Almost optimal control design for discrete-time nonlinear time-delay systems

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Abstract: This paper proposes an optimal control design method for a class of discrete-time nonlinear time-delay systems, subject to mixed control-state constraints. The delays in the systems are in state and/or in control input. An optimal control is designed based on the measure theory, functional analysis and linear programming by optimising a definite objective function and transferring the difference system equation into a new metric space in which the number of the sample times are the same but their values and properties may differ. In fact, by an embedding process the problem is nearly the same as in a continuous case. The problem is first transferred to a new optimal measure problem and then this new problem is replaced with one in which minimising a linear form over a subset of linear equalities is required. It is shown that this minimisation is global and the embedding procedure allows one to develop a computational method to find a solution using a finite–dimensional linear programming problem. A numerical example is also presented to illustrate the procedure of the method.

Keywords: Time-delay systems, optimal control, nonlinear systems, control design, linear programming.

1. INTRODUCTION

The dynamics of many control systems may be expressed by time-delay equations. The delay(s) may appear in the system state, control input and/or output (Levaggi and Punta 2006; Fridman *et al.*, 2005). Delays in the control input arise in many chemical processes and radiation problems in physics. Delays may also appear because of physical properties of equipment used in the system, signal transmission or measurement of system variables. For example, actuators, sensors and field networks which are involved in feedback loops may establish delays. Time-delay systems are also used to model several different mechanisms in the dynamics of epidemics.

Many problems such as incubation periods, mechanics, viscoelasticity, physics, physiology, population dynamics, communication, information technologies and stability of networked controlled, maturation times, age structure, blood transfusions and interactions across spatial distances or through complicated paths have been modelled by the introduction of time-delay systems (Busenberg and Cooke 1992; Boukas and Liu 2002).

Many approaches have been developed during the last two decades to control linear and nonlinear time-delay systems with and without uncertainties including sliding mode control (Koshkouei and Zinober 1996; Niu *et al.* 2005; Luo *et al.* 1997), backstepping techniques (Mazenc and Bliman 2006; Hua *et al.* 2005; Jankovic 2009), H_2/H_∞ approach (Fridman *et al.* 2005; Wu and Zheng 2009), output tracking (Shtessel *et al.* 2003), neural networks and adaptive control (Ge *et al.* 2003).

LMI techniques may also be used to design a control/observer or to prove the stability of a time-delay system (Koshkouei and Burnham 2009). However, the control of a nonlinear time-delay system is relatively complicated, particularly when there is an uncertainty in the system. Therefore, to simplify the problem usually it is assumed that the system is linearisable and/or the uncertainty is matched (Shin *et al.* 2006).

Optimal control (regulator) problem for linear system states has been solved in 1960s (Fleming and Rishel 1975; Kwakernaak and Sivan 1972). However, many attempts have been done to solve optimal control problems for many classes of linear /nonlinear time-delay systems. These approaches have been proposed based on the type of delays, the structure of the systems and assumptions (Boukas and Liu 2002; Dion 2001; Kolmanovskii and Shaikhet 1996; Kolmanovskii and Myshkis 1996; Malek-Zavarei and Jamshidi 1987), using the maximum principle (Kharatashvili 1967) or the dynamic programming method (Oguztoreli 1966). Most earlier studies focused on the time-optimal criterion: for linear systems (Oguztoreli 1963) or the quadratic (Delfour 1986; Eller *et al.* 1969; Uchida 1988) or by applying the duality principle to the solution of the optimal filtering problem.

In this paper, the optimal control for discrete time nonlinear system with delays in state and control variables subject to mixed control-state constraints are considered. An almost discrete-time optimal control is designed using functional analysis and linear programming theories and optimising an appropriate cost function. By applying an embedding process, the problem is first transferred to a new optimal measure problem, which is solvable by minimising a linear form over a subset of linear equalities. This method is an extension of the work by Rubio (1986) to a class of nonlinear discrete time-delay systems. These methods have been used for designing/finding an optimal control/solution of various systems including designing an optimal control for multidimensional diffusion equation (Kamayad *et al.* 1991), optimal control of linear wave equation (Farahi *et al.* 1996), optimal control of diffusion equation (Rubio 1995, 2000), shortest path problems (Zamirian *et al.* 2007) and optimal nonlinear control problems (Kamayad *et al.* 2007). The obtained control is almost optimal and no extra initial condition is required to be assumed. The most important advantage of this method is that the optimal control, and therefore the optimal trajectory, is estimated without employing any iteration technique. In addition, this method is systematic and straightforward and can be applied or extended to many classes of discrete-time nonlinear systems.

