Communications in Statistics—Theory and Methods, 37: 1525–1542, 2008

ISSN: 0361-0926 print/1532-415X online
DOI: 10.1080/03610920801893897

Taylor & Francis
Taylor & Francis Group

Ordered Data Analysis

Nonparametric Confidence Intervals and Tolerance Limits Based on Minima and Maxima

M. RAZMKHAH, J. AHMADI, AND B. KHATIB

School of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran

Let $X_{i,j}$ $(1 \le i \le k, 1 \le j \le n_i)$ be independent random variables and for a fixed i, $X_{i,j}$'s, $(1 \le j \le n_i)$ be identically distributed random variables with survival function $\overline{F}_i = \overline{F}^{\alpha_i}$, where α_i is a known positive constant. Also, suppose M_i and M_i', respectively, denote the maximum and minimum of the ith sample. This article investigates the nonparametric confidence intervals for an arbitrary quantile of the distribution F and tolerance limits based on these statistics. Various cases have been studied and in each case, the nonparametric confidence intervals are obtained and exact expressions for the confidence coefficients of these confidence intervals are derived. A data set representing the time of successive failures of the air conditioning system on Boeing 720 jet aircraft is used to illustrate the results. Finally, the accuracy of the proposed procedure has been investigated, when α_i's are unknown via a simulation study.

Keywords Coverage probability; Multisampling plan; Order statistics; Proportional hazard model; Record values.

Mathematics Subject Classification Primary 62G30; Secondary 62G15, 62E15.

1. Introduction

There are many experiments in which only the lowest or highest observations are recorded. Some experiments which have been done in different periods of times, only *maxima* are applied, for example scientific competitions and in some other experiments only *minima* are exerted, like speed sports. Sometimes, both *minima* and *maxima* are used, say the lowest and highest temperature during a weak or a month. Therefore, it is worthwhile to use the extreme order statistics in a multisampling

Received November 29, 2006; Accepted September 27, 2007

The second author is a member of the Ordered and Spatial Data Center of Excellence of Ferdowsi University of Mashhad.

Address correspondence to J. Ahmadi, School of Mathematical Sciences, Ferdowsi University of Mashhad, P.O. Box 91775-1159 Mashhad, Iran; E-mail: ahmadi-j@ferdowsi.um.ac.ir

plan to do inference about various characteristics of the parent distribution, i.e., mean, standard deviation, quantiles, and so on. The goal of this article is to develop the procedures for constructing nonparametric confidence intervals for quantiles of a parent distribution and tolerance limits.

For any univariate distribution function F, the population quantile ξ_p of order p $(0 < p < 1)$ is defined

$$\xi_p = \inf\{x : F(x) \geq p\}.$$

The *quantiles* are used to construct the concentrations and deviations criteria. The order statistics play an important role in the inferences related to the quantiles; interested reader may refer to the books of Serfling (1980), Arnold et al. (1992), and David and Nagaraja (2003). In recent years, several articles had been published on nonparametric confidence intervals for quantiles based on usual order statistics. See, for example, Krewski (1976), Sathe and Lingras (1981), Hutson (1999), and Zielinski (2005). Similar results are obtained based on ranked set samples, see for example, Chen (2000), Ozturk and Deshpande (2006), Balakrishnan and Li (2006), Deshpande et al. (2006), and references therein. Also, in this direction, a little work based on record values have been developed in the literature; see Gulati and Padgett (1994), Ahmadi and Arghami (2003), and Ahmadi and Balakrishnan (2004, 2005).

Suppose that we would like to make a decision about the warranty of a product in present year (F) and the available data are related not only to the present year, but also to the $k-1$ previous years (F_i, $1 \leq i \leq k-1$) such that in each year the experiments are performed randomly with different sample sizes and the minimum and maximum are recorded. Moreover, assume that there exists a hierarchical relationship between them, for example the hazard rate function of the ith population is a proportion of the hazard rate function of the present population, i.e.,

$$h_i(t) = \alpha_i h(t),$$

where the proportionality rate, α_i, is a known positive constant, $h_i(\cdot)$ is the hazard rate function of the ith population, and $h(\cdot)$ is the hazard rate function of the present population. The aforementioned identity is well known in the literature as the proportional hazard rate model (PHRM) and is equivalent to $\overline{F}_i(t) = [\overline{F}(t)]^{\alpha_i}$, where $\overline{F}_i(\cdot)$ and $\overline{F}(\cdot)$ are the survival functions of the ith and the present population, respectively. Constructing the distribution-free confidence intervals for quantiles of F and tolerance limits based on these statistics is the aim of this article.

Section 2 contains some preliminary results. Section 3 is focused on confidence intervals for quantiles of F based on extreme order statistics in a multisampling plan. Various cases have been studied and in each case, the nonparametric confidence intervals are obtained and exact expressions for the confidence coefficients of these confidence intervals are derived. Such intervals are distribution-free in that the corresponding coverage probabilities are known without any assumptions about the present population other than that its distribution function is continuous. In Sec. 4, we derive tolerance intervals and limits in terms of maxima and minima. We illustrate our proposed procedure on a real data set in Sec. 5. At the end, we present some remarks including how the results of the article can be used when α_i's are unknown. The effect of estimated α_i's on the coverage probabilities via a simulation study is investigated.

2. Preliminaries

P: Consider the following data set:

$$\begin{array}{cccc} X_{1,1} & X_{1,2} & \cdots & X_{1,n_1} \\ X_{2,1} & X_{2,2} & \cdots & X_{2,n_2} \\ \vdots & \vdots & \vdots & \vdots \\ X_{k,1} & X_{k,2} & \cdots & X_{k,n_k} \end{array}$$

where $X_{i,j}$ $(1 \le i \le k, 1 \le j \le n_i)$ are independent. Moreover, for a fixed i, $X_{i,j}$'s $(1 \le j \le n_i)$ are identically distributed random variables with cumulative distribution function (cdf)

$$F_i(x) = 1 - [1 - F(x)]^{\alpha_i} = 1 - [\overline{F}(x)]^{\alpha_i}, \tag{1}$$

where $F(\cdot)$, cdf of the present population, is an absolutely continuous distribution function and α_i is a known positive constant, such that $\alpha_k = 1$ and so $F_k = F$. As mentioned in Sec. 1, the identity (1) is equivalent to the PHRM. For example in the Olympic games, n_i is the population size of the athletes of the world at the ith Olympic and α_i's is the proportionality rate of the ith Olympic with respect to the present Olympic, $i = 1, 2, \ldots, k$.

Let $M_i, i = 1, 2, \ldots, k$ be the maximum of the random sample of size n_i; that is, $M_i = \max\{X_{i,1}, X_{i,2}, \ldots, X_{i,n_i}\}$ and M_i' is the corresponding minimum. Then M_i's $(1 \le i \le k)$ are independent random variables with cdf

$$F_{M_i}(x) = (1 - [\overline{F}(x)]^{\alpha_i})^{n_i}, \quad i = 1, 2, \ldots, k. \tag{2}$$

Also, M_i''s $(1 \le i \le k)$ are independent random variables with cdf

$$F_{M_i'}(x) = 1 - [\overline{F}(x)]^{n_i \alpha_i}, \quad i = 1, 2, \ldots, k. \tag{3}$$

In the present article, we will attempt to construct nonparametric confidence intervals for quantiles of F based on M_i's and M_i''s by assuming that α_i's are known positive constants. First, we present some known lemmas, which will be used to prove the new results in this article, also they maybe independent interest.

