Bulletin of the Iranian Mathematical Society Vol. 35 *No. 1* (*2009*), *pp 129-146.*

CONVOLUTION AND HOMOGENEOUS SPACES

R. A. KAMYABI-GOL AND N. TAVALLAEI*

Communicated by Fereidoun Ghahramani

ABSTRACT. Let G be a locally compact Hausdorff topological group and H be a compact subgroup of G. Then, the homogeneous space G/H possesses a specific Radon measure, which is called a relatively invariant measure. We show that the concepts of convolution and involution can be extended to the integrable functions defined on this homogeneous space. We study the properties of convolution and prove that the space of integrable functions is an involutive Banach algebra with an approximate identity. We also find a necessary and sufficient condition on a closed subspace of this Banach algebra to make it a left ideal.

1. **Introduction**

In functional analysis, convolution is an operator which maps two integrable functions f and g into a third function that represents the amount of overlap between f and a reversed and translated version of g. More precisely, if $f, g \in L^1(\mathbb{R})$, then $f * g$ is defined by:

$$(f * g)(x) = \int_{\mathbb{R}} f(y)\, g(x-y)\, dy \quad \text{(almost all } x \in \mathbb{R}).$$

MSC(2000): 43A15, 43A85
Keywords: Homogeneous space, rho-function, strongly quasi-invariant measure, convolution, involutive Banach algebra, relatively invariant measure, integrable function.
Received: 9 March 2008. Accepted: 24 July 2008
∗Corresponding author

In harmonic analysis, where G is a locally compact topological group, we can define the convolution of two integrable functions on G, say f and g, by,

$$(f * g)(x) = \int_G f(y)\, g(y^{-1}x)\, dy \quad (\text{almost all } x \in G),$$

where dy is the left Haar measure on G. Recall that the convolution and involution make $L^1(G)$ an involutive Banach algebra with an approximate identity. But in the case where H is a nontrivial subgroup of G, there is neither an inverse map nor a left Haar measure on the homogeneous space G/H, in general. We focus on the homogeneous spaces of the form G/H, where G is a locally compact topological group, H is a compact subgroup of G, and G/H is attached to a relatively invariant Radon measure. Then, we define a convolution and a specific involution on $L^1(G/H)$, induced by those in $L^1(G)$ with the same effect.

This paper consists of 4 sections. Section 2 is devoted to fix some notations and get some elementary results on homogeneous spaces which will be used here. We then discuss conditions which enable us to define a convolution of two integrable functions. Section 3 is concerned with the properties of homogeneous spaces of the form G/H, where H is a compact subgroup of a locally compact topological group G. We introduce a specific dense subset of $L^p(G/H)$, $1 \leq p \leq +\infty$, which is the main key to get the result. In Section 4, we introduce the convolution of two integrable functions on this kind of homogeneous space as a generalized linear combination of the left translations of one of them. Theorems 4.4 and 4.5 assert that $L^1(G/H)$ is an involutive Banach algebra which has an approximate identity with the similar structure of that in $L^1(G)$. Finally, a necessary and sufficient condition on a closed subspace of $L^1(G/H)$ is given to make it a left ideal.

2. Notations and preliminary results

We start our work with fixing some useful notations. Let X be a locally compact Hausdorff space and μ be a Radon measure on it. We occasionally encounter the following spaces: $C(X)$ is the space of all continuous complex-valued functions on X, $C_c(X)$ consists of all functions in $C(X)$ with compact supports, and $L^1(X)$ denotes the set of all equivalent classes of μ-almost everywhere defined integrable functions on X.

From now on, we suppose that G is a locally compact topological group with identity e and the left Haar measure dx, H is a closed subgroup of G, and Δ_G and Δ_H are modular functions on G and H respectively. Also, G/H is considered as a homogeneous space that G acts on it from the left and $q : G \longrightarrow G/H$ denotes the canonical mapping. For a function f on G, we define the functions $\check{f}$ by $\check{f}(x) = f(x^{-1})$, $x \in G$, and $\tilde{f} = \bar{\check{f}}$. It is known that $C_c(G/H)$ consists of all Pf functions, where $f \in C_c(G)$ and

$$Pf(xH) = \int_H f(x\xi)\, d\xi \quad (x \in G).$$

Moreover, $P : C_c(G) \to C_c(G/H)$ is a surjective bounded linear operator which is not injective (cf. [2], Subsection 2.6).

Let μ be a Radon measure on G/H. For all $x \in G$, we define the translation μ_x of μ by $\mu_x(E) = \mu(xE)$, where E is a Borel subset of G/H. Then, μ is said to be *G-invariant* if $\mu_x = \mu$, for all $x \in G$, and is said to be *strongly quasi-invariant* provided that a continuous function $\lambda : G \times G/H \longrightarrow (0, +\infty)$ exists which satisfies,

$$d\mu_x(yH) = \lambda(x, yH)\, d\mu(yH) \quad (x, y \in G).$$

If the functions $\lambda(x, .)$ reduce to constants, then μ is called *relatively invariant* under G. Also, we mean by a rho-function for the pair (G, H), a continuous function $\rho : G \longrightarrow (0, +\infty)$ which satisfies,

$$\rho(x\xi) = \frac{\Delta_H(\xi)}{\Delta_G(\xi)}\, \rho(x) \quad (x \in G,\ \xi \in H).$$

As proved, (G, H) admits a rho-function and for each rho-function ρ there is a strongly quasi-invariant measure μ on G/H such that

$$\int_{G/H} Pf(xH)\, d\mu(xH) = \int_G f(x)\, \rho(x)\, dx \quad (f \in C_c(G)),$$

and

$$\frac{d\mu_x}{d\mu}(yH) \;=\; \frac{\rho(xy)}{\rho(y)} \quad (x, y \in G). \tag{2.1}$$

Moreover, all strongly quasi-invariant measures on G/H arise from rho-functions in this manner, and all of these measures are strongly equivalent (cf. [2], Subsection 2.6).

