Capacity of a More General Class of Relay Channels

Ghosheh Abed Hodtani and Mohammad Reza Aref

Department of Electrical Engineering, Sharif University of Technology, Tehran, Iran

hodtani@yahoo.com , aref@sharif.edu

Abstract – Capacity has been found for degraded, reversely degraded, full feedback, semi-deterministic, orthogonal relay channels, also for a class of deterministic relay channels and a class of modulo sum relay channels. We indicate what the relay decodes and forwards with one auxiliary random variable having bounded cardinality and attempt to define a more general class of relay channels in order to unify most of known capacity theorems into one capacity theorem by considering additional assumptions imposed to the definition of those channels. In other words, the relay channel inputs are dependent as in multiple access channel with arbitrarily correlated sources and here we do for the relay channel the same as Cover, El Gamal and Salehi has done for the multiple access channel. Certainly, our theorem includes only all of the relay channels which satisfy the constraints of our definition.

I. INTRODUCTION

The discrete and memoryless relay channel (Fig.1) consists of four finite sets χ_1,χ_2,y, y_1 and a collection of probability distributions $p(.\,,.\,|\,x_1\,x_2)$ on $y \times y_1$, one for every $(x_1, x_2) \in \chi_1 \times \chi_2$;y and y_1 are the channel outputs and are received by the receiver and the relay, respectively; x_1 and x_2 are the channel inputs and are sent by the transmitter and the relay, respectively.

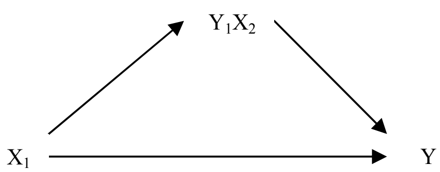

Fig .1 .the relay channel

An $(2^{nR}, n)$ code for the relay channel consists of a set of integers $\mathcal{M}=\{1, 2, \ldots, 2^{nR}\}$, an encoding function that maps each message w$\in\mathcal{M}$ into a code word $x_1, x_1 : \mathcal{M} \rightarrow x_1^n$, a set of relay functions $\{f_i\}_{i=1}^{n}$ such that $x_{2i} = f_i\{y_{11}, y_{12}, \ldots, y_{1i-1}\}, 1 \le i \le n$ and a decoding function $g : y^n \rightarrow \mathcal{M}$. A rate R is achievable if there exists a sequence of $(2^{nR}, n)$ codes with $P_e^{(n)} = P\{\hat{w} \neq w\} \rightarrow 0$ as $n \rightarrow \infty$. Channel capacity C is defined as the supremum over the set of achievable rates.

This work was supported in part by Iran Communications Research centre under contract T500-20613.

The relay channel was first introduced by Van der Meulen [1].In [2] the capacity of degraded and reversely degraded relay channels and the capacity of the relay channel with feedback as well as upper and lower bounds on the capacity of the general relay channel were established. In [3] the capacity of semi deterministic relay channel, in [4] and [5] the capacity of relay channel with orthogonal components, in [6] the capacity of modulo-sum relay channel and in [7] the capacity of a class of deterministic relay channels have been determined .The capacity of general relay channel is still unknown, therefore, one challenge on the relay channel is to work about the problem of the capacity.

Most of known capacity theorems ([2, degraded and full feedback relay channels], [3, semi-deterministic relay channel], [4, orthogonal relay channel]) have their achievability part using decode- and- forward (DAF) strategy [2],[8] and have their cut-set bound achieving converse under the restrictions imposed to the definition of the special channel. All of these capacities can not be achieved via estimate- and- forward (EAF) strategy [2],[8] and among those, we have only the reversely degraded relay channel the capacity of which can be achieved by both DAF and EAF strategies.

In multiple access channel with correlated sources, the channel inputs are dependent and in [9] this dependence has been indicated by three auxiliary random variables. Also, in the relay channel, the channel inputs are dependent and here we indicate this dependence with one auxiliary random variable and define Almost General Relay Channel with Decode- and- Forward strategy (AGRCDAF) and determine its capacity. Then we show that all of the relay channels with known capacity are special cases of it. Here, we have attempted to unify the previous capacity theorems into one theorem towards clarifying the helping and cooperating role of the relay in the relay channel and in all of these theorems. Also, we show that our capacity theorem is validated by its consistency with the capacity regions for broadcast channel with degraded message sets and multiple access channel with partially cooperating encoders (in a special case).