Lemma 2.1. *Let X_i, $i = 1, \ldots, k$ be independent and non identical random variables such that X_i have F_i distribution and denote the ith order statistic of X_j's $(1 \le j \le k)$ by $X_{i:k}$. Then*

$$P(X_{i:k} \le x) = \sum_{r=i}^{k} \sum_{\Gamma_{r,k}} \prod_{s=1}^{r} F_{X_{t_s}}(x) \prod_{s=r+1}^{k} \overline{F}_{X_{t_s}}(x),$$

where the summation index $\Gamma_{r,k}$ extends over all permutations $(t_1, \ldots, t_k)$ of $\{1, \ldots, k\}$ for which $t_1 < \cdots < t_r$ and $t_{r+1} < \cdots < t_k$.

Lemma 2.2. *Under the assumptions of Lemma* 2.1, *the joint pdf of* $(X_{r:k}, X_{s:k})$, $s > r$, *is as follows*

$$f_{r,s:k}(x, y) = [(r-1)!(s-r-1)!(k-s)!]^{-1} \times \sum_{\Delta_k} \left\{ \prod_{i=1}^{r-1} F_{t_i}(x) f_{t_r}(x) \prod_{i=r+1}^{s-1} [F_{t_i}(y) - F_{t_i}(x)] f_{t_s}(y) \prod_{i=s+1}^{k} \overline{F}_{t_i}(y) \right\},$$

where the summation index Δ_k *extends over all permutations* $(t_1, \ldots, t_k)$ *of* $\{1, \ldots, k\}$.

Lemma 2.3. *Under the assumptions of Lemma* 2.1, *the pdf of* $W_{r,s:k} = X_{s:k} - X_{r:k}$, $(s > r)$ *is given by*

$$f_{W_{r,s:k}}(w) = [(r-1)!(s-r-1)!(k-s)!]^{-1} \times \sum_{\Delta_k} \int \left\{ \prod_{i=1}^{r-1} F_{t_i}(x) f_{t_r}(x) \prod_{i=r+1}^{s-1} [F_{t_i}(x+w) - F_{t_i}(x)] f_{t_s}(x+w) \times \prod_{i=s+1}^{k} \overline{F}_{t_i}(x+w) \right\} dx,$$

where the summation index Δ_k *is defined in Lemma* 2.2.

See David and Nagaraja (2003) for proofs of Lemmas 2.1, 2.2, and 2.3.

Lemma 2.4. *Let* $X_1, \ldots, X_n$ *be iid random variables with continuous distribution function F and let* $X_{r:n}$ *denote the rth order statistic in a sample at size n. Then* $F(X_{r:n}) \stackrel{\mathrm{d}}{=} U_{r:n}$, *where* $U_{r:n}$ *is the rth order statistic in a sample of size n from a uniform distribution over* $(0, 1)$ *and* $\stackrel{\mathrm{d}}{=}$ *means identical in distribution.*

The proof of Lemma 2.4 is obvious. By Lemma 2.4 we get the following result.

Lemma 2.5. *Let* $X_{i,1}, X_{i,2}, \ldots, X_{i,n_i}$, $i = 1, 2, \ldots, k$ *be iid random variables from cdf* $1 - [\overline{F}(x)]^{\alpha_i}$, *then*

$$F(M_i) \stackrel{\mathrm{d}}{=} 1 - (1 - U_{n_i:n_i})^{1/\alpha_i} \stackrel{\mathrm{d}}{=} 1 - (U_{1:n_i})^{1/\alpha_i}, \tag{4}$$

$$F(M_i') \stackrel{\mathrm{d}}{=} 1 - (1 - U_{1:n_i})^{1/\alpha_i} \stackrel{\mathrm{d}}{=} 1 - (U_{n_i:n_i})^{1/\alpha_i}, \tag{5}$$

where $M_i = \max\{X_{i,1}, \ldots, X_{i,n_i}\}$ *and* $M_i' = \min\{X_{i,1}, \ldots, X_{i,n_i}\}$.

3. Confidence Intervals

The interval (L, U) is a confidence interval for $F^{-1}(p) = \xi_p$ with confidence coefficient γ if $P(L \le \xi_p \le U) = \gamma$. Setting $L = -\infty$ $[U = \infty]$, U $[L]$ is upper [lower] confidence limit for ξ_p with level γ. By assumptions **P** which are indicated in Sec. 2, we intend to show that how one can use M_i's and M_i''s $(1 \le i \le k)$ in order to construct confidence interval for $F^{-1}(p) = \xi_p$ with coverage probability free of F, as long as it is continuous.

3.1. *Confidence Intervals Based on Minima and Maxima*

For $k = 1$, in our plan, $\alpha_1 = 1$. So, by using Lemma 2.5, we immediately have

$$\begin{aligned} P(M_1' \le \xi_p \le M_1) &= P(X_{1:n_1} \le \xi_p \le X_{n_1:n_1}) \\ &= P(X_{1:n_1} \le \xi_p) - P(X_{n_1:n_1} \le \xi_p) \\ &= P\big(F(X_{1:n_1}) \le p\big) - P\big(F(X_{n_1:n_1}) \le p\big) \\ &= P(U_{1:n_1} \le p) - P(U_{n_1:n_1} \le p) \\ &= 1 - (1-p)^{n_1} - p^{n_1}. \end{aligned} \tag{6}$$

Now, consider $k \ge 2$. Arrange M_i's and M_i''s $(1 \le i \le k)$ in ascending order and denote the rth order statistic of the set $\{M_1', M_1, \ldots, M_k', M_k\}$ by $V_{r:2k}$. The problem is to determine the coverage probability of the event $(V_{i:2k} \le \xi_p \le V_{j:2k})$, $j > i$. First, we obtain the cdf of $V_{j:2k}$, which is stated in the following theorem.

Theorem 3.1. *Let M_i' and M_i be corresponding minima and maxima of the ith sample with distribution F_i, respectively. Then the distribution function of $V_{j:2k}$ is*

$$P(V_{j:2k} \le x) = \sum_{r=j}^{2k} \eta(r, x),$$

where

$$\eta(r, x) = \sum_{i=0}^{c} \sum_{A_{\zeta_r - i, \xi_r + i, k}} \prod_{s=1}^{\zeta_r - i} F_{M_{t_s}}(x) \prod_{s=\zeta_r - i + 1}^{\xi_r + i} \big[F_{M_{t_s}'}(x) - F_{M_{t_s}}(x)\big] \prod_{s=\xi_r + i + 1}^{k} \overline{F}_{M_{t_s}'}(x), \tag{7}$$

and $c = min\{\zeta_r, k - \xi_r\}$. For any positive integer a from 1 to k if $r = 2a$ then $\zeta_r = \xi_r = \frac{r}{2}$ and if $r = 2a + 1$ then $\zeta_r = \frac{r-1}{2}$ and $\xi_r = \frac{r+1}{2}$. $A_{i_1, i_2, k}$ extends over all permutations of $(t_1, \ldots, t_k)$ from $\{1, \ldots, k\}$ such that $t_1 < \cdots < t_{i_1}$, $t_{i_1+1} < \cdots < t_{i_2}$ and $t_{i_2+1} < \cdots < t_k$.

