x,\alpha) + \eta h^2(x,\alpha)\end{aligned} \quad (12)$$

Where u_i and v are are lagrange multiplier. The following result provides the basis by virtue of which the AL penalty function can be classified as an *exact penalty function*.

Algorithm

The method of multipliers is an approach for solving nonlinear programming problems by using the augmented lagrangian penalty function in a manner that combines the algorithmic aspects of both Lagrangian duality methods and penalty function methods.

Initialization Step:

Select some initial Lagrangian multipliers $u = \bar{u}$ and $v = \bar{v}$ and positive values μ_i for each $i = 1,\ldots,m+n+2$ and η for the penalty parameters. Let (x^0,α_0) be a null vector, and denoted $VIOL(x^0,\alpha_0) = \infty$, where for any $x \in \Re^n$ and $\alpha \in \Re$,

$$VIOL(x,\alpha) = \max\{|h(x,\alpha)|, g_i(x,\alpha), i \in I = \{i : g_i(x,\alpha) > 0\}\}$$

is a measure of constraint violations. Put $k = 1$ and proceed to the "inner loop" of the algorithm.

Inner Loop: (penalty function minimization)

Solve the unconstrained problem to

$$\text{Minimize} \quad F_{AL}(x,\alpha,\bar{u},\bar{v}),$$

and let (x^k,α_k) denote the optimal solution obtained. If $VIOL(x^k,\alpha_k) = 0$, then stop with (x^k,α_k) as a KKT point, (Practically, one would terminate if $VIOL(x^k,\alpha_k)$ is lesser than some tolerance $\varepsilon > 0$). Otherwise, if

$$VIOL(x^k,\alpha_k) \le \frac{1}{4}VIOL(x^{k-1},\alpha_{k-1}),$$

proceed to the outer loop. On the other hand, if $VIOL(x^k,\alpha_k) > \frac{1}{4}VIOL(x^{k-1},\alpha_{k-1})$, then, for each constraint $i = 1,\ldots,m+n+2$ for which $g_i(x^k,\alpha_k) > \frac{1}{4}VIOL(x^{k-1},\alpha_{k-1})$ and $|h(x^k,\alpha_k)| > \frac{1}{4}VIOL(x^{k-1},\alpha_{k-1})$, replace the corresponding penalty parameter μ_i by $10\mu_i$ and $\eta = 10\eta$, respectively, and repeat this inner loop step.

Outer Loop: (Lagrange Multiplier Update)

Replace $\bar{u}$ by $\bar{u}_{new}$, where for each $i = 1,\ldots,m+n+2$

$$(\bar{u}_{new})_i = \bar{u}_i + \max\{2\mu g_i(x^k,\alpha_k), -\bar{u}_i\},$$

And also $\bar{v}$ by $\bar{v}_{new}$, where

$$(\bar{v}_{new}) = \bar{v} + 2\mu h(x^k,\alpha_k).$$

Increment k by 1, and return to the inner loop.

4. Numerical Example

Solve the optimization problem

$$\begin{aligned}\text{Maximize} \quad & 2x_1 + 3x_2 \\ \text{Subject to} \quad & \tilde{1}x_1 + \tilde{2}x_2 \le 4 \\ & \tilde{3}x_1 + \tilde{1}x_2 \le 6 \\ & x_1, x_2 \ge 0\end{aligned} \quad (13)$$

Which take fuzzy parameters as $\tilde{1}=L(1,1)$, $\tilde{2}=L(3,2)$, $\tilde{3}=L(3,2)$ and $\tilde{1}=L(1,3)$, as used by shaocheng [6]. That is, $(a_{ij})=\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}$, $(d_{ij})=\begin{bmatrix}1 & 3\\ 2 & 3\end{bmatrix}$, $(a_{ij}+d_{ij})=\begin{bmatrix}2 & 5\\ 5 & 4\end{bmatrix}$. For example, $\tilde{1}=L(1,1)$ is as:

$$\mu_{\tilde{1}}(x)=\begin{cases}1 & , \quad x<1\\ \dfrac{1+1-x}{1} & , \quad 1\le x\le 1+1\\ 0 & , \quad x\ge 1+1.\end{cases}$$

For solving this problem we must solve the following two subproblems:

$$\begin{aligned} f_1 = \text{Maximize} \quad & 2x_1+3x_2\\ \text{Subject to} \quad & x_1+2x_2\le 4\\ & 3x_1+x_2\le 6\\ & x_1,x_2\ge 0\end{aligned}$$