The paper is organized as follows: In section II, to date the best achievable rates by DAF and EAF strategies are referenced. Then, we define AGRCDAF, interpret the definition, and introduce its properties. In section III, we determine the capacity of the AGRCDAF. Then, in section IV, we show that the Markovity conditions used in the definition are satisfied by the known capacity relay channels and our theorem unifies all previously known capacity results. And also, the consistency of the theorem with previous results concerning broadcast and multiple access channels is pointed out. Finally, section V provides a conclusion.

II: ALMOST GENERAL RELAY CHANNEL WITH DECODE- AND- FORWARD STRATEGY

a. The relay channels with DAF strategy

DAF strategy as one of many possible coding schemes (although the best scheme up to now) gives the best rate when the channel between the sender and the relay is a good one. To date the best rate of DAF strategy is the rate in [2, theorem 7, $\hat{Y}_1 = \phi, V = X_2, U = (X_2, U)$].

b. The relay channels with EAF strategy

EAF strategy gives the best rate when the channel between the relay and the receiver has a better quality. To date the best rate obtained by this strategy is the rate in [2, theorem 6].The less rate has been obtained for the noisier relay-receiver channel in [14].The rate in [15] is achieved by EAF strategy with block internally dependent channel inputs.

c. The definition of AGRCDAF and its interpretation

By considering broadcast and multiple access sub-channels in the relay channel, we describe the essential role of the relay with Marcovity conditions on one auxiliary random variable and channel input-outputs.

Definition: A discrete and memoryless relay channel $p(yy_1|x_1x_2)$ (Fig.1) is said to be Almost General Relay Channel with Decode- and- Forward strategy (AGRCDAF) if there exists joint distribution $p(x_1x_2u)$ such that

$$\begin{cases} X_1 \to X_2U \to Y_1 & \textbf{(1),} \\ U \to X_2Y_1 \to Y & \textbf{(2),} \\ Y_1 \to X_1X_2U \to Y \quad , \quad U \neq X_1 & \textbf{(3),} \end{cases}$$

where

$$\mathbf{p(x_1x_2uyy_1) = p(x_1x_2u)p(yy_1|x_1x_2)} \qquad \textbf{(4)},$$

and U is one auxiliary random variable in **all** (1)-(3) and represents the **information decodable** by the relay, of X_1 through Y_1 and X_2. And $\|U\| \leq \min\{\|X_1\| \cdot \|X_2\|, \|Y_1\|, \|Y\|\}$.

Remark: In this definition, U =*constant* does not **represent** the **decodable information** , therefore, it is not and need not be permissible because : U =*constant*, carries no **information** and according to (1), it is not **decodable** $(X_1 \to X_2 \to Y_1)$ and hence, the relay can not and need not **decode and forward** anything, while every AGRCDAF must decode something (U) of X_1 and **(2)** means that in this decoding, the relay is better than the receiver.

The interpretation of the definition

1. The relay channel aims to send X_1 to Y and the receiver needs the cooperation of the relay to find X_1, otherwise the relay is useless.

2. The relay generally does not understand X_1 fully (the channel is not a degraded one in general) and only understands U of X_1, through X_2Y_1 and nothing. Mathematically ($X_1 \to U \to X_2Y_1$ or necessarily **(1)** in the definition).

3.The relay must be better than the receiver in finding U, otherwise the relay is useless. Mathematically ($U \to X_2Y_1 \to Y$, **(2)**). In other words by $X_2 = \phi$, the broadcast sub-channel ($X_1 \to Y, Y_1$) must be less noisy regarding U , (sufficiently $U \to Y_1 \to Y$).

4.In the useless cases of the relay, the receiver is itself better than the relay in finding U and X_1 (sufficiently $UX_1 \to Y \to Y_1X_2$ or necessarily from (6) below $X_1 \to X_2UY \to Y_1$, or sufficiently **(3)** accompanying **(1)**)

d. Some properties of AGRCDAF

For Markov chains on arbitrary random variables of X , W , Z and Y according to the relation:

$$I(X;Z,Y|W) = I(X;Z|W) + I(X;Y|Z,W) = I(X;Y|W) + I(X;Z|Y,W) \qquad \textbf{(5)}$$

And the nonnegativity of mutual information, it is readily shown that.

$$X \to W \to (Z,Y) \Leftrightarrow \left\{ \begin{array}{c} X \to W \to Z \\ \text{and} \\ X \to (W,Z) \to Y \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} X \to W \to Y \\ \text{and} \\ X \to (W,Y) \to Z \end{array} \right\} \quad \textbf{(6)}$$

Lemma (1): For the AGRCDAF, we have:

p₁ : $X_1 \to X_2UY \to Y_1$ **(7)**

p₂ : If $Y_1 = f(X_1, X_2)$, then U can be a function of X_1 and X_2

p₃ : $I(U;Y_1|X_2) = I(X_1;Y_1|X_2)$ **(8-a)**

$I(X_1;Y|X_2U) = I(X_1;Y|X_2Y_1)$ **(8-b)**

Proof of lemma 1: The proof is omitted for brevity.