Proof. We can write

$$\begin{aligned} P(V_{j:2k} \le x) &= P(\text{at least } j \text{ of } M_i \text{ or } M_i', i = 1, \ldots, k \text{ are at most } x) \\ &= \sum_{r=j}^{2k} \eta(r, x), \end{aligned}$$

where

$$\eta(r, x) = P(\text{exactly } r \text{ of } M_i \text{ or } M_i', i = 1, \ldots, k \text{ are at most } x).$$

It is considerable that by extracting k independent samples, we have $2k$ statistics as $M_1' < M_1, M_2' < M_2, \ldots, M_k' < M_k$. Now, by a careful scrutiny of the details and a simplification of computations follows

$$\eta(r, x) = \begin{cases} \eta_1(r, x), & \text{if } r \text{ is odd,} \\ \eta_2(r, x), & \text{if } r \text{ is even,} \end{cases}$$

where

$$\eta_1(r,x)=\sum_{i=0}^{c_1}\sum_{A_{\frac{r-1}{2}-i,\frac{r+1}{2}+i,k}}\prod_{s=1}^{\frac{r-1}{2}-i}F_{M_{t_s}}(x)\prod_{s=\frac{r-1}{2}-i+1}^{\frac{r+1}{2}+i}\left[F_{M'_{t_s}}(x)-F_{M_{t_s}}(x)\right]\prod_{s=\frac{r+1}{2}+i+1}^{k}\overline{F}_{M'_{t_s}}(x),$$

and

$$\eta_2(r,x)=\sum_{i=0}^{c_2}\sum_{A_{\frac{r}{2}-i,\frac{r}{2}+i,k}}\prod_{s=1}^{\frac{r}{2}-i}F_{M_{t_s}}(x)\prod_{s=\frac{r}{2}-i+1}^{\frac{r}{2}+i}\left[F_{M'_{t_s}}(x)-F_{M_{t_s}}(x)\right]\prod_{s=\frac{r}{2}+i+1}^{k}\overline{F}_{M'_{t_s}}(x),$$

where $c_1=min\left\{\frac{r-1}{2},k-\frac{r+1}{2}\right\}$ and $c_2=min\left\{\frac{r}{2},k-\frac{r}{2}\right\}$. The result follows by combining the above expressions. □

Similar to (6), by using the expression in Eq. (7) and Lemma 2.5, we have

$$\begin{aligned}\delta(i,j;p)&=P(V_{i:2k}\le\xi_p\le V_{j:2k}),\quad j>i\\&=\sum_{r=i}^{j-1}\pi(r;p),\end{aligned}\tag{8}$$

where

$$\begin{aligned}\pi(r;p)=\sum_{i=0}^{c}\sum_{A_{\zeta_r-i,\xi_r+i,k}}\prod_{s=1}^{\zeta_r-i}(1-q^{\alpha_{t_s}})^{n_{t_s}}\prod_{s=\zeta_r-i+1}^{\xi_r+i}\left(1-q^{\alpha_{t_s}n_{t_s}}-(1-q^{\alpha_{t_s}})^{n_{t_s}}\right)\\\times q^{\sum_{s=\xi_r+i+1}^{k}\alpha_{t_s}n_{t_s}},\end{aligned}\tag{9}$$

such that $q=1-p$ and c, ζ_r, ξ_r and $A_{i_1,i_2,k}$ are defined in Theorem 3.1. Thus, we have a confidence interval $(V_{i:2k},V_{j:2k})$ for $\xi_p=F^{-1}(p)$ whose confidence coefficient is free of F.

If p, α_r, n_r $(r=1,\ldots,k)$ and the desired confidence level γ are specified, we can choose i and j so that $\delta(i,j;p)$ achieve to γ. Notice that the choice of i and j is not unique, the one that minimizes the expected width of the confidence interval appears reasonable. By Theorem 3.1, we have:

$$\begin{aligned}E(V_{j:2k}-V_{i:2k})&=\int_0^\infty[P(V_{j:2k}\ge x)-P(V_{j:2k}\le -x)]dx\\&\quad-\int_0^\infty[P(V_{i:2k}\ge x)-P(V_{i:2k}\le -x)]dx\\&=\int_0^\infty[P(V_{i:2k}\le x)-P(V_{j:2k}\le x)]dx\\&\quad+\int_0^\infty[P(V_{i:2k}\le -x)-P(V_{j:2k}\le -x)]dx\\&=\sum_{r=i}^{j-1}\int_0^\infty[\eta(r,x)+\eta(r,-x)]dx,\end{aligned}$$

where $\eta(r,x)$ is defined in (7). Because of the fact that the above expression is a step function of $j-i$, one can choose i and j as close together as possible to achieve a

specified confidence level. For given p, α_r's, and n_r's, the value of $\delta(i, j; p)$ can be computed by numerical methods.

3.1.1. *Special Case* ($\alpha_i = 1$ *and* $n_i = n$, $1 \le i \le k$). Suppose that $\alpha_i = 1$ and $n_i = n$ for $i = 1, \ldots, k$; i.e., k independent samples at size n are extracted from the same population F. Let $M_i^{*'} = F(M_i')$, $M_i^* = F(M_i)$, and $V_{r:2k}^* = F(V_{r:2k})$, then we have the following results.

Lemma 3.2. *Under the above assumptions, we have*

$$V_{r:2k}^* \overset{\mathrm{d}}{=} 1 - V_{2k-r+1:2k}^*, \quad \textit{for } 1 \le r \le k. \tag{10}$$

Proof. In view of Lemmas 2.4 and 2.5, $M_i^* = F(M_i) \overset{\mathrm{d}}{=} U_{n:n}$ and $M_i^{*'} = F(M_i') \overset{\mathrm{d}}{=} U_{1:n}$, $i = 1, \ldots, k$, respectively. Therefore, $M_1^*, \ldots, M_k^* \overset{\text{iid}}{\sim} Beta(n, 1)$ and $M_1^{*'}, \ldots, M_k^{*'} \overset{\text{iid}}{\sim} Beta(1, n)$. So, $F_{M_i^*}(x) = x^n$ and $F_{M_i^{*'}}(x) = 1 - (1 - x)^n$, $i = 1, \ldots, k$, for $0 < x < 1$. Notice that $V_{r:2k}^*$ is identical in distribution with the rth order statistic of the set $\{M_1^{*'}, M_1^*, M_2^{*'}, M_2^*, \ldots, M_k^{*'}, M_k^*\}$. By Theorem 3.1, we get

$$\begin{aligned} P(V_{j:2k}^* \le x) = \sum_{r=j}^{2k} \sum_{i=0}^{c} \binom{k}{\zeta_r - i,\ \xi_r - \zeta_r + 2i,\ k - \xi_r - i} x^{n(\zeta_r - i)} \\ \times \left(1 - (1-x)^n - x^n\right)^{\xi_r - \zeta_r + 2i} (1-x)^{n(k - \xi_r - i)}, \end{aligned}$$

where c, ζ_r, and ξ_r are defined in Theorem 3.1. By the aforementioned identity, one can verify (10), for a fixed k. For $k = 1$, it is obvious; we prove for $k = 2$.