This paper is organised as follows: In Section 2 the system formulations and background of the problem are presented. Section 3 addresses the almost optimal method for a class of nonlinear time-delay systems using measure theory. In Section 4, an example is given to illustrate the theoretical results. Conclusions are expressed in Section 5.

2. SYSTEM DESCRIPTION

Consider the following discrete- time delay system

$$x(k+1) = f(x(k)) + g(x(k-\tau_1)) + h(x(k))u(k-\tau_2) \qquad (1)$$

where, $x \in A \subseteq \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$, while A and U are closed subsets, τ_1 and τ_2 are constant known positive integer numbers, $f, g \in \mathbb{R}^n$ and $h \in \mathbb{R}^{n\times m}$ are bounded vector fields. It is assumed that

$$x(k) = \phi(k), \quad -\tau_1 \le k \le k_0,$$

$$u(k) = \theta(k), \quad -\tau_2 \le k \le k_0$$

where k_0 is the initial time and, $\phi(k)$ and $\theta(k)$ are known bounded vector functions. One may consider

$$u(k) = \alpha, \quad -\tau_2 \le k \le k_0 \ .$$

Let k_f be the final time. The aim is to design an optimal discrete control u such that the state trajectories starting from $x(k_0) = x_0$ reach a given point $x(k_f) = x_f$. It is also assumed that $u(k) = [u_1(k) \quad u_2(k) \quad \dots \quad u_m(k)]^T$ and $x(k) = [x_1(k) \quad x_2(k) \quad \dots \quad x_n(k)]^T$ are bounded functions for $I = \{k \in \mathbb{Z}: k_0 \le k \le k_f\}$ where $\mathbb{Z}$ denotes the integer numbers, i.e.

$$\alpha_{1_i} = u_{m_i}(k) \le u_i(k) \le u_{M_i}(k) = \alpha_{2_i}, \qquad i = 1, 2, \cdots, m$$

$$\beta_{1_j} = x_{m_j}(k) \le x_j(k) \le x_{M_j}(k) = \beta_{2_j}, \qquad j = 1, 2, \cdots, n.$$

$$(2)$$

where, α_{1_i}, α_{2_i}, β_{1_j} and β_{2_j} are appropriate known real numbers. This assumption is necessary for the existence of a discrete optimal control using the method of this paper. Consider the cost function

$$J(x(\cdot), u(\cdot)) = \sum_{i=k_0}^{i=k_f} \left(f_x^T(x(i)) f_x(x(i)) + f_u^T(u(i)) f_u(u(i))\right) \qquad (3)$$

where the vector field functions $f_x \in \mathbb{R}^n$ and $f_u \in \mathbb{R}^m$ are known bounded functions. It is desired to minimise the functional (3) over the set of admissible pairs $(x(\cdot), u(\cdot))$.

3. VARIATIONAL FORMULATION

The aim is to minimise (3), subject to the discrete-time delay system (1) with the boundary conditions (2). To achieve the objective, the system (1) is first converted to a scalar integral equation problem using an arbitrary auxiliary nonzero bounded n-vector function. Then by applying an embedding process, the problem is transferred to a new optimal measure problem, which is solvable by minimising a linear form over a subset of linear equalities.

Define $\tau = \max\{\tau_1, \tau_2\}$ and select the auxiliary function

$$\eta(k) = [\eta_1(k) \quad \eta_2(k) \quad \dots \quad \eta_n(k)]^T$$

where $\eta_j(k)$, $-\tau \le k \le k_f$ are arbitrary nonzero bounded functions. Then from (1)

$$\eta^T(k+1)x(k+1) = \eta^T(k+1) f(x(k)) \qquad (4)$$

$$-\eta^T(k+1) g(x(k-\tau_1)) + \eta^T(k+1) h(x(k)) u(k-\tau_2).$$

Subtracting $\eta^T(k)x(k)$ from the both sides of (4) yields

$$\eta^T(k+1)x(k+1) - \eta^T(k)x(k) = \eta^T(k+1) f(x(k)) +$$

$$\eta^T(k+1) g(x(k-\tau_1)) + \eta^T(k+1) h(x(k)) u(k-\tau_2)$$

$$-\eta^T(k)x(k) \qquad (5)$$

Note that both $\eta(k)$ and $x(k)$ are n-vector functions, and the multiplications in the above relation, are in fact inner product operations.

