Interaction of Topological solitons with defects: Collective coordinate method

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Abstract
By including a potential into the flat metric, we study the interaction of sine-Gordon soliton with different potentials. We will show numerically that while the soliton-barrier system shows fully classical behaviour, the soliton-well system demonstrates non-classical behaviour. In particular, solitons with low velocities are trapped in the well and radiate energy. Also for narrow windows of initial velocity, soliton reflects back from a potential well.

Keywords: Solitons, Nonlinear field theory, Metric of space-time

1. INTRODUCTION

Toplogical solitons are important objects in nonlinear field theories. They are stable against dispersive effects, and "live" similar to classical point-like particles.
Scattering of solitons from potentials (which generally come from medium properties) have been studied in many papers by different methods [1-7]. The effects of medium disorders and impurities can be added to the equation of motion as perturbative terms [1, 2]. These effects also can be taken into account by making some parameters of the equation of motion to be function of space or time [3, 4]. There still exists another interesting method which is mainly suitable for working with topological solitons [5, 6]. In this method, one can add such effects to the Lagrangian of the system by introducing a suitable nontrivial metric for the back ground space-time, without missing the topological boundary conditions [5-7]. In other words, the metric carries the information of the medium. By adding the potential to the metric, the total energy of "soliton + potential" and also topological charge of the soliton will remain conserved during the soliton-potential interaction [7]. This method can be used for objects that their equation of motion results from a Lorentz invariant action, such as sine-Gordon model, φ^4 theory, CP^N model, Skyrme model, Faddeev-Hopf equation, chiral quark-soliton model, Gross-Neveu model, nonlinear Klein-Gordon models and so on. In this paper, we have used a nontrivial metric for coupling the kink of sin-Gordon model to a potential. We find that the method is very powerful and several important properties of soliton-potential interactions come out easily.

It was pointed out in [9] that the scattering of non-topological solitons from defects show some non-classical properties at the low velocities. Motivated by this, it is natural to search for non-classical behaviour in topological solitons. Baby Skyrme model which contains topological solitons has been used for searching such behaviour [10]. In [10], the potential barrier and well has been simulated by adding an extra field to the Lagrangian. In the present paper, however we are interested in the method of adding the potential through the metric [6, 7]. Using this method, we study numerically the scattering of sine-Gordon solitons with the potential and search for the non-classical behaviours.

2. Potentials and the metric

Space dependent potentials can be added to the Lagrangian of a system, through the metric of background space-time. So the metric includes characteristics of the medium. The general form of the action in an arbitrary metric is:

$$I = \int \ell(\varphi, \partial_\mu \varphi)\sqrt{-g}\, d^n x dt \qquad (1)$$

Where "g" is determinant of the metric $g^{\mu\nu}(x)$. Energy density of the "field + potential" can be found by varying "both" the field and the metric (See [7]). Simulations show that the "total" energy is conserved during the field-potential interaction. Here we are interested in the evolution of energy of the field itself. So we have to calculate the soliton energy, by varying only the field, in (1).

Sine-Gordon model is a well-known equation which contains topological solitons. Lagrangian of the sine-Gordon model is:

$$\ell = \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - U(\varphi) \qquad (2)$$

With $U(\varphi) = 1 - \cos\varphi$.

The equation of motion for the Lagrangian (2) is [8]:

$$\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\partial_\mu \partial^\mu \varphi + \partial_\mu \varphi \partial^\mu (\sqrt{-g})\right) + \frac{\partial U(\varphi)}{\partial \varphi} = 0 \qquad (3)$$

The suitable metric in the presence of a weak potential V(x) is [5, 6, and 7]:

$$g_{\mu\nu}(x) \cong \begin{pmatrix} 1+V(x) & 0 \\ 0 & -1 \end{pmatrix} \qquad (4)$$

The equation of motion (3) (describes by Lagrangian (2)) in the background space-time (4), is [8]:

$$\begin{aligned} &(1+V(x))\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{2|1+V(x)|}\frac{\partial V(x)}{\partial x}\frac{\partial \varphi}{\partial x} \\ &+ \frac{\partial U(\varphi)}{\partial \varphi} = 0 \end{aligned} \qquad (5)$$

The field energy density is:

$$\aleph = g^{00}(x)\left(\frac{1}{2}g^{00}(x)\dot{\varphi}^2 + \frac{1}{2}\varphi'^2 + U(\varphi)\right) \qquad (6)$$

and the topological charge density is:

$$Q = \int_{-\infty}^{\infty} \frac{\partial \varphi}{\partial x} dx\,. \qquad (7)$$

Equation (7) shows that topological charge is independent of the metric and consequently from the potential.

Localized solutions of the sine-Gordon equation in flat space time are:

$$\varphi(x) = 4\tan^{-1}\left(\pm \exp\left(\frac{(x-x_0) - ut}{\sqrt{1-u^2}}\right)\right) \qquad (8)$$

The plus (minus) sign is soliton (anti soliton) which moves with speed "u" from the initial place x_0. We can use these solutions as initial condition for solving (5), if soliton is located far from the center of the potential. A potential of the form of $V(x) = ae^{-b(x-c)^2}$ has been chosen in the simulations. Parameter "a" controls the strength of the potential, "b" represents its range, and "c" indicates center of the potential. If $a > 0$, the potential shows a barrier and for $a < 0$ the potential shows a well. I have performed simulations using 4th order Runge-Kutta method for time derivatives. Space derivatives were expanded by using finite difference method. I have used grid spacing h=0.01, 0.02 and sometimes h=0.001. Time step has been chosen as $\frac{1}{4}$ of the space step "h".

Simulations have been setup with fixed boundary conditions and solitons have been kept far from the boundaries. We have controlled the results of simulations by checking the conserved quantities: total energy and topological charge, during the evolution. It is clear that the energy of soliton "itself" changes in time. But the energy of soliton + potential remains unchanged. Here we interested in the energy of soliton and its evolution during the interaction. If we subtract the soliton energy density from total energy density, we find the shape of the potential. Also we can place a static soliton at different places and calculate its total energy. This tells us what the potential is like as seen by the soliton [10].

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