$$\begin{aligned} P\left(V_{1:4}^* \le x\right) &= 1 - (1-x)^{2n} \\ &= P\left(1 - V_{4:4}^* \le x\right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} P\left(V_{2:4}^* \le x\right) &= 1 + 2(1-x)^n\left(x^n - 1\right) + (1-x)^{2n} \\ &= P\left(1 - V_{3:4}^* \le x\right). \end{aligned}$$

Hence the result is followed for $k = 1$ and 2, for others, (10) can be deduced similarly. □

Theorem 3.3. *Let* $\alpha_i = 1$ *and* $n_i = n$ $(i = 1, \cdots, k)$, *then*:

(i) $[V_{r:2k}, V_{s:2k}]$ *is the confidence interval for* ξ_p *with confidence coefficient* γ *if and only if* $[V_{2k-s+1:2k}, V_{2k-r+1:2k}]$ *is the confidence interval for* ξ_{1-p} *with confidence coefficient* γ, *i.e.*,

$$P\left(V_{r:2k} \le \xi_p \le V_{s:2k}\right) = \gamma \Leftrightarrow P\left(V_{2k-s+1:2k} \le \xi_{1-p} \le V_{2k-r+1:2k}\right) = \gamma.$$

(ii) *$V_{r:2k}$ is the upper confidence limit for ξ_p with confidence coefficient γ if and only if $V_{2k-r+1:2k}$ is the lower confidence limit for ξ_{1-p} with confidence coefficient γ, i.e.,*

$$P\big(\xi_p \le V_{r:2k}\big) = \gamma \Leftrightarrow P\big(V_{2k-r+1:2k} \le \xi_{1-p}\big) = \gamma.$$

Proof. Using (10), the result in (i) can be established as follows:

$$\begin{aligned} &P(V_{r:2k} \le \xi_p \le V_{s:2k}) = \gamma \\ &\quad \Leftrightarrow P\big(V^*_{r:2k} \le p \le V^*_{s:2k}\big) = \gamma \\ &\quad \Leftrightarrow P\big(1 - V^*_{2k-r+1:2k} \le p \le 1 - V^*_{2k-s+1:2k}\big) = \gamma \\ &\quad \Leftrightarrow P\big(V^*_{2k-s+1:2k} \le 1 - p \le V^*_{2k-r+1:2k}\big) = \gamma \\ &\quad \Leftrightarrow P(V_{2k-s+1:2k} \le \xi_{1-p} \le V_{2k-r+1:2k}) = \gamma. \end{aligned}$$

The proof for part (ii) is similar to that of part (i). □

3.2. *Confidence Intervals Based on Maxima*

Suppose that we observe just the maxima, i.e., M_i's. Denote the rth order statistic of the set $\{M_1, M_2, \ldots, M_k\}$ by $M_{r:k}$. In this subsection, we find the confidence interval for the ξ_p whose endpoints are $M_{r:k}$'s. By Lemma 2.1, for $j > i$, we have

$$\begin{aligned} \varphi(i, j; p) &= P(M_{i:k} \le \xi_p \le M_{j:k}) \\ &= \sum_{r=i}^{j-1} \sum_{\Gamma_{r,k}} \prod_{s=1}^{r} (1 - q^{\alpha_{t_s}})^{n_{t_s}} \prod_{s=r+1}^{k} \big(1 - \big(1 - q^{\alpha_{t_s}}\big)^{n_{t_s}}\big), \end{aligned} \tag{11}$$

where $\Gamma_{r,k}$ is defined in Lemma 2.1. Thus, we have a confidence interval $(M_{i:k}, M_{j:k})$, $j > i$, for ξ_p whose confidence coefficient is free of F.

For special case if $\alpha_i = 1$ and $n_i = n$ $(i = 1, \ldots, k)$, the identity (11) is simplified as follows:

$$\varphi(i, j; p) = \sum_{r=i}^{j-1} \binom{k}{r} p^{nr} \big(1 - p^n\big)^{k-r}. \tag{12}$$

In this case, the results are similar to Arnold et al. (1992, p. 183) putting p^n instead of p.

It is evident from (12) that the confidence coefficients only based on *maxima* (φ_0) can not exceed $1 - p^{nk} - (1 - p^n)^k$ for given k and n. Therefore, confidence intervals for ξ_p based on just *maxima* are not possible for $\varphi_0 > 1 - p^{nk} - (1 - p^n)^k$.

3.3. *Confidence Intervals Based on Minima*

It is well known that the *minima* provides more information than *maxima* about the left tail of the distributions when the underlying distribution is skewed to right. It is, therefore, reasonable to use *minima* to construct confidence intervals for ξ_p, if $p < 0.5$. In this subsection, we present the similar results as previous subsection, in terms of *minima*. Let $M'_{r:k}$ denote the rth order statistic of the set $\{M'_1, M'_2, \ldots, M'_k\}$.

The problem is to determine the coverage probability of the event $(M'_{i:k} \le \xi_p \le M'_{j:k})$, $j > i$. By Lemma 2.1, we have

$$\begin{aligned}\psi(i, j; p) &= P(M'_{i:k} \le \xi_p \le M'_{j:k}) \\ &= \sum_{r=i}^{j-1} \sum_{\Gamma_{r,k}} \prod_{s=1}^{r} \left(1 - q^{\alpha_{t_s} n_{t_s}}\right) q^{\sum_{s=r+1}^{k} \alpha_{t_s} n_{t_s}}, \end{aligned} \tag{13}$$

where $\Gamma_{r,k}$ is defined in Lemma 2.1. Thus, we have a confidence interval $(M'_{i:k}, M'_{j:k})$, $j > i$ for ξ_p whose confidence coefficient is free of F.

It may be noted that for $\alpha_i = 1$ and $n_i = n$ $(i = 1, \ldots, k)$, Eq. (13) can be expressed as follows:

$$\psi(i, j; p) = \sum_{r=i}^{j-1} \binom{k}{r} \left(1 - q^n\right)^r q^{n(k-r)}. \tag{14}$$

From (14), it is obvious that the confidence coefficients only based on *minima* (ψ_0) can not exceed $1 - q^{nk} - (1 - q^n)^k$, for given k and n. Thus, confidence intervals for ξ_p based on just *minima* are not possible for $\psi_0 > 1 - q^{nk} - (1 - q^n)^k$.

4. Tolerance Limits and Intervals

The interval (A_1, A_2) is called a $100\beta\%$ tolerance interval with probability level v, if

$$P\big(F(A_2) - F(A_1) > \beta\big) = v,$$

where the endpoints A_1 and A_2 are called tolerance limits. Setting $A_1 = -\infty$ or $A_2 = \infty$, we get upper or lower tolerance limits, respectively. Usually, the standard nonparametric tolerance intervals are based on the smallest and the largest observations in the sample. Now, under the assumptions **P** (presented in Sec. 2), we determine the tolerance intervals for population F, based on extreme order statistics of k independent and non identical samples. With this in mind, we consider the following three cases.