It can be shown that G/H has a G-invariant Radon measure if and only if the constant function $\rho(x) = 1$ is a rho-function for the pair (G, H), or equivalently $\Delta_G|_H = \Delta_H$ (cf. [2], Theorem 2.49). Also, one can show that the existence of a homomorph rho-function $\rho : G \to (0, +\infty)$ for the pair (G, H) is a necessary and sufficient condition for the existence of a relatively invariant measure on G/H. Because, if μ is a relatively invariant measure and $\lambda : G \to (0, +\infty)$ is the continuous function such that $d\mu_x = \lambda(x)\, d\mu$, then,

$$\begin{aligned}\int_G f(y)\,\rho(xy)\,dy &= \int_{G/H} P(L_x f)(yH)\,d\mu(yH)\\ &= \lambda(x)\int_{G/H} Pf(yH)\,d\mu(yH)\\ &= \lambda(x)\int_G f(y)\,\rho(y)\,dy,\end{aligned}$$

where $x \in G$ and $f \in C_c(G)$. Thus, for a fixed $x \in G$ we have,

$$\int_G f(y)\,(\rho(xy) - \lambda(x)\,\rho(y))\,dy = 0,$$

for all $f \in C_c(G)$. This leads to:

$$\frac{\rho(xy)}{\rho(y)} = \lambda(x) \quad (x, y \in G).$$

Obviously, if ρ is a homomorphism, then by (2.1) we get $d\mu_x = \rho(x)\,d\mu$, for all $x \in G$, which shows that μ is relatively invariant under G. More precisely, every relatively invariant measure on G/H is a positive constant multiple of another one, which arises from a homomorphism rho-function (cf. [6]). Observe that if μ is the relatively invariant measure which arises from a rho-function ρ, then by using (2.1), we achieve,

$$\rho(xy) = \frac{\rho(x)\,\rho(y)}{\rho(e)} \quad (x, y \in G) \tag{2.2}$$

(cf. [6]).

It should be mentioned that if μ is the strongly quasi-invariant measure on G/H arising from a rho-function ρ, then the following holds for all $f \in L^1(G)$.

(a) There exists a subset E of G/H of zero measure such that the mapping $\xi \mapsto \frac{f(x\xi)}{\rho(x\xi)}$ is in $L^1(H)$, for all $x \in G$ that $q(x) \notin E$.

(b) The function $xH \mapsto \int_H \frac{f(x\xi)}{\rho(x\xi)}\, d\xi$, defined almost everywhere on G/H, is integrable.

(c) $\int_{G/H} \int_H \frac{f(x\xi)}{\rho(x\xi)}\, d\xi\, d\mu(xH) = \int_G f(x)\, dx$.

Moreover, the mapping $T : L^1(G) \mapsto L^1(G/H)$ defined by

$$(2.3) \qquad Tf(xH) = \int_H \frac{f(x\xi)}{\rho(x\xi)}\, d\xi \quad (\mu - \text{almost all } xH \in \frac{G}{H})$$

is a surjecive bounded linear operator with $\|T\| \leq 1$ (cf. [5], Subsection 3.4). More precisely, for all $\varphi \in L^1(G/H)$, there exists some $f \in L^1(G)$ such that $\varphi = Tf$ and $f \geq 0$ almost everywhere, provided that $\varphi \geq 0$ almost everywhere. It can be shown that

$$(2.4) \qquad \|\varphi\|_1 = \inf\{\|f\|_1 : f \in L^1(G),\, \varphi = Tf\},$$

for all $\varphi \in L^1(G/H)$ and

$$(2.5) \qquad \|\varphi\|_1 = \inf\{\|f\|_1 : f \in C_c(G),\, \varphi = Tf\},$$

where $\varphi \in C_c(G/H)$ (cf. [5], Subsection 3.4). Throughout the rest of the paper, we suppose that G/H has a relatively invariant Radon measure μ which arises from a rho-function ρ.

For all $f, g \in L^1(G)$, the convolution of f and g, $f * g$, is defined as an element of $L^1(G)$ by:

$$f * g(x) = \int_G f(y)\, g(y^{-1}x)\, dy \quad (\text{almost all } x \in G).$$

It is known that $L^1(G)$ is an involutive Banach algebra on which the involution operator assigns to each $f \in L^1(G)$ the function f^* that is defined by $f^*(x) = \frac{\overline{f(x^{-1})}}{\Delta_G(x)}$, $x \in G$ (cf. [1, 2]). It is worthwhile mentioning that if H is a closed normal subgroup of G and μ is the invariant measure on the topological group G/H, which arises from the constant rho-function $\rho(x) = 1$, then $T : L^1(G) \to L^1(G/H)$, defined by (2.3), is an algebra homomorphism; i.e., $T(f * g) = (Tf) * (Tg)$, for all $f, g \in L^1(G)$ (cf. [5], Theorem 3.5.4). From this point of view, for all $\varphi, \psi \in L^1(G/H)$, we will define a convolution $\varphi * \psi$ by $\varphi * \psi = T(f * g)$, where $f, g \in L^1(G)$, $\varphi = Tf$, $\psi = Tg$. Obviously, $\varphi * \psi$ is well-defined just when $T(f * g) = 0$, provided that one of Tf or Tg vanishes. For all $f, g \in L^1(G)$ and $x \in G$, we have,

$$(2.6) \qquad T(f * g)(xH) \;=\; \rho(e) \int_G \frac{f(y)}{\rho(y)}\, Tg(y^{-1}xH)\, dy.$$