The summation from $k = k_0$ to $k_f - 1$ in (5) yields

$$\sum_{k=k_0}^{k=k_f-1} \left(\eta^T(k+1)x(k+1) - \eta^T(k)x(k)\right)$$

$$= \sum_{k=k_0}^{k=k_f-1} [\eta^T(k+1) f(x(k)) +$$

$$\eta^T(k+1) g(x(k-\tau_1)) + \eta^T(k+1) h(x(k)) u(k-\tau_2)$$

$$-\eta^T(k)x(k))] \qquad (6)$$

Define $\Delta_j\eta = x_j(k_f)\eta_j(k_f) - x_j(k_0)\eta_j(k_0)$ and $\Delta\eta = \sum_{j=1}^{j=n} \Delta_j\eta$. Then the left hand side of (6) is

$$\eta^T(k_f)x(k_f) - \eta^T(k_0)x(k_0) = \sum_{j=1}^{j=n} [\eta_j(k_f)x_j(k_f)$$

$$-\eta_j(k_0)x_j(k_0)] = \sum_{j=1}^{j=n} \Delta_j\eta \qquad (7)$$

Thus using (4) and (7), the variational form associated to the delay system (1) is

$\sum_{k=k_0}^{k=k_f-1}[\eta^T(k+1)f(x(k)) + \eta^T(k+1)$
$g(x(k-\tau_1)) + \eta^T(k+1)h(x(k))u(k-\tau_2) - \eta^T(k)x(k)]$
$= \Delta\eta$ (8)

Now the optimal control problem with delays in the state and control variables is converted into minimise (3) subject to (8).

The difference equation (8) can be expressed as

$$\sum_{k=k_0}^{k=k_f-1} f_\eta(k) + \sum_{k=k_0}^{k=k_f-1} g_{d_\eta}(k) + \sum_{k=k_0}^{k=k_f-1} h_{d_\eta}(k) - \sum_{k=k_0}^{k=k_f-1} x_\eta(k) = \Delta\eta,$$

where

$f_\eta(k) = \eta^T(k+1)f(x(k)),$

$g_{d_\eta}(k) = \eta^T(k+1)g(x(k-\tau_1)),$

$h_{d_\eta}(k) = \eta^T(k+1)\,h(x(k))\,(u(k-\tau_2)),$

$x_\eta(k) = \eta^T(k)x(k).$

Note that $g_{d_\eta}(k)$, and $h_{d_\eta}(k)$ are defined for all $k_0 \le k \le k_f$ and each multiplication on the right hand side of the above relations is in fact an inner product. The operating region of (k, x, u) is Ω, where

$$\Omega = I \times \prod_{j=1}^{j=n}[\beta_{1_j}, \beta_{2_j}] \times \prod_{i=1}^{i=m}[\alpha_{1_i}, \alpha_{2_i}] = I \times A \times U.$$

Definition 1: It is said that a trajectory control pair $p = (x(\cdot), u(\cdot))$ is *admissible*, if the following conditions hold:

(i) Trajectory function $x(\cdot)$ satisfies $x(k) \in A$, for $k \in I$, and is bounded on I;
(ii) The control function $u(\cdot)$ takes values within the closed set U, and is bounded on I;
(iii) The boundary conditions $x(k_0) = x_0$ and $x(k_f) = x_f$ are satisfied;
(iv) The pair p satisfies the discrete-time equation (1) for all $k_0 \le k < k_f$.

Let W be the set of admissible pairs. The discrete time-delay optimal control problem now is to find an admissible pair $p \in W$ such that

$J(p) = \text{Min}\, J(x(\cdot), u(\cdot)) = \text{Min} \sum_{k=k_0}^{k=k_f}(f_x^T(x(k))f_x(x(k)) + f_u^T(u(k))f_u(u(k))$ (9)

Consider the mapping

$\Lambda_p : F \in C(\Omega) \rightarrow \sum_{k=k_0}^{k=k_f} F(k, x(k), u(k)) \in \mathbb{R}$ (10)

where $C(\Omega)$ indicates the set of continuous functions on Ω. The functional Λ_p has the following properties:

(i) well-defined;
(ii) linear, i.e. $\Lambda_p(\alpha F + \beta G) = \alpha\Lambda_p(F) + \beta\Lambda_p(G)$, for any $\alpha, \beta \in \mathbb{R}$;
(iii) nonnegative, i.e. if $F(k, x, u) \ge 0$, for any $(k, x, u) \in \Omega$, then $\Lambda_p(F(k, x, u)) \ge 0$;
(iv) bounded.