4.1. *Tolerance Intervals in Terms of Maxima*

Here, we determine a $100\beta\%$ tolerance interval with probability level v, based on the set $\{M_{1:k}, \ldots, M_{k:k}\}$. The problem is to determine the interval $(M_{r:k}, M_{s:k})$, $s > r$, such that

$$P(F(M_{s:k}) - F(M_{r:k}) > \beta) = v.$$

By (4), $F(M_{r:k})$ is identically distributed as the rth order statistic of Z_i's, where $Z_i = 1 - (U_{1:n_i})^{1/\alpha_i}$, $i = 1, \ldots, k$. Therefore,

$$P(F(M_{s:k}) - F(M_{r:k}) > \beta) = P(W_{r,s:k} > \beta),$$

where $W_{r,s:k} = Z_{s:k} - Z_{r:k}$. Thus, by Lemma 2.3

$$P(F(M_{s:k}) - F(M_{r:k}) > \beta) = \sum_{\Delta_k} \int_{\beta}^{1} \int_{0}^{1-w} g(w)dxdw,$$

such that summation index Δ_k is defined in Lemma 2.2 and

$$\begin{aligned} g(w) = [(r-1)!(s-r-1)!(k-s)!]^{-1} &\times \prod_{i=1}^{r-1} \left(1-(1-x)^{\alpha_{t_i}}\right)^{n_{t_i}} \\ &\times \alpha_{t_r} n_{t_r} (1-x)^{\alpha_{t_r}-1} \left(1-(1-x)^{\alpha_{t_r}}\right)^{n_{t_r}-1} \\ &\times \prod_{i=r+1}^{s-1} \left[\left(1-(1-x-w)^{\alpha_{t_i}}\right)^{n_{t_i}} - \left(1-(1-x)^{\alpha_{t_i}}\right)^{n_{t_i}}\right] \\ &\times \alpha_{t_s} n_{t_s} (1-x-w)^{\alpha_{t_s}-1} \left(1-(1-x-w)^{\alpha_{t_s}}\right)^{n_{t_s}-1} \\ &\times \prod_{i=s+1}^{k} \left[1-\left(1-(1-x-w)^{\alpha_{t_i}}\right)^{n_{t_i}}\right]. \end{aligned}$$

4.2. *Tolerance Intervals in Terms of Minima*

By proceeding similarly to the previous subsection, we can find the tolerance intervals based on minima. By Eq. (5), $F(M'_{r:k})$ is identically distributed as the rth order statistic of Z'_i's where $Z'_i = 1 - (U_{n_i:n_i})^{1/\alpha_i}$, $i = 1, \ldots, k$. So, the confidence coefficient of a $100\beta\%$ tolerance interval in terms of ordered minima, $(M'_{r:k}, M'_{s:k})$, $s > r$, is

$$P(F(M'_{s:k}) - F(M'_{r:k}) > \beta) = \sum_{\Delta_k} \int_{\beta}^{1} \int_{0}^{1-w} g'(w)dxdw,$$

where summation index Δ_k is defined as in Lemma 2.2 and

$$\begin{aligned} g'(w) = [(r-1)!(s-r-1)!(k-s)!]^{-1} &\times \prod_{i=1}^{r-1} \left(1-(1-x)^{\alpha_{t_i} n_{t_i}}\right) \\ &\times \alpha_{t_r} n_{t_r} (1-x)^{\alpha_{t_r} n_{t_r}-1} \times \prod_{i=r+1}^{s-1} \left((1-x)^{\alpha_{t_i} n_{t_i}} - (1-x-w)^{\alpha_{t_i} n_{t_i}}\right) \\ &\times \alpha_{t_s} n_{t_s} (1-x-w)^{\alpha_{t_s} n_{t_s}-1} \times (1-x-w)^{\sum_{i=s+1}^{k} \alpha_{t_i} n_{t_i}}. \end{aligned}$$

4.3. *Upper and Lower Tolerance Limits in Terms of Minima and Maxima*

In this part, the $100\beta\%$ upper and lower tolerance limits with probability level v in terms of $V_{j:2k}$'s are desired, where $V_{j:2k}$ is the jth order statistic of the set $\{M'_1, M_1, \ldots, M'_k, M_k\}$. To construct a $100\beta\%$ upper tolerance limit, we seek to determine the interval $(-\infty, V_{j:2k})$, such that

$$P(F(V_{j:2k}) > \beta) = v.$$

By considerations in Subsecs. 4.1 and 4.2, $F(V_{j:2k})$ is identically distributed as the jth order statistic among Z_i's and Z_i''s ($i = 1, \ldots, k$). By Theorem 3.1, we have

$$P(F(V_{j:2k}) > \beta) = 1 - \sum_{r=j}^{2k} \pi(r; \beta),$$

where $\pi(r; \beta)$ is defined in (9). Similarly, $V_{i:2k}$ is a $100\beta\%$ lower tolerance limit with confidence coefficient

$$P(F(V_{i:2k}) < 1 - \beta) = \sum_{r=i}^{2k} \pi(r; 1 - \beta).$$

For special case, $\alpha_i = 1$ and $n_i = n$ ($i = 1, \ldots, k$), we have the following result.

Theorem 4.1. *Let $\alpha_i = 1$ and $n_i = n$, $i = 1, \ldots, k$, also suppose that $\beta > 0$ and $v \leq 1$. Then $[V_{j:2k}, \infty)$ is the lower tolerance interval that covers β proportion of the population with the confidence coefficient v if and only if $(-\infty, V_{2k-j+1:2k}]$ is the upper tolerance interval that covers β proportion of the population with the same confidence coefficient, i.e.,*

$$P(F(V_{j:2k}) > \beta) = v \Leftrightarrow P(1 - F(V_{2k-j+1:2k}) > \beta) = v.$$

Proof. Using the fact that $F(V_{j:2k}) = V^*_{j:2k}$, the result is a direct application of Lemma 3.2. □

5. Application to a Real Data Set

We illustrate the results of this article on a real data set which consist of the time (in hours) of successive failures of the air conditioning system in seven Boeing 720 jet aircraft. See Proschan (1963) for a detailed description of the data set. He tested and accepted the hypothesis that the successive failure times were iid exponential for each plane, but with different failure rates. Therefore, the assumptions **P** (cited in Sec. 2) hold. The summary descriptive statistics of the data are reported in Table 1.

Suppose that we intend to do inference about quantiles of the lifetime of the plane 7 from Boeing 720 jet aircrafts based on available data in Table 1, such that its survival function is related to the first six planes via a PHRM. Toward this end, we assume that α_i's are known and for $i = 1, \ldots, 7$ are equal to 0.85, 1.75, 1.43, 0.97, 1.25, 1.32, 1, respectively.

From Eqs. (8), (11), (13) and using the data in Table 1, we obtain the values of $\varphi(i, j; p)$, $\psi(i, j; p)$, and $\delta(i, j; p)$ for some choices of i, j, and p when $k = 7$.

Table 1
Summary descriptions of Boeing 720 jet aircraft data

	Plane 1	Plane 2	Plane 3	Plane 4	Plane 5	Plane 6	Plane 7
n_i	6	23	29	15	14	30	27
M_i'	15	7	10	12	15	1	1
M_i	194	447	310	502	320	261	216

Table 2
Values of $\varphi(i, j; p)$ for $k = 7$

		p										
i	j	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
1	6	0.000	0.000	0.000	0.000	0.002	0.009	0.036	0.150	0.509	0.956	0.877
	7	0.000	0.000	0.000	0.000	0.002	0.009	0.036	0.150	0.509	0.959	**0.983**
2	6	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.008	0.124	0.755	0.864
	7	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.008	0.124	0.758	**0.970***
3	7	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.015	0.424	0.896

These are tabulated in Tables 2, 3, and 4, respectively, and help us to choose the appropriate confidence intervals for the pth quantile of F.