Since

$$\begin{aligned}
T(f*g)(xH) &= \int_H \frac{f*g(x\varepsilon)}{\rho(x\varepsilon)}\,d\varepsilon \\
&= \int_H \int_G f(y)\,g(y^{-1}x\varepsilon)\,\frac{\rho(e)}{\rho(y)\,\rho(y^{-1}x\varepsilon)}\,d\varepsilon\,dy \\
&= \int_G \frac{\rho(e)}{\rho(y)}\,f(y)\,\int_H \frac{g(y^{-1}x\varepsilon)}{\rho(y^{-1}x\varepsilon)}\,d\varepsilon\,dy \\
&= \rho(e)\int_G \frac{f(y)}{\rho(y)}\,Tg(y^{-1}xH)\,dy,
\end{aligned}$$

Then, $Tg = 0$ implies that $T(f*g) = 0$, for all $f \in L^1(G)$. But in the following we give a counterexample which shows $Tf = 0$ does not generally imply $T(f*g) = 0$, for all $g \in L^1(G)$. It means that, in general case, the mapping $* : L^1(G/H) \times L^1(G/H) \to L^1(G/H)$ is not well-defined. To reach the goal, we offer some conditions on (G, H) to obtain $T(f*g) = 0$ from $Tf = 0$, for all $g \in L^1(G)$. To do this, we will analyze some properties of integrable functions.

Proposition 2.1. *For all $f \in C_c(G)$ the following statements are equivalent:*

- **(a)** $T(f*g) = 0$, *for all* $g \in L^1(G)$.
- **(b)** $T(R_x f) = 0$, *for all* $x \in G$, *where* $R_x f$ *is defined by* $R_x f(y) = f(yx)$, $y \in G$.

Proof. It is clear that $T(f*g) = 0$, for all $g \in L^1(G)$ just when $T(f*g) = 0$, for all $g \in C_c(G)$. Now, for all $g \in C_c(G)$, by using (2.2) and (2.6) we get

$$\begin{aligned}
T(f*g)(xH) &= \rho(e)\int_G \frac{f(y)}{\rho(y)}\,\psi(y^{-1}xH)\,dy \\
&= \frac{\rho^2(e)}{\rho(x)}\int_G \frac{(L_{x^{-1}}f)(y)}{\rho(y)}\,\psi o\, q(y^{-1})\,dy \\
&= \frac{\rho(e)}{\rho(x)}\int_G \frac{(L_{x^{-1}}f)\check{}\,(y)}{\Delta_G(y)}\,\rho(y)\;\psi o\, q(y)\,dy \\
&= \frac{\rho(e)}{\rho(x)}\int_G \eta_x(y)\,\psi o\, q(y)\,dy \\
&= \int_{G/H} T(\eta_x)(yH)\,\psi(yH)\,d\mu(yH),
\end{aligned}$$

where $x \in G$, $\psi = Tg$, and $\eta_x = \frac{(L_{x^{-1}}f)\check{}\,\rho}{\Delta_G} \in C_c(G)$. This shows that $T(f*g) = 0$, for all $g \in C_c(G)$, if and only if $T(\eta_x) = 0$, for all $x \in G$. On the other hand,

$$\begin{aligned}
T(\eta_x)(yH) &= \frac{1}{\Delta_G(y)}\int_H \frac{f(x\xi^{-1}y^{-1})}{\Delta_G(\xi)}\,d\xi \\
&= \frac{\rho(e)}{\Delta_G(y)}\int_H \frac{f(x\xi y^{-1})}{\rho(\xi)}\,d\xi \\
&= \frac{\rho(x)}{\Delta_G(y)}\,T(R_{y^{-1}}f)(xH),
\end{aligned}$$

for all $y \in G$. Hence, $T(\eta_x) = 0$, for all $x \in G$, if and only if $T(R_y f) = 0$, for all $y \in G$. □

Therefore, we need some conditions on which $T(R_y f) = 0$ is obtained from $Tf = 0$, for all $f \in C_c(G)$ and $y \in G$. Note that this is equivalent to get $P(R_y f) = 0$, for all $f \in C_c(G)$ with $Pf = 0$ and all $y \in G$. The next lemma shows that this is equivalent to get $P(\check{f}) = 0$ from $P(f) = 0$, for all $f \in C_c(G)$, where H is a unimodular subgroup of G. And Example 2.3 shows that it may not hold in general.

Lemma 2.2. *Let H be a closed subgroup of a locally compact topological group G. Consider the following statements:*

- **(a)** *$P\check{f} = 0$ provided that $Pf = 0$, for all $f \in C_c(G)$.*
- **(b)** *$Pf = 0$ implies that $P(R_x f) = 0$, where $x \in G$ and $f \in C_c(G)$.*

Then, **(a)** *necessitates* **(b)**, *and they are also equivalent in the case where H is a unimodular subgroup of G.*

Proof. Evidently, $Pf = 0$ is equivalent to $P(L_x f) = 0$, where $f \in C_c(G)$ and $x \in G$. So, we can deduce **(b)** from **(a)**, because $R_x f = (L_x \check{f})\check{}$, for all $f \in C_c(G)$ and $x \in G$. The reverse implication becomes true when H is a unimodular subgroup of G, since then $P\check{f}(xH) = P(R_{x^{-1}} f)(eH)$, for all $x \in G$ and $f \in C_c(G)$. □