III. THE CAPACITY OF THE AGRCDAF

Theorem: The capacity of the AGRCDAF is given by:

$$C = \sup_{p(x_1 x_2 u)} \min \{ I(X_1 X_2 ; Y),\ I(U ; Y_1 \mid X_2) + I(X_1 ; Y \mid X_2 U) \} \quad \textbf{(11)},$$

where **supremum** is taken over **all** $p(x_1 x_2 u)$ for which (4) satisfies (1)-(3) .

Proof of the theorem:

Achievability:

It can be directly proved using random coding and random binning but here the proof is omitted, because it can be seen in [4] (Theorem2.4) and in [3] or in [2,theorem7] by the substitution of $\hat{Y}_1 = \phi, V = X_2, U = (X_2, Q)$ and renaming Q by U .

Converse:

From **(8-a,b)** both for $U = X_1$ and $U \neq X_1$, the achievability result coincides with the max flow-min cut upper bound in [2] and the converse proof is completed. Or we can prove the converse, in both cases ,using Fano's inequality [10] and the following inequalities the proof of which is omitted for brevity :

$$I(w;\underline{Y}) \leq \sum_{i=1}^{n} I(X_{1i} X_{2i} ; Y_i)$$ and

$$I(w;\underline{Y}) \leq \sum_{i=1}^{n} I(U_i ; Y_{1i} | X_{2i}) + I(X_{1i} ; Y_i | X_{2i} U_i)$$, where

w is the message to be sent from the sender to the receiver and $\underline{Y}$ is the received sequence at the receiver in Fig.1

IV. THE RESULTS OF THE THEOREM

A. Relay Channel ($U \neq \phi$)

To now , for the relay channel, the capacity achievable by decode-and-forward strategy has been found in the following special cases:

1.Degraded relay channel (and Gaussian degraded relay channel) in [2].

2.Reversely degraded relay channel in [2].

3.The relay channel with full feedback in [2].

4.Semi-deterministic relay channel in [3]

5.The relay channel with orthogonal components in [4] and [5].

Now, we prove that the capacity of all of the above special relay channels is derived from our capacity theorem:

1. If $U = X_1$ (or $U = g(X_1)$ and g is reversible) we have only two conditions **(1)** and **(2)**, then there exists $p(x_1 x_2 u) = p(x_1 x_2)$ such that **(1)** is obvious and from **(4)** we can have:

$$p(x_1 x_2 y y_1) = p(x_1 x_2) p(y_1 | x_1 x_2) p(y | x_2 y_1) \rightarrow \textbf{(2)},$$

in accordance with **(2)** , the AGRCDAF is a degraded relay channel and **(11)** gives **(12)** in **[2**,theorem1**]** .

2. If $U = X_2$, then there exists $p(x_1 x_2 u) = p(x_1 x_2)$ such that **(2)** is obvious and from **(4)** we can have :

$$p(x_1 x_2 y y_1) = p(x_1 x_2) p(y | x_1 x_2) p(y_1 | x_2) \rightarrow \textbf{(3)}$$

and **(1)** and

(3) , **(1)** $\xrightarrow{(6)} (X_1 \rightarrow X_2 Y \rightarrow Y_1)$ or from **(7)** by $U = X_2$ we have $(X_1 \rightarrow X_2 Y \rightarrow Y_1)$,

then, the AGRCDAF is a reversely degraded relay channel the capacity of which is obtained from **(11)**.

Remark: The capacity of reversely degraded relay channel can also be achieved by EAF strategy

3. The capacity of the relay channel with full feedback **(**theorem 3 in [2]**)** is obtained from **(11)** by $U = X_1$ and $Y_1 \rightarrow (Y, Y_1)$.

4. If $Y_1 = f(X_1 X_2)$, then Y_1 is known at the transmitter (assuming that the transmitter knows the first symbol of X_2) and according to the property **(p_2)**, U is also a function of X_1 and X_2 and we can put $U = Y_1$, then there exists $p(x_1 x_2 u) = p(x_1 x_2 y_1)$ such that in this case, (1)-(3) are obvious for every (4) and **(11)** reduces to the capacity of semi deterministic relay channel in **[**3,corrolary(6)**]**.