And

$$\begin{aligned} f_2 = \text{Maximize} \quad & 2x_1+3x_2\\ \text{Subject to} \quad & 2x_1+5x_2\le 4\\ & 5x_1+4x_2\le 6\\ & x_1,x_2\ge 0\end{aligned}$$

Optimal solution of these subproblems are $f_1=6.8$ and $f_2=3.06$. By using these optimal values, problem (13) can be reduced to the following equivalent non-linear programming problem:

$$\begin{aligned} \text{Maximize} \quad & 2x_1+3x_2\\ \text{Subject to} \quad & 2x_1+3x_2-3.74\alpha=3.06\\ & (1+\alpha)x_1+(2+3\alpha)x_2\le 4\\ & (3+2\alpha)x_1+(1+3\alpha)x_2\le 6 \qquad (14)\\ & 0\le\alpha\le 1, \quad x_1,x_2\ge 0\end{aligned}$$

Let's solve problem (14) by using the augmented lagrangian penalty function method.We first formulate it in the form

$$\begin{aligned} \text{Minimize} \quad & -2x_1-3x_2\\ \text{Subject to} \quad & h=-2x_1-3x_2+3.74\alpha+3.06=0\\ & g_1=(1+\alpha)x_1+(2+3\alpha)x_2-4\le 0\\ & g_2=(3+2\alpha)x_1+(1+3\alpha)x_2-6\le 0\\ & g_3=-x_1\le 0\\ & g_4=-x_2\le 0\\ & g_5=-\alpha\le 0\\ & g_6=\alpha-1\le 0\end{aligned}$$

Select initial Lagrangian multipliers and positive values for the penalty parameters
$\bar{u}_i=0$, $\bar{\mu}_i=0.1$, $1\le i\le 6$ and $\eta=0.1$, $v=0$

The starting point is taken as $x^0=(1 \quad 1 \quad 1)$ and $\varepsilon=0.00001$. since $VIOL(x^0,\alpha^0)=3>\varepsilon$, we going to inner loop. The augmented lagrangian penalty function is as

$$\begin{aligned} F_{AL}(x_1,x_2,\alpha,\bar{u},\bar{v}) &= -2x_1-3x_2\\ &+\frac{1}{10}[-2x_1-3x_2-\frac{187}{50}\alpha-\frac{153}{50}]^2\\ &+\frac{1}{10}[(1+\alpha)x_1+2(2+3\alpha)x_2-4]^2\\ &+\frac{1}{10}[(3+2\alpha)x_1+(1+3\alpha)x_2-6]^2\end{aligned}$$

With solving problem minimize F_{AL}, then we obtained

$(x^1,\alpha^1)=(1.10272471 \quad 1.66081313 \quad 0.19700472)$

And $VIOL(x^1,\alpha_1)=3.39109117>\varepsilon$ and also $VIOL(x^1,\alpha_1)>\frac{1}{4}VIOL(x^0,\alpha_0)$, hence we have $\mu_{new}=(1 \quad 0.1 \quad 0.1 \quad 0.1 \quad 0.1 \quad 0.1)$ and $\eta_{new}=1$. So repeat this inner loop step.

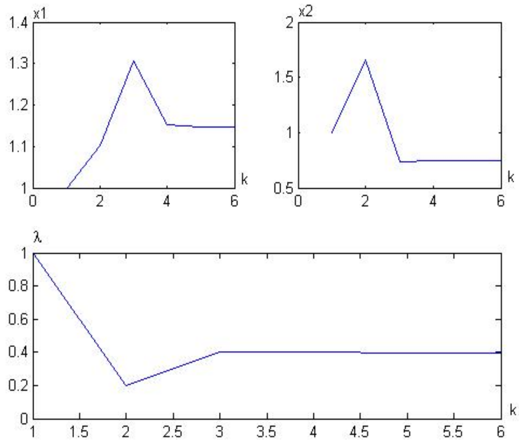
Since $VIOL(x^5,\alpha_5)=2.58968018\times 10^{-6}<\varepsilon$, so vector

$$(x_1^*,x_2^*)=(1.14730009 \quad 0.75069884)$$

is a solution to the problem (13) which has best membership grad

$$\alpha^*=0.39751314.$$

The progress of the algorithm of the method of the augmented lagrangian penalty function of example is depicted in the following figure.



Now we solve problem (14) by using the fuzzy decisive set method. We obtain value of λ at the thirty three iteration by using the fuzzy decisive set method.

$$\lambda = 0.39752688.$$

Note that, the optimal value of . found at the six iteration of the augmented lagrangian penalty function method is approximately equal to the optimal value of calculated at the thirty three iteration of the fuzzy decisive set method.

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