Note: In Tables 2, 3, and 4, the confidence intervals with the shortest width and confidence coefficient more than 0.95 are indicated by *.

We note from these tables that although confidence intervals only based on maxima or minima are not accurate enough, they are more precise considering both of them, together. It is also clear that maxima are better for constructing confidence intervals for upper quantiles and minima for lower quantiles. We also observe that:

1. From Table 2, no intervals for lower quantiles based on maxima have confidence coefficient higher than 0.2%. But for example, each of (194, 502) and (216, 502) can be considered as an confidence interval with confidence level 95% for $\xi_{0.95}$, whereas by considering the shortest interval length criterion, the second one is accepted.

2. From Table 3, no intervals for upper quantiles based on minima have confidence coefficient higher than 0.9%, but for example, (1, 15) is an confidence interval with confidence level 95% for $\xi_{0.05}$.

3. From Table 4, considering minima and maxima jointly, improve the results. Confidence intervals have much better coverage probability for lower and upper quantiles. For example, (1, 15), (15, 194), (15, 216), and (216, 502) are exact 96.2, 99.8, 99, and 97.1% confidence intervals for $\xi_{0.05}$, $\xi_{0.4}$, $\xi_{0.7}$, and $\xi_{0.95}$, respectively.

Table 3
Values of $\psi(i, j; p)$ for $k = 7$

		p										
i	j	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
1	6	0.751	0.263	0.020	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	7	**0.956**	0.751	0.362	0.169	0.075	0.029	0.009	0.002	0.000	0.000	0.000
2	6	0.749	0.263	0.020	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	7	**0.954***	0.751	0.362	0.169	0.075	0.029	0.009	0.002	0.000	0.000	0.000
3	7	0.933	0.751	0.362	0.169	0.075	0.029	0.009	0.002	0.000	0.000	0.000

Table 4
Values of $\delta(i, j; p)$ for $k = 7$

		p										
i	j	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
1	6	0.759	0.264	0.020	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	7	**0.964**	0.752	0.362	0.169	0.075	0.029	0.009	0.002	0.000	0.000	0.000
	8	**1.000**	**1.000**	**1.000**	**1.000**	**0.998**	**0.991**	**0.964**	0.851	0.491	0.044	0.001
	9	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.992**	0.876	0.245	0.014
	10	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.985**	0.579	0.087
	11	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.999**	0.852	0.297
	12	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.970**	0.616
	13	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.997**	0.879
	14	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.984**
2	6	0.757	0.264	0.020	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	7	**0.962***	0.752	0.362	0.169	0.075	0.029	0.009	0.002	0.000	0.000	0.000
	8	**0.998**	**1.000**	**1.000**	**1.000**	**0.998**	**0.991**	**0.964**	0.851	0.491	0.044	0.001
	9	**0.998**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.992**	0.876	0.245	0.014
	10	**0.998**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.985**	0.579	0.087
	11	**0.998**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.999**	0.852	0.297
	12	**0.998**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.970**	0.616
	13	**0.998**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.997**	0.879
	14	**0.998**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.984**
3	6	0.737	0.264	0.020	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	7	0.941	0.752	0.362	0.169	0.075	0.029	0.009	0.002	0.000	0.000	0.000
	8	**0.977**	**1.000**	**1.000**	**1.000**	**0.998**	**0.991**	**0.964**	0.851	0.491	0.044	0.001
	9	**0.977**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.992**	0.876	0.245	0.014
	10	**0.977**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.985**	0.579	0.087
	11	**0.977**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.999**	0.852	0.297
	12	**0.977**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.970**	0.616
	13	**0.977**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.997**	0.879
	14	**0.977**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.984**
4	7	0.836	0.750	0.362	0.169	0.075	0.029	0.009	0.002	0.000	0.000	0.000
	8	0.872	**0.997**	**1.000**	**1.000**	**0.998**	**0.991**	**0.964**	0.851	0.491	0.044	0.001
	9	0.872	**0.997**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.992**	0.876	0.245	0.014
	10	0.872	**0.997**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.985**	0.579	0.087
	11	0.872	**0.997**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.999**	0.852	0.297
	12	0.872	**0.997**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.970**	0.616
	13	0.872	**0.997**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.997**	0.879
	14	0.872	**0.997**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.984**
5	7	0.561	0.712	0.362	0.169	0.075	0.029	0.009	0.002	0.000	0.000	0.000
	8	0.597	**0.960***	**1.000**	**1.000**	**0.998**	**0.991**	**0.964**	0.851	0.491	0.044	0.001
	9	0.597	**0.960**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.992**	0.876	0.245	0.014
	10	0.597	**0.960**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.985**	0.579	0.087
	11	0.597	**0.960**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.999**	0.852	0.297
	12	0.597	**0.960**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.970**	0.616
	13	0.597	**0.960**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.997**	0.879
	14	0.597	**0.960**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.984**

(continued)

6. Concluding Remarks

In this article, we developed the nonparametric inferential procedures based on extreme order statistics from k independent random samples, where the observations of the ith sample coming from $\overline{F}_i = (\overline{F})^{\alpha_i}$ such that, α_i is a known positive constant

Table 4
Continued

		p										
i	*j*	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
6	7	0.205	0.489	0.342	0.167	0.075	0.029	0.009	0.002	0.000	0.000	0.000
	8	0.241	0.736	**0.980***	**0.998***	**0.998***	**0.991**	**0.964**	0.851	0.491	0.044	0.001
	9	0.241	0.736	**0.980**	**0.999**	**1.000**	**1.000**	**1.000**	**0.992**	0.876	0.245	0.014
	10	0.241	0.736	**0.980**	**0.999**	**1.000**	**1.000**	**1.000**	**1.000**	**0.985**	0.579	0.087
	11	0.241	0.736	**0.980**	**0.999**	**1.000**	**1.000**	**1.000**	**1.000**	**0.999**	0.852	0.297
	12	0.241	0.736	**0.980**	**0.999**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.970**	0.616
	13	0.241	0.736	**0.980**	**0.999**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.997**	0.879
	14	0.241	0.736	**0.980**	**0.999**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**0.984**
7	8	0.036	0.248	0.638	0.831	0.924	**0.962***	**0.955***	0.849	0.491	0.044	0.001
	9	0.036	0.248	0.638	0.831	0.925	**0.971**	**0.990**	**0.990***	0.876	0.245	0.014
	10	0.036	0.248	0.638	0.831	0.925	**0.971**	**0.991**	**0.998**	**0.985***	0.579	0.087
	11	0.036	0.248	0.638	0.831	0.925	**0.971**	**0.991**	**0.998**	**0.999**	0.852	0.297
	12	0.036	0.248	0.638	0.831	0.925	**0.971**	**0.991**	**0.998**	**1.000**	**0.970***	0.616
	13	0.036	0.248	0.638	0.831	0.925	**0.971**	**0.991**	**0.998**	**1.000**	**0.997**	0.879
	14	0.036	0.248	0.638	0.831	0.925	**0.971**	**0.991**	**0.998**	**1.000**	**1.000**	**0.984**
8	13	0.000	0.000	0.000	0.000	0.002	0.009	0.036	0.149	0.509	0.953	0.878
	14	0.000	0.000	0.000	0.000	0.002	0.009	0.036	0.149	0.509	0.955	**0.984**
9	13	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.008	0.124	0.752	0.865
	14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.008	0.124	0.755	**0.971***
10	14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.015	0.421	0.897

and $\overline{F}_k$ show the present population, i.e., $\alpha_k = 1$. We obtained confidence intervals for quantiles and tolerance limits of the population F in three cases. In each case, the exact expressions for the coverage probabilities are derived. An illustrative example presented to implement the proposed inferential technique. The proposed procedure can be extended to some other cases:

- In the assumptions **P**, distribution-free confidence intervals and tolerance limits have been obtained by assuming that α_i's are known positive constants. If the α_i's are unknown, we can estimate them nonparametrically. Suppose that the complete samples are available and let F and f be the cdf and pdf of the present population, respectively. Since $\overline{F}_i(x) = [\overline{F}(x)]^{\alpha_i}$, the likelihood function of the sample $X_{i,1}, \ldots, X_{i,n_i}$ from the ith population is given by

$$L(\alpha_i) = \alpha_i^{n_i} \prod_{j=1}^{n_i} f(x_{i,j}) \prod_{j=1}^{n_i} [\overline{F}(x_{i,j})]^{\alpha_i - 1}.$$

Therefore, the likelihood equation is as follows

$$\frac{\partial}{\partial \alpha_i} \log L(\alpha_i) = \frac{n_i}{\alpha_i} + \sum_{j=1}^{n_i} \log \overline{F}(X_{i,j}) = 0.$$

So, the maximum likelihood estimator (MLE) of α_i $(i = 1, \ldots, k-1)$ is

$$\mathrm{MLE}(\alpha_i) = \frac{-n_i}{\sum_{j=1}^{n_i} \log \overline{F}(X_{i,j})}. \tag{15}$$

To estimate α_i nonparametrically, we can substitute $\widehat{\overline{F}}$, the empirical estimator of $\overline{F}$, instead of $\overline{F}$ in Eq. (15), i.e., the nonparametric MLE (NMLE) of α_i is

$$\text{NMLE}(\alpha_i) = \frac{-n_i}{\sum_{j=1}^{n_i} \log \hat{\overline{F}}(X_{i,j})}. \tag{16}$$

In order to investigate the effect of the estimated α_i's on the coverage probabilities and to compare them with the results in Sec. 5 (with known α_i's), a simulation study has been done as follows:

Table 5
Mean and MSE of $\hat{\delta}(i, j; p)$ for $k = 7$

		p								
i	*j*	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
4	7	[a]0.6586	0.2744	0.1193	0.0524	0.0217	0.0078	0.0021	0.0003	0.0000
		[b]0.0252	*0.0255*	*0.0103*	*0.0033*	*0.0009*	*0.0002*	*0.0000*	*0.0000*	*0.0000*
	8	0.9969	0.9963	0.9855	0.9586	0.8983	0.7748	0.5578	0.2688	0.0429
		0.0001	*0.0010*	*0.0040*	*0.0120*	*0.0341*	*0.0881*	*0.1629*	*0.1070*	*0.0074*
	9	0.9973	1.0000	1.0000	0.9996	0.9955	0.9716	0.8756	0.6083	0.1765
		0.0000	*0.0000*	*0.0000*	*0.0000*	*0.0006*	*0.0066*	*0.0464*	*0.1579*	*0.0493*
	10	0.9973	1.0000	1.0000	1.0000	0.9998	0.9969	0.9727	0.8385	0.3945
		0.0000	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0005*	*0.0071*	*0.0677*	*0.1223*
	11	0.9973	1.0000	1.0000	1.0000	1.0000	0.9997	0.9952	0.9490	0.6395
		0.0000	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0009*	*0.0159*	*0.1299*
	12	0.9973	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.9884	0.8413
		0.0000	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0001*	*0.0024*	*0.0573*
	13	0.9973	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9985	0.9585
		0.0000	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0002*	*0.0087*
	14	0.9973	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9980
		0.0000	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*
5	7	0.6281	0.2740	0.1193	0.0524	0.0217	0.0078	0.0021	0.0003	0.0000
		0.0196	*0.0254*	*0.0103*	*0.0033*	*0.0009*	*0.0002*	*0.0000*	*0.0000*	*0.0000*
	8	0.9664*	0.9959	0.9854	0.9586	0.8983	0.7748	0.5578	0.2688	0.0429
		0.0010	*0.0010*	*0.0040*	*0.0120*	*0.0341*	*0.0881*	*0.1629*	*0.1070*	*0.0074*
	9	0.9668	0.9996	1.0000	0.9996	0.9955	0.9716	0.8756	0.6083	0.1765
		0.0009	*0.0000*	*0.0000*	*0.0000*	*0.0006*	*0.0066*	*0.0464*	*0.1579*	*0.0493*
	10	0.9668	0.9996	1.0000	1.0000	0.9998	0.9969	0.9727	0.8385	0.3945
		0.0009	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0005*	*0.0071*	*0.0677*	*0.1223*
	11	0.9668	0.9996	1.0000	1.0000	1.0000	0.9997	0.9952	0.9490	0.6395
		0.0009	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0009*	*0.0159*	*0.1299*
	12	0.9668	0.9996	1.0000	1.0000	1.0000	1.0000	0.9993	0.9884	0.8413
		0.0009	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0001*	*0.0024*	*0.0573*
	13	0.9668	0.9996	1.0000	1.0000	1.0000	1.0000	0.9999	0.9985	0.9585
		0.0009	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0002*	*0.0087*
	14	0.9668	0.9996	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9980
		0.0009	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*

(continued)