5. If $X_1 = (X_D, X_R)$ and $U = X_R$, from **(4)** we can have:

$$p(x_2 x_R x_D y y_1) = p(x_2) p(x_R | x_2) p(x_D | x_2) p(y_1 | x_2 x_R) p(y | x_2 x_D)$$

(12)

Then , the AGRCDAF becomes a relay channel with orthogonal components in [5] and the capacity in **(11)** is readily reduced to the capacity in [5].

6. The feedback from the relay to the transmitter does not increase the capacity of the AGRCDAF because the achievable rate in [theorem 3,[11]] by $\hat{Y}_1 = \phi$ and $\hat{V} = U$ coincides with the upper bound in [theorem 1, [11]] by $V = U$ and in accordance with **(7).**

B. Relay Channel ($U = \phi$)

In accordance with (1)and (3), the definition considers every AGRCDAF with *U=constant* or ϕ or X_2 as a reversely degraded relay channel the capacity of which can also be achieved by **EAF** strategy.

B_1 . About the capacity of deterministic relay channel [7]:

According to [7], the channel uses EAF strategy or hash and forward strategy and hence, our theorem dose nothing to say about this channel, but we can establish a kind of equivalency between DAF in our theorem and EAF in [7] by $U = \phi$.

B_2 . About Modulo-Sum Relay Channel (MSRC) in [6]:

It is easily shown that MSRC is not AGRCDAF in general.

C. Consistency of the theorem with previous results

We can validate our theorem by its consistency of the terms in (11) with previous results (the capacity region for broadcast channel with degraded message sets [12] and multiple access channel with partially cooperating encoders (in a special case) [13]) .

V. CONCLUSION

We have defined Almost General Relay Channel with Decode-and-Forward strategy (AGRCDAF) and showed that all of relay channels with known capacity are special cases of it .Also, the capacity regions for broadcast channel with degraded message sets and multiple access channel with partially cooperating encoders(in a special case) are consistent with the terms in the theorem.. We claim that the capacity of general relay channel might be described by one auxiliary random variable or more variables depending on how we define the role of the relay.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their invaluable comments.

REFERENCES

[1] E.C. Van der Meulan, “Three–Terminal Communication Channels”, Adv. Appl. prob. vol.3, 1971.

[2] T. M. Cover and A. El Gamal, “Capacity Theorems for the Relay Channel", *IEEE Trans*. Inform. theory, vol.25, no.5, Sep1979.

[3] A. El Gamal and M. Aref, “The Capacity of Semi deterministic Relay Channel”, *IEEE Trans*. inform. theory, vol. 28, no .3, May 1982.

[4] S. Zahedi , On Reliable Communications Over Relay Networks, Ph.D. Thesis, Stanford University, Stanford CA , Aug 2005.

[5] A. El Gamal and S. Zahedi, “Capacity of A Class of Relay Channels with Orthogonal Components,” *IEEE Trans*. Inform. theory, vol.51 , no.3 ,May 2005.

[6] M. Aleksic, P. Razaghi and w.Yu, “ Capacity of a Class of Modulo-Sum Relay Channels,” ISIT 2007.

[7] T. M. Cover and y.-Han Kim, “Capacity of Deterministic Relay Channels,” ISIT 2007.

[8] G. Kramer, M. Gastpar and P. Gupta , “Cooperative Strategies and Capacity Theorems for Relay Networks”, *IEEE Trans*. Inform. theory, vol.51,no.9,sep.2005

[9] T. Cover, A. El Gamal and M. Salehi, “Multiple Access Channels with Arbitrarily Correlated Sources,” *IEEE Trans*. Inform. theory, vol.26, no.6, sep.1980

[10] T. Cover and J. Thomas, Elements of Information Theory, Wiley, Newyork,1991.

[11] S.I. Bross , “A Converse Result For The Discrete Memoryless Relay Channel With Relay-Transmitter Feedback,” IEEE.Trans.Inform.Theory,vol.52, No.8, Aug.2006

[12] J. Korner and K. Marton, “General Broadcast Channels with Degraded Message Sets,” IEEE Trans.Inform.Theory,vol.23,no.1,1977.

[13] F.J. Willems, “The Discrete Memoryless Multiple Access Channel with Partially Cooperating Encoders,” IEEE Trans. Inform. Theory, vol.29, no.3, 1983.

[14] R. Dabora and S.D. Servetto, “On the role of Estimate and Forward with Time sharing in Coopentive Communications,” Submitted to the IEEE Trans. Inform. Theory, Oct 2006.

[15] W. Kang and S. Ulukas, “A New Achievability Scheme for the Relay Channel,” IEEE ITW, 2007.