Example 2.3. Let G be the affine group, $G = \mathbb{R} \times' \mathbb{R}^+$, which is the cartesian product of $\mathbb{R}$ and $\mathbb{R}^+$ endowed with the usual topology and the operation,

$$(b_1, a_1).(b_2, a_2) = (b_1 + a_1 b_2\, ,\, a_1 a_2).$$

In this case, $b\,a$ will be another presentation for $(b, a) \in G$. Suppose that $H = \mathbb{R}^+$ and define $f_1 : \mathbb{R}^+ \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ by:

$$f_1(x) = \begin{cases} 0 & x \leq 1 \\ x\, sin(2\pi x) & 1 < x \leq 2 \\ 0 & 2 < x \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} f_1(\frac{x}{2}) & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Let $f : G \to \mathbb{R}$ be so that

$$f(t, \xi) = f_1(\xi) f_2(t) \quad (t \in \mathbb{R}\, ,\, \xi \in \mathbb{R}^+).$$

Trivially, f_1 and f_2 are continuous with compact supports and so is f. $Pf = 0$, since for all $x \in G$,

$$\begin{aligned} Pf(xH) &= \int_{\mathbb{R}^+} f(x\xi)\, \frac{1}{\xi}\, d\xi \\ &= \int_{\mathbb{R}^+} f(th\xi)\, \frac{1}{\xi}\, d\xi \\ &= f_2(t) \int_{\mathbb{R}^+} f_1(h\xi)\, \frac{1}{\xi}\, d\xi \\ &= f_2(t) \int_{\mathbb{R}^+} f_1(\xi)\, \frac{1}{\xi}\, d\xi \\ &= f_2(t) \int_1^2 sin(2\pi\xi)\, d\xi \\ &= 0, \end{aligned}$$

where $x = th$, $t \in \mathbb{R}$, and $h \in \mathbb{R}^+$. But $P(R_y f) \neq 0$, for $y = 2 \in \mathbb{R}$. Indeed,

$$\begin{aligned} P(R_y f)(eH) &= \int_{\mathbb{R}^+} f((0,\xi).(2,1))\,\frac{1}{\xi}\,d\xi \\ &= \int_{\mathbb{R}^+} f(2\xi,\xi)\,\frac{1}{\xi}\,d\xi \\ &= \int_{\mathbb{R}^+} f_1(\xi)^2\,\frac{1}{\xi}\,d\xi \\ &= \int_1^2 \xi^2\,sin^2(2\pi\xi))\,\frac{1}{\xi}\,d\xi \\ &= \frac{3}{4}. \end{aligned}$$

3. **On an special kind of homogeneous space**

Throughout this section, let G be a locally compact topological group and H be a compact subgroup of G. It is clear that, in this case $\Delta_G\mid_H = \Delta_H = 1$ and for all rho-functions ρ and $r \in \mathbb{R}$,

$$\rho^r(x\xi) \;=\; \rho^r(x) \quad (x \in G,\, \xi \in H). \tag{3.1}$$

This guarantees the existence of a continuous function η on G/H so that $\eta(xH) = \rho^r(x)$, for all $x \in G$. Also, Δ_G is a homomorphism rho-function and hence G/H has a relatively invariant Radon measure μ which arises from a rho-function ρ. We define $F(G,H)$ as follows:

$$F(G,H) = \{f \in C_c(G) : \forall x \in G\ \forall \xi \in H;\ f(x\xi) = f(x)\},$$

and $F^+(G,H)$ will denote the set of its positive elements. In the next lemma, we show that $C_c(G/H)$ consists of all functions of the form Tf, where $f \in C_c(G)$ is constant on the left cosets of H.

Lemma 3.1. *For all $\varphi \in C_c(G/H)$ there is some $f \in F(G,H)$ such that $\varphi = Tf$ and* supp $(f) = KH$*, where K is a compact subset of G for which* supp $(\varphi) = q(K)$*. Moreover, we can take $f \in F^+(G,H)$ where $\varphi \in C_c^+(G/H)$.*

Proof. Let $f = \frac{(\varphi\, o\, q)\,\rho}{m(H)}$, where $m(H) = \int_H d\xi$. Obviously, $f \in C_c(G)$ and if φ is positive, then so is f. Also, $\mathrm{supp}\,(f) = KH$, $f(x\xi) = f(x)$, and

$$\begin{aligned} Tf(xH) &= \int_H \frac{f(x\xi)}{\rho(x\xi)}\, d\xi \\ &= \frac{1}{m(H)} \int_H \frac{\varphi(xH)\,\rho(x)}{\rho(x)}\, d\xi \\ &= \varphi(xH), \end{aligned}$$

where $x \in G$ and $\xi \in H$. □

For all $1 \leq p \leq +\infty$, there exists a left module action of $L^1(G)$ on $L^p(G/H)$, defined by,

$$f *_p \psi(xH) = \rho(e)^{\frac{1}{p}} \int_G \frac{f(y)}{\rho(y)^{\frac{1}{p}}}\, \psi(y^{-1}xH)\, dy \quad (\mu - \text{almost all } xH \in \frac{G}{H}),$$

where $f \in L^1(G)$ and $\psi \in L^p(G/H)$. This action makes $L^p(G/H)$, $1 \leq p < +\infty$, a Banach left $L^1(G)$-module with an approximate identity (cf. [7]). We need an special kind of approximate identity for $L^p(G/H)$ that we introduce as follows.

Lemma 3.2. *Let $\mathcal{U}$ be a neighborhood base at e in G. For all $U \in \mathcal{U}$, there exists a function $h_U \in C_c^+(G)$ such that $\tilde{h}_U \in F(G, H)$, $\mathrm{supp}\,(h_U) \subseteq U$, and $\| \frac{h_U}{\rho^{\frac{1}{p}}} \|_1 = \rho(e)^{-\frac{1}{p}}$. Moreover, $\| h_U *_p \varphi - \varphi \|_p \to 0$ as $U \to \{e\}$ if $1 \leq p < +\infty$ and $\varphi \in L^p(G/H)$, or if $p = +\infty$ and φ is left uniformly continuous.*

Proof. Without loss of generality, suppose that each U in $\mathcal{U}$ is symmetric with compact closure. For all $U \in \mathcal{U}$, there exists an open set V_U with compact closure such that $H\, V_U \subseteq U$. Let $k_U \in C_c^+(G)$, whose support is contained in V_U. By using Lemma 3.1, we can take some $s_U \in F^+(G, H)$ so that $\mathrm{supp}\,(s_U) \subseteq V_U H$ and $Tk_U = Ts_U$. For all $1 \leq p \leq +\infty$, let $h_U = \frac{\rho^{\frac{1}{p}}\, \tilde{s}_U}{\rho(e)^{\frac{1}{p}}\, \|\tilde{s}_U\|_1}$. Obviously, $h_U \in C_c^+(G)$, $\tilde{h}_U = \frac{\rho(e)^{\frac{1}{p}}\, s_U}{\|\tilde{s}_U\|_1\, \rho^{\frac{1}{p}}} \in F(G, H)$, $\| \frac{h_U}{\rho^{\frac{1}{p}}} \|_1 = \rho(e)^{-\frac{1}{p}}$, and

$$\mathrm{supp}\,(h_U) \subseteq HV_U \subseteq U.$$

It has been shown that such a family of functions with properties supp $(h_U) \subseteq U$, and $\|\frac{h_U}{\rho^{\frac{1}{p}}}\|_1 = \rho(e)^{-\frac{1}{p}}$ is an approximate identity for $L^p(G/H)$ as a Banach left $L^1(G)$-module, where $1 \leq p < +\infty$. This is due to the fact that

$$\begin{aligned}
\| h_U *_p \varphi - \varphi \|_p &= (\int_{G/H} | h_U *_p \varphi(xH) - \varphi(xH)|^p \, d\mu(xH))^{\frac{1}{p}} \\
&= (\int_{G/H} | \int_G (\tfrac{\rho(e)}{\rho(y)})^{\frac{1}{p}} h_U(y) \, (\varphi(y^{-1}xH) - \varphi(xH)) \, dy|^p \, d\mu(xH))^{\frac{1}{p}} \\
&\leq \int_G (\tfrac{\rho(e)}{\rho(y)})^{\frac{1}{p}} h_U(y) \, (\int_{G/H} | (\varphi(y^{-1}xH) - \varphi(xH)) |^p \, d\mu(xH))^{\frac{1}{p}} dy \\
&\leq \sup\{ \|L_y\varphi - \varphi\|_p : y \in U\},
\end{aligned}$$

where $\varphi \in L^p(G/H)$. Also, for all $\varphi \in L^\infty(G/H)$, $\| h_U *_\infty \varphi - \varphi \|_\infty \to 0$ as $U \to \{e\}$ if φ is left uniformly continuous (cf. [7]). □

From now on, let

$$P(\frac{G}{H}) = \{\varphi \in C_c(\frac{G}{H}) : \exists \eta \in C(\frac{G}{H}) \; \forall x \in G; \; \varphi(x^{-1}H) = \eta(xH) \, \overline{\varphi(xH)}\}.$$

We denote by $\langle P(G/H) \rangle$ the linear span of $P(G/H)$ and show that each continuous function with compact support is the uniform limit of a sequence in this space.

Proposition 3.3. *$\langle P(G/H) \rangle$ is a dense subset of $L^p(G/H)$, where $1 \leq p \leq +\infty$.*

Proof. Without loss of generality, suppose that $m(H) = 1$. We first prove that for all $f \in F(G,H)$, $\tilde{f} *_1 \varphi \in P(G/H)$, where $\varphi = Tf$. Then, we show that the linear space $\langle P(G/H) \rangle$ contains all functions of the form $\tilde{f} *_1 \psi$, and by using this, we conclude that $\langle P(G/H) \rangle$ contains all functions of the form $\tilde{f} *_p \psi$, where $f \in F(G,H)$ and $\psi \in C_c(G/H)$, for $1 \leq p \leq +\infty$. It turns out that $\langle P(G/H) \rangle$ is dense in $L^p(G/H)$ by using an approximate identity. For details, let η be the continuous function on G/H for which $\eta(xH) = \frac{\rho^2(x)}{\rho^2(e)}$, $x \in G$. Recall that (3.1) guarantees that η is well-defined. If $f \in F(G,H)$ and $\varphi = Tf$, then for all $x \in G$, we have,

$$\begin{aligned}\tilde{f} *_1 \varphi(xH) &= \int_G \frac{\rho(e)}{\rho(y)} \tilde{f}(y)\, \varphi(y^{-1}xH)\, dy \\ &= \int_G \tilde{f}(y) \int_H \frac{f(y^{-1}x)}{\rho(x\xi)}\, d\xi\, dy \\ &= \frac{m(H)}{\rho(x)}\, (\tilde{f} * f)(x),\end{aligned}$$

and hence,

$$\begin{aligned}\tilde{f} *_1 \varphi(x^{-1}H) &= \frac{m(H)\,\rho(x)}{\rho^2(e)}\, \overline{(\tilde{f} * f)(x)} \\ &= \eta(xH)\, \overline{\tilde{f} *_1 \varphi(xH)}.\end{aligned}$$

This shows that

$$\{\tilde{f} *_1 \varphi :\ f \in F(G,H),\, \varphi = Tf\} \subseteq P(\frac{G}{H}).$$

It is easy to check that for all $f \in F(G,H)$ and $\psi \in C_c(G/H)$, we have,

$$\tilde{f} *_1 \psi \ = \ \frac{1}{4}\, \Sigma_{n=1}^{4} i^n\, (g + i^n\, f)\tilde{\ } *_1 (\psi + i^n\, Tf),$$

where $g \in F(G,H)$ and $\psi = Tg$, and so we have $\tilde{f} *_1 \psi \in \langle P(G/H)\rangle$. It can easily be seen that, for all $1 \le p \le +\infty$, $f \in F(G,H)$, $\psi \in C_c(G/H)$, and $x \in G$, we have,

$$\tilde{f} *_p \psi \ = \ (\frac{\rho^{\frac{1}{p}-1} f}{\rho(e)^{\frac{1}{p}-1}})\tilde{\ } *_1 \psi \quad \text{and} \quad \frac{\rho^{\frac{1}{p}-1} f}{\rho(e)^{\frac{1}{p}-1}} \in F(G,H).$$

This implies $\tilde{f} *_p \psi \in \langle P(G/H)\rangle$ if $f \in F(G,H)$ and $\psi \in C_c(G/H)$. Now, for all $1 \le p \le +\infty$, let $\{h_U\}_{U\in\mathcal{U}}$ be a family of functions in $C_c(G/H)$, as introduced in Lemma 3.2. Then, for all $\psi \in C_c(G/H)$, we have $\| h_U *_p \psi - \psi \|_p \to 0$ as $U \to \{e\}$. With the argument above, for all $U \in \mathcal{U}$, $h_U *_p \psi \in \langle P(G/H)\rangle$. This completes the proof. □

Theorem 3.4. *Let H be a compact subgroup of G. Then, for all $f \in C_c(G)$, $Pf = 0$ if and only if $P\check{f} = 0$.*

Proof. Since $(\check{f})\check{\ } = f$, then we only need to prove that $Pf = 0$ implies $P\check{f} = 0$, where $f \in C_c(G)$. For this, let μ be the invariant measure on

G/H, which arises from $\rho(x) = 1$. Also, let $f \in C_c(G)$ and $Pf = 0$. For all $\psi \in P(G/H)$, there exists a continuous function η on G/H such that for all $x \in G$, $\psi(x^{-1}H) = \eta(xH)\,\overline{\psi(xH)}$. Also, $\Delta_G(x^{-1}) = \gamma\, o\, q(x)$, for some continuous function γ on G/H. Therefore, we have,

$$\begin{aligned}\int_{G/H} P(\check{f})(xH)\ \psi(xH)\ d\mu(xH) &= \int_G f(x)\ \psi\, o\, q(x^{-1})\ \Delta_G(x^{-1})\ dx \\ &= \int_G f(x)\ \eta\, o\, q(x)\ \overline{\psi\, o\, q(x)}\ \gamma(xH)\ dx \\ &= \int_{G/H} P(f)(xH)\,\eta(xH)\,\overline{\psi(xH)}\ \gamma(xH)\ d\mu(xH), \\ &= 0.\end{aligned}$$

for all $\psi \in P(G/H)$ and hence $P(\check{f}) = 0$. □

4. **Main results**

A group acts on itself by its own multiplication. A left translation of a function defined on a group G is defined via the action of G on itself. The convolution of two integrable functions f and g on G can be introduced as a generalized linear combination of the left translations of g. Through this point of view, we can extend the concept of convolution to more general cases.

If H is a closed subgroup of G, then G/H may be considered as a homogeneous space on which G acts on G/H by $x(yH) = (xy)H$, where $x, y \in G$. Thus, a type of the left translation $L_x\varphi$ of a function φ on G/H may be defined by $L_x\varphi(yH) = \varphi(x^{-1}yH)$, where $x, y \in G$. It is easy to check that for all $\varphi \in C_c(G/H)$, $L_x\varphi \in C_c(G/H)$ and $L_x\varphi = P(L_xf)$, where $\varphi = Pf$ and $x \in G$. The left translations of integrable functions on a homogeneous space can be introduced in a similar way. More precisely, for all $\varphi \in L^1(G/H)$ and $x \in G$, there is an element $L_x\varphi$ of $L^1(G/H)$, called the left translation of φ by x, which satisfies,

$$L_x\varphi(yH) = \varphi(x^{-1}yH) \quad (\mu - \text{almost all } yH \in \frac{G}{H}).$$

For all $\varphi \in L^1(G/H)$, the mapping from G into $L^1(G/H)$, defined by $x \mapsto L_x\varphi$, is continuous. Also, $\|L_x\varphi\|_1 = \frac{\rho(x)}{\rho(e)}\|\varphi\|_1$, where $x \in G$ and

$\varphi \in L^1(G/H)$ (cf. [7]).

From now on, we suppose that H is a compact subgroup of G and μ is a relatively invariant measure on G/H, which arises from a rho-function ρ. For all $\varphi, \psi \in C_c(G/H)$, we define $\varphi * \psi$, the *convolution* of φ and ψ, by $\varphi * \psi = T(f * g)$, in which $f, g \in C_c(G)$ are so that $\varphi = Tf$ and $\psi = Tg$. Then, a glance at Proposition 2.1, Lemma 2.2, and Theorem 3.4 persuade us that $\varphi * \psi$ is well-defined. The following corollaries easily follow from the definition.

Corollary 4.1. *The following identities hold for all* $\varphi, \varphi_1, \varphi_2, \psi, \psi_1, \psi_2 \in C_c(G/H)$ *and* $c \in \mathbb{C}$*:*

- **(a)** $(\varphi_1 + c\,\varphi_2) * \psi = (\varphi_1 * \psi) + c\,(\varphi_2 * \psi)$.
- **(b)** $\varphi * (\psi_1 + c\,\psi_2) = (\varphi * \psi_1) + c\,(\varphi * \psi_2)$.
- **(c)** $\varphi * \psi \in C_c(G/H)$ *where* $\varphi, \psi \in C_c(G/H)$*, and* $\varphi * \psi \in C_c^+(G/H)$ *if* $\varphi, \psi \in C_c^+(\frac{G}{H})$.

Proposition 4.2. *For all* $\varphi, \psi \in C_c(G/H)$*,* $\|\varphi * \psi\|_1 \leq \|\varphi\|_1 \,.\, \|\psi\|_1$.

Proof. Let $\varphi, \psi \in C_c(G/H)$. Then, for all $f, g \in C_c(G)$ with $\varphi = Tf$ and $\psi = Tg$, we have,

$$\begin{aligned} \|\varphi * \psi\|_1 &= \|T(f * g)\|_1 \\ &\leq \|f * g\|_1 \\ &\leq \|f\|_1 . \|g\|_1. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\varphi * \psi\|_1 &\leq \inf\{\|f\|_1 : f \in C_c(G), \varphi = Tf\} \,.\, \inf\{\|g\|_1 : g \in C_c(G), \psi = Tg\} \\ &= \|\varphi\|_1 \,.\, \|\psi\|_1. \end{aligned}$$

□

Proposition 4.2 helps us define the convolution of two elements of $L^1(G/H)$ by using the unique extension of the convolution on $C_c(G/H)$:

$$\begin{cases} * : C_c(\frac{G}{H}) \times C_c(\frac{G}{H}) \longrightarrow L^1(\frac{G}{H}), \\ \qquad (\varphi, \psi) \longmapsto \varphi * \psi. \end{cases}$$

In other words, for all $\varphi, \psi \in L^1(G/H)$, $\varphi * \psi$ can be defined as the limit of $\{\varphi_n * \psi_n\}_{n \in \mathbb{N}}$ in $L^1(G/H)$, where $\{\varphi_n\}_{n \in \mathbb{N}}$ and $\{\psi_n\}_{n \in \mathbb{N}}$ are two sequences in $C_c(G/H)$ which approach φ and ψ, respectively, in $L^1(G/H)$.

Theorem 4.3. *For all* $\varphi, \psi \in L^1(G/H)$, $\varphi * \psi = T(f * g)$, *where* $f, g \in L^1(G)$ *such that* $\varphi = Tf$ *and* $\psi = Tg$.

Proof. For all $\varphi, \psi \in L^1(G/H)$, fix two elements $f, g \in L^1(G)$ so that $\varphi = Tf$ and $\psi = Tg$. There exist sequences $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ in $C_c(G)$ converging to f and g, respectively, in $L^1(G)$. Then, we can write,

$$\begin{aligned}\varphi * \psi &= lim_{n\to+\infty}(Tf_n) * (Tg_n) \\ &= lim_{n\to+\infty}T(f_n * g_n) \\ &= T(f * g).\end{aligned}$$

□

As indicated before, the convolution of two integrable functions f and g, $f * g$, is usually introduced as a generalized linear combination of the left translations of g. When H is a compact subgroup of G, the convolution of two integrable functions φ and ψ, defined on $L^1(G/H)$, can be considered as a generalized linear combination of the left translations of ψ. Since by using (2.6) and Theorem 4.3, we have,

$$\varphi * \psi(xH) = \rho(e) \int_G \frac{f(y)}{\rho(y)}\, \psi(y^{-1}xH)\, dy \quad (for\ \mu-almost\ all\ xH \in \frac{G}{H}) \tag{4.1}$$

where $f \in L^1(G)$ and $\varphi = Tf$.

Now, for all $\varphi \in L^1(G/H)$, we introduce φ^* as $T(f^*)$, where $f \in L^1(G)$ is such that $\varphi = Tf$. This operator is well-defined, since $T(f^*) = 0$ if $T(f) = 0$. To show this, let $f \in L^1(G)$ and $T(f) = 0$. For all $\psi \in P(G/H)$, there exists a continuous function η on G/H for which $\psi(x^{-1}H) = \eta(xH)\,\overline{\psi(xH)}$, $x \in G$. Therefore, we have,

$$\begin{aligned}\int_{G/H} T(f^*)(xH)\,\psi(xH)\,d\mu(xH) &= \int_G \frac{\overline{f(x^{-1})}}{\Delta_G(x)}\,\psi\, o\, q(x)\,dx \\ &= \int_G \overline{f(x)}\,\eta\, o\, q(x)\,\overline{\psi\, o\, q(x)}\,dx \\ &= \int_{G/H} \overline{Tf(xH)}\,\eta(xH)\,\overline{\psi(xH)}\,d\mu(xH) \\ &= 0,\end{aligned}$$

for all $\psi \in P(G/H)$. Hence, $T(f^*) = 0$, where $f \in L^1(G)$ and $Tf = 0$.

It is easy to show that the linear operator $\varphi \mapsto \varphi^*$, $\varphi \in L^1(G/H)$, is an involution on $L^1(G/H)$. Our main result is that the convolution and the involution defined on $L^1(G/H)$, make it an involutive Banach algebra with an approximate identity with a similar structure to $L^1(G)$.

Theorem 4.4. *$L^1(G/H)$ is an involutive Banach algebra.*

Proof. The associativity of convolution "$*$" on $L^1(G/H)$ is obtained from Theorem 4.3 and the associativity of convolution on $L^1(G)$. Also, it is easily observed that the equalities in Corollary 4.1 remain valid for all functions in $L^1(G/H)$. A conclusion similar to the proof of Theorem 4.1 proves that:

$$\|\varphi * \psi\|_1 \leq \|\varphi\|_1.\|\psi\|_1 \quad (\varphi, \psi \in L^1(\frac{G}{H})).$$

Moreover, for all $\varphi \in L^1(G/H)$, we have,

$$\begin{aligned} \|\varphi^*\|_1 &= \inf\{\|g\|_1 : g \in L^1(G),\, \varphi^* = Tg\} \\ &= \inf\{\|f^*\|_1 : f \in L^1(G),\, \varphi = Tf\} \\ &= \inf\{\|f\|_1 : f \in L^1(G),\, \varphi = Tf\} \\ &= \|\varphi\|_1. \end{aligned}$$

□

Theorem 4.5. *Let $\mathcal{U}$ be a neighborhood base at e in G. For all $U \in \mathcal{U}$, take $\psi_U = T(h_U)$, where $h_U \in C_c^+(G)$ is such that* $\text{supp}\,(h_U) \subseteq U$, *$\check{h}_U = h_U$, and $\int_G h_U(x)\,dx = 1$. Then, for all $\varphi \in L^1(G/H)$, $\|\varphi * \psi_U - \varphi\|_1 \to 0$ and $\|\psi_U * \varphi - \varphi\|_1 \to 0$ as $U \to \{e\}$.*

Proof. Let $\mathcal{U}$ be a neighborhood base at e. For all $U \in \mathcal{U}$, there exists a function $h_U \in C_c^+(G)$ so that $\text{supp}\,(h_U) \subseteq U$, $\check{h}_U = h_U$, and $\int_G h_U(x)\,dx = 1$. Then, $\{h_U\}_{U \in \mathcal{U}}$ is an approximate identity for $L^1(G)$ (cf. [2], Proposition 2.42). Now, let $\psi_U = T(h_U)$, where $U \in \mathcal{U}$, and suppose that $\varphi \in L^1(G/H)$. Then, $\varphi = Tf$, for some $f \in L^1(G)$ and hence,

$$\begin{aligned} \|\varphi * \psi_U - \varphi\|_1 &= \|T(f * h_U) - Tf\|_1 \\ &\leq \|f * h_U - f\|_1. \end{aligned}$$

This implies that $\|\varphi * \psi_U - \varphi\|_1 \to 0$ as $U \to \{e\}$. In a similar way, it can be shown that $\|\psi_U * \varphi - \varphi\|_1 \to 0$ as well, where $U \to \{e\}$. □

Finally, we give a necessary and sufficient condition on a closed subspace of $L^1(G/H)$ to make it a left ideal. But, first we point out that one can show by an easy calculation that

$$L_x(\varphi * \psi) = (L_x\varphi) * \psi,$$

for all $\varphi, \psi \in L^1(G/H)$ and $x \in G$.

Theorem 4.6. *Let $\mathcal{I}$ be a closed subspace of $L^1(G/H)$. Then, $\mathcal{I}$ is a left ideal if and only if it is closed under left translations.*

Proof. Suppose that $\mathcal{I}$ is a left ideal, $\{\psi_U\}_{U\in\mathcal{U}}$ is an approximate identity, and $\varphi \in \mathcal{I}$. Then, for all $x \in G$ we can write,

$$\begin{aligned} L_x\varphi &= lim_{U\to\{e\}} L_x(\psi_U * \varphi) \\ &= lim_{U\to\{e\}} (L_x\psi_U) * \varphi, \end{aligned}$$

which shows that $L_x\varphi \in \mathcal{I}$. For the converse, suppose that $\mathcal{I}$ is closed under left translations. It follows from (4) that for all $\varphi, \psi \in C_c(G/H)$, $\varphi * \psi$ is in the closed linear span of the left translations of ψ, which shows that $\varphi * \psi$ belongs to $\mathcal{I}$. Using the fact that $C_c(G/H)$ is dense in $L^1(G/H)$, we achieve $\varphi * \psi \in \mathcal{I}$ for all $\varphi, \psi \in L^1(G/H)$. □

It is easy to see that if H is the trivial subgroup $\{e\}$, then $T(f * g)(xH) = f * g(x)$, $x \in G$, where $(G/H, \mu)$ is considered as a measure space with the G-invariant measure μ which arises from the constant function $\rho(x) = 1$.

References

[1] J. Dixmier, *C*-Algebras*, translated from French by Francis Jellet, North-Holland Mathematical Library, Vol. 15, North-Holland Publishing Co., New York, 1977.

[2] G.B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, Boca Raton, 1995.

[3] G.B. Folland, *Real Analysis, Modern Techniques and Their Applications*, 2nd Ed., John Wiley and Sons, New York, 1999.

[4] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis*, Vol. 1 and 2, Springer-Verlag, Berlin, 1963.

[5] H. Reiter and J.D. Stegeman, *Classical Harmonic Analysis*, 2nd Ed., Oxford University Press, New York, 2000.

[6] R.A. Kamyabi-Gol and N. Tavallaei, Wavelet transforms via generalized quasi-regular representations, J. Appl. Comp. Harmonic Anal. **26** (3) (2009) 291–300.
[7] R.A. Kamyabi-Gol and N. Tavallaei, On the function spaces related to homogeneous spaces, Preprint.

R.A. Kamyabi-Gol, Center of Excellence in Analysis on Algebraic Structures, Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran.
Email: kamyabi@ferdowsi.um.ac.ir

N. Tavallaei,
School of Mathematics and Computer Science, Damghan University of Basic Sciences, Damghan, Iran.
Email: tavallaie@dubs.ac.ir