Table 5
Continued

		p								
i	j	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
6	7	0.4528	0.2587	0.1180	0.0523	0.0217	0.0078	0.0021	0.0003	*0.0000*
		0.0052	*0.0214*	*0.0101*	*0.0033*	*0.0009*	*0.0002*	*0.0000*	*0.0000*	*0.0000*
	8	0.7911	0.9806*	0.9841*	0.9585*	0.8983	0.7748	0.5578	0.2688	0.0429
		0.0135	*0.0012*	*0.0039*	*0.0120*	*0.0341*	*0.0881*	*0.1629*	*0.1070*	*0.0074*
	9	0.7914	0.9844	0.9987	0.9995	0.9955	0.9716	0.8756	0.6083	0.1765
		0.0136	*0.0003*	*0.0000*	*0.0000*	*0.0006*	*0.0066*	*0.0464*	*0.1579*	*0.0493*
	10	0.7914	0.9844	0.9987	0.9999	0.9998	0.9969	0.9727	0.8385	0.3945
		0.0136	*0.0003*	*0.0000*	*0.0000*	*0.0000*	*0.0005*	*0.0071*	*0.0677*	*0.1223*
	11	0.7914	0.9844	0.9987	0.9999	1.0000	0.9997	0.9952	0.9490	0.6395
		0.0136	*0.0003*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0009*	*0.0159*	*0.1299*
	12	0.7914	0.9844	0.9987	0.9999	1.0000	1.0000	0.9993	0.9884	0.8413
		0.0136	*0.0003*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0001*	*0.0024*	*0.0573*
	13	0.7914	0.9844	0.9987	0.9999	1.0000	1.0000	0.9999	0.9985	0.9585
		0.0136	*0.0003*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0002*	*0.0087*
	14	0.7914	0.9844	0.9987	0.9999	1.0000	1.0000	1.0000	1.0000	0.9980
		0.0136	*0.0003*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0000*
7	8	0.3383	0.7219	0.8661	0.9062	0.8766	0.7670	0.5556	0.2685	0.0429
		0.0255	*0.0241*	*0.0099*	*0.0099*	*0.0301*	*0.0852*	*0.1617*	*0.1068*	*0.0074*
	9	0.3387	0.7256	0.8806	0.9472	0.9738*	0.9638*	0.8735	0.6080	0.1764
		0.0258	*0.0255*	*0.0103*	*0.0033*	*0.0012*	*0.0063*	*0.0460*	*0.1578*	*0.0493*
	10	0.3387	0.7256	0.8807	0.9476	0.9781	0.9891	0.9706*	0.8381	0.3945
		0.0258	*0.0255*	*0.0103*	*0.0033*	*0.0009*	*0.0006*	*0.0070*	*0.0676*	*0.1223*
	11	0.3387	0.7256	0.8807	0.9476	0.9783	0.9919	0.9930	0.9486	0.6395
		0.0258	*0.0255*	*0.0103*	*0.0033*	*0.0009*	*0.0002*	*0.0009*	*0.0159*	*0.1299*
	12	0.3387	0.7256	0.8807	0.9476	0.9783	0.9922	0.9972	0.9881*	0.8413
		0.0258	*0.0255*	*0.0103*	*0.0033*	*0.0009*	*0.0002*	*0.0001*	*0.0024*	*0.0573*
	13	0.3387	0.7256	0.8807	0.9476	0.9783	0.9922	0.9978	0.9981	0.9585*
		0.0258	*0.0255*	*0.0103*	*0.0033*	*0.0009*	*0.0002*	*0.0000*	*0.0002*	*0.0087*
	14	0.3387	0.7256	0.8807	0.9476	0.9783	0.9922	0.9979	0.9997	0.9980
		0.0258	*0.0255*	*0.0103*	*0.0033*	*0.0009*	*0.0002*	*0.0000*	*0.0000*	*0.0000*
8	13	0.0300	0.0723	0.0994	0.1318	0.1912	0.3067	0.5063	0.7659	0.9232
		0.0001	*0.0010*	*0.0040*	*0.0120*	*0.0341*	*0.0881*	*0.1628*	*0.1057*	*0.0125*
	14	0.0300	0.0723	0.0994	0.1318	0.1912	0.3067	0.5064	0.7674	0.9627*
		0.0001	*0.0010*	*0.0040*	*0.0120*	*0.0341*	*0.0881*	*0.1629*	*0.1070*	*0.0072*

[a] and [b] stand for the mean and MSE of $\hat{\delta}(i, j; p)$.

* stands for the confidence intervals with the shortest width and confidence coefficient more than 95%.

(i) Let $\overline{F}(x) = e^{-x}$ and α_i's and n_i's to be the same as in Sec. 5.

(ii) Seven data sets with sample sizes n_i are generated from $\overline{F}_i = \overline{F}^{\alpha_i}$ $(i = 1, \ldots, 7)$.

(iii) The empirical estimator of F is computed based on the data set extracted from the seventh population.

(iv) The $\hat{\alpha}_i$'s $(i = 1, \ldots, 6)$ are calculated from Eq. (16).

(v) The coverage probabilities, $\hat{\delta}(i, j; p) = P(V_{i:2k} \le \xi_p \le V_{j:2k})$, are computed from Eqs. (8)–(9) by substituting $\hat{\alpha}_i$ for α_i $(i = 1, \ldots, 6)$.

(vi) The squared deviations $\left(\hat{\delta}(i, j; p) - \delta(i, j; p)\right)^2$ are computed for some choices of i, j, and p where $\delta(i, j; p)$ stands for known α_i's.

(vii) The steps (ii)–(vi) are repeated 10^4 times and the mean and MSE (Mean Square Error) of estimated coverage probabilities are computed by averaging the $\hat{\delta}(i, j; p)$'s and their squared deviations, respectively, over the 10^4 repetitions.

The computational results are displayed in Table 5. It seems that the results of the simulation study are close to the theoretical computations, when α_i's are unknown, which confirms the accuracy of the proposed procedure.

- Let $Y_{i,j}(1 \le i \le k, 1 \le j \le n_i)$ be independent random variables. Moreover for a fixed i, $Y_{i,j}$'s, $(1 \le j \le n_i)$ be identically distributed with cdf $G_i(x) = [G(x)]^{\theta_i}$, where $G(x)$ is an absolutely continuous distribution function and θ_i is a known positive constant. The aforementioned identity is well-known in the lifetime experiments literature as proportional reversed hazard model (see for example Lawless, 2003). In this case the results of this article hold with obvious modifications.

Acknowledgment

The authors would like to thank the anonymous referee for his/her useful comments and careful reading the manuscript that definitely improved the article.

Partial support from the Ordered and Spatial Data Center of Excellence of Ferdowsi University of Mashhad is acknowledged.

References

Ahmadi, J., Arghami, N. R. (2003). Nonparametric confidence and tolerance intervals from record values data. *Statist. Papers* 44:455–468.

Ahmadi, J., Balakrishnan, N. (2004). Confidence intervals for quantiles in terms of record range. *Statist. Probab. Lett.* 68:395–405.

Ahmadi, J., Balakrishnan, N. (2005). Distribution-free confidence intervals for quantile intervals based on current records. *Statist. Probab. Lett.* 75:190–202.

Arnold, B. C., Balakrishnan, N., Nagaraja, H. N. (1992). *A First Course in Order Statistics.* New York: Wiley.

Balakrishnan, N., Li, T. (2006). Confidence intervals for quantiles and tolerance intervals based on ordered ranked set samples. *Ann. Instit. Statist.* 58:757–777.

Chen, Z. (2000). On ranked-set sample quantiles and their applications. *J. Statist. Plann. Infer.* 83:125–135.

David, H. A., Nagaraja, H. N. (2003). *Order Statistics*. 3rd ed. New York: Wiley.

Deshpande, J. V., Frey, J., Ozturk, O. (2006). Nonparametric rank-set sampling confidence intervals for quantiles of a finite population. *Environ. Ecol. Stat.* 13:25–40.

Gulati, S., Padgett, W. J. (1994). Nonparametric quantile estimation from record-breaking data. *Austral. J. Statist.* 36:211–223.

Hutson, A. D. (1999). Calculating nonparametric confidence intervals for quatiles using fractional order statistics. *J. Appl. Statist.* 26:343–353.

Krewski, D. (1976). Distribution-free confidence intervals for quantile intervals. *J. Amer. Statist. Assoc.* 71:420–422.

Lawless, J. L. (2003). *Statistical Models and Methods for Lifetime Data*. 2nd ed. New York: Wiley.

Ozturk, O., Deshpande, J. V. (2006). Ranked-set sample nonparametric quantile confidence intervals. *J. Statist. Plann. Infer.* 136:570–577.

Proschan, F. (1963). Theoretical explanation of observed decreasing failure rate. *Technometrics* 5:375–383.

Sathe, Y. S., Lingras, S. R. (1981). Bounds for the confidence coefficient of outer and inner confidence intervals for quantile interval. *J. Amer. Statist. Assoc.* 76:473–475.

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*. New York: Wiley.

Zielinski, R., Zielinski, W. (2005). Best exact nonparametric confidence intervals for quantiles. *Statistics* 39:67–71.