Now consider the transformation $\phi_p : p \rightarrow \Lambda_p$, of an admissible pair into a positive linear functional. Then the following proposition shows that ϕ_p is an injection.

Proposition 1. The transformation $\phi_p : p \rightarrow \Lambda_p$ which maps the admissible pairs in W into the linear mapping Λ_p defined as in (10) is an injection.

Proof: The proof is similar to that one in Rubio (1986) and is omitted.

Therefore, a pair p can be identified with the linear functional Λ_p. Now the discrete time-delay optimal control problem (1) can then be written using the definition of the linear functional as:

Minimise $J[p] = \Lambda_p(f_0)$

subject to

$$\Lambda_p\left(f_\eta(k) + g_{d_\eta}(k) + h_{d_\eta}(k) - x_\eta(k)\right) = \Delta\eta \qquad (11)$$

where, $f_0 = f_x^T(x(k))f_x(x(k)) + f_u^T(u(k))f_u(u(k))$.

The image of the set of all admissible pairs W under the transformation $\phi_p : p \rightarrow \Lambda_p$ is within the set of all those linear functionals on $C(\Omega)$ which satisfy (11). Thus the problem is now converted into the following form:

Among those linear functionals on $C(\Omega)$ of the type Λ_p, it is required to find the one which minimises $\Lambda_p(f_0)$.

Let $D(I_d)$ be the set of all auxiliary functions defined on $I_d = \{k : k_0 \le k \le k_f\}$ and zero value at the initial point k_0 and the terminal point k_f, i.e. $\psi(k_0) = \psi(k_f) = 0$. Select $\eta(k) = \psi(k)$ where $\psi(k) \in D(I_d)$. Then $\Delta\eta = \Delta\psi(k) = 0$ and substituting ψ into the condition (11) yields

$$\Lambda_p\left(f_\psi(k) + g_{d_\psi}(k) + h_{d_\psi}(k) + x_\psi(k)\right) = 0$$

Now the optimisation problem (11) is converted into the following form:

Minimise $J[p] = \Lambda_p(f_0)$

subject to

$$\begin{aligned} &\Lambda_p\left(f_\psi(k) + g_{d_\psi}(k) + h_{d_\psi}(k) - x_\psi(k)\right) = 0, \\ &\Lambda_p\left(f_\eta(k) + g_{d_\eta}(k) + h_{d_\eta}(k) - x_\eta(k)\right) = \Delta\eta, \end{aligned} \qquad (12)$$

where $\eta(k_0) \ne 0$ or $\eta(k_f) \ne 0$. The optimisation problem (12) is a linear programming problem. The discrete time-delay optimal control problem is now to find an appropriate Λ_p on the set $C(\Omega)$ such that the functional $\Lambda_p(f_0)$ takes its minimum value.

To find such Λ_p it is required to *enlarge* the domain $C(\Omega)$ and consider all positive linear functionals Λ on $C(\Omega)$ which are satisfied the constraints (12), and minimise the continuous function $\Lambda \rightarrow \Lambda(f_0)$ over the enlarged domain $C(\Omega)$). These requirements guarantee the existence of an optimal solution. To this end, a Borel measure equivalent to Λ is first introduced. This result can be driven from Rosenblom (1952) and it is presented as a Proposition.

Proposition 2. Let Λ be a positive functional on $C(\Omega)$. Then there exists a positive Borel measure μ on Ω such that for any $F \in C(\Omega)$

$$\Lambda(F) = \sum_{k=k_0}^{k=k_f} F\big(k, x(k), u(k)\big) = \mu(F).$$

Assume that $M^+(\Omega)$ denotes the space of all positive Borel measures on Ω. Using these definitions and facts, the non-classical problem (12) is now converted into its definitive form, which is used latter. The extended problem can be now presented as follows:

Find a positive measure μ^* within the space $M^+(\Omega)$ which minimises the functional

$$\mu \in M^+(\Omega) \rightarrow \mu(f_0) \in \mathbb{R} \qquad (13)$$

subject to the following constraints:

(i) For any $\eta \in C \quad (I)$
$$\mu(f_\eta(k) + g_{d_\eta}(k) + h_{d_\eta}(k) + x_\eta(k) = \Delta\eta,$$
(ii) For any $\psi \in D(I)$
$$\mu\Big(f_\psi(k) + g_{d_\psi}(k) + h_{d_\psi}(k) - x_\psi(k)\Big) = 0, \qquad (14)$$

Define the set of all positive Radon measures satisfying (14) as Q, and consider the space $M^+(\Omega)$ by the weak*-topology. One can prove the existence of an optimal measure in the set Q for the functional $\mu \rightarrow \mu(f_0)$ under the imposed conditions (see Rubio (1986)).

Recall the optimisation problem (13)-(14). As stated, this problem is a Linear Programming (LP) in which all the functions in (13) and (14) are linear in variable μ, and the measure μ is required to be positive even if the original problem (1) with the cost functional (3) are nonlinear. Converting a nonlinear problem into a linear counterpart is an important result which is applicable to a wide range of practical problems.

Note that the LP problem (13)-(14) is an infinite dimensional linear programming problem. Many attempts have been done to extend the work and apply to many classes of systems without delays (Vershik 1970; Anderson and Nash 1987; Taksar 1997; Fattorini 2005). In this paper an approximation method is extended to time-delay discrete nonlinear systems First, assume that $I = \lfloor k_0, k_f \rfloor$ and the set consisting of functions η_i, i.e. $C_t = \{\eta_i: \ i \in \mathbb{N}\}$, is total in $C^1(I)$, that is, the linear combinations of the functions η_i are uniformaly dense in $C^1(I)$. Also assume that $D_t = \{\psi_j: j \in \mathbb{N}\}$ is total in $D(I)$. Under these conditions the following proposition can be proven.

Proposition 3. Let $C_{M_1} = \{\eta_i \in C_t: \ i = 1, 2, \dots, M_1\}$ and $D_{M_2} = \{\psi_j \in D_t: \ j = 1, 2, \dots, M_2\}$. Consider the linear program consisting in minimising the functional

$$\mu \rightarrow \mu(f_0) \qquad (15)$$

over the set $Q_{(M1,M2)}$ of measures in $M^+(\Omega)$ satisfying

$$\mu\big(f_{\eta_i}(.) + g_{d_{\eta_i}}(.) + h_{d_{\eta_i}}(.) - x_{\eta_i}(.)\big) = \Delta\eta_i, \quad i = 1, 2, \dots, M_1,$$
$$\mu\Big(f_{\psi_j}(.) + g_{d_{\psi_j}}(.) + h_{d_{\psi_j}}(.) - x_{\psi_j}(.)\Big), \ j = 1, 2, \dots, M_2 \qquad (16)$$

If M_1 and M_2 simultaneously tend to infinity, then $\inf \mu_{Q_{(M1,M2)}}(f_0) \rightarrow \mu^* = \inf \mu_Q(f_0)$.

Now it is possible to replace the above problem with a finite dimensional linear program scheme by minimising $\mu \rightarrow \mu(f_0)$ over the set of constraints (16). This fact can be driven from the results presented in Royden (1970). The following proposition presents the structure and properties of the optimal measure obtaining from a finite dimensional linear program scheme.

Proposition 4. The measure μ^* in the set $Q_{(M1,M2)}$ at which the functional $\mu \rightarrow \mu(f_0)$ attains its minimum has the form

$$\mu^* = \sum_{h=1}^{h=M_1+M_2} \alpha_h^* \, \delta(Z_h^*) \qquad (17)$$

where $Z_h^* \in \Omega$, the coefficients $\alpha_h^* \geq 0$ and $\delta(Z_h^*) \in M^+(\Omega)$, is the unitary atomic measure with the singleton set $\{Z_h^*\}$ as its support and characterised by $\delta(Z_h^*)F = F(Z_h^*)$, for $F \in C(\Omega)$, and $Z_h^* \in \Omega$.

Using (17) one can conclude that the measure-theoretical optimisation (15)-(16) is equivalent to a nonlinear optimisation problem in which the unknown parameters are the coefficients $\alpha_h^* \geq 0$, and supports $\{Z_h^*: h = 1, 2, \dots, M_1 + M_2\}$, while they are in Ω. If ω is a dense subset of Ω, and $N \gg M_1 + M_2$, and $\{z_i: i = 1, 2, \dots, N\} \subseteq \omega$ where z_i; $i = 1, 2, \dots, N$ are known points in ω, then the nonlinear optimisation problem (15)-(16) can be approximated by the following linear programming statement:

$$\text{Minimise} \quad \sum_{h=1}^{N} \alpha_h \, f_0(z_h) \qquad (18)$$

subject to the following constraints:

$$\sum_{h=1}^{N} \alpha_h \left(f_{\eta_i} + g_{d_{\eta_i}} + h_{d_{\eta_i}} - x_{\eta_i}\right)(z_h) = \Delta\eta_i, \quad i = 1, 2, \dots, M_1$$
$$\sum_{h=1}^{N} \alpha_h \left(f_{\psi_j} + g_{d_{\psi_j}} + h_{d_{\psi_j}} - x_{\psi_j}\right)(z_h) = 0, \quad j = 1, 2, \dots, M_2$$
$$\alpha_h \geq 0, \qquad z_h \in \omega \subseteq \Omega \qquad (19)$$

The set ω will be covered with a grid, where the grid will be defined by taking all points in ω as:

$$z_h = (h, x_h, u_h), \qquad h = 1, 2, \dots, N.$$

The points in the grid are numbered sequentially from 1 to N. We used a home-made Revised Simplex to solve the linear programming problem (18)-(19). The analysis of constructing control and trajectories follows from Rubio(1986).

In the next section, this computational algorithm is used to solve a time-delay optimal control problem.

4. NUMERICAL RESULT

Consider the following discrete time-delay nonlinear system

$x(k+1) = x(k-1)u(k-2)$

with

$x(k) = \phi(k) = 1, \quad k = -1, 0,$

$u(k) = \theta(k) = 1, \quad k = -2, -1, 0.$

Therefore, the state and control input delays are $\tau_1 = 1$ and $\tau_2 = 2$, respectively. Assume that $k_0 = 0$ and $k_f = 30$. Consider the cost functional

$$J = \sum_{0}^{30} (x^2(k) + u^2(k))$$

Now the optimal control problem for this system can be summarised as follows:

Minimize $J = \sum_0^{30} (x^2(k) + u^2(k))$

Subject to:

$x(k+1) = x(k-1)u(k-2),$

$x(k) = \phi(k) = 1, \quad k = -1, 0,$

$u(k) = \theta(k) = 1, \quad k = -2, -1, 0.$

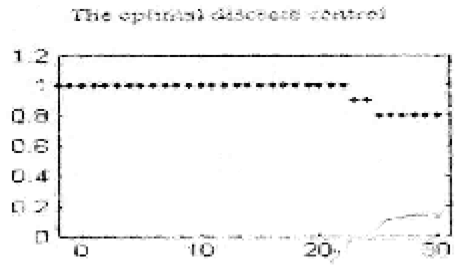
So $x(0) = 1$. Select the final state value $x(k_f) = 0.6$. The optimal problem is to design the control u such that a trajectory which starts form $x(0)$ reaches the final point $x(k_f) = 0.6$.

In this example, $I = [0, 30]$ and it is assumed $0 \le x(t) \le 1$ and $-1 \le u(t) \le 1$. So A=[0, 1], $U = [-1,1]$. The set $\Omega = I \times A \times U$ is covered with a grid, where the grid is defined by taking points $z_h = (h, x_h, u_h)$ in Ω. The points in the grid are numbered sequentially from 1 to N.

Figure 1 shows the obtained discrete optimal control and the system state behaviour. Note that the control was defined $u = 1$ for all $-\tau_2 \le k \le 0$]. Therefore, the system equation yields $x(k) = 1$ for $1 \le k \le \tau_2 + 1$.

5. CONCLUSIONS

In this paper, the embedding method (embedding an admissible set into a subset of measures), successfully used for designing an almost discrete optimal control for a class of time-delay nonlinear systems subject to mixed control-state constrains. It has been shown that by considering a measure theoretical non-classical problem (say (15)-(16)), it is possible to solve a modified time-delay system (see (6) or (8)). The functional $\mu \to \mu(f_0)$ in (15), as well as the functions appearing in the left-hand side of (16), are linear in their arguments and the measure. Therefore, this is a fact that the involved functions are linear even if the classical discrete time-delay problem is non-linear. The optimal control design method as described in this paper is systematic and straightforward in comparison with the traditional and established methods (see for example Göllmann *et al.* (2008)). The obtained control is almost optimal and no extra initial solution is required to be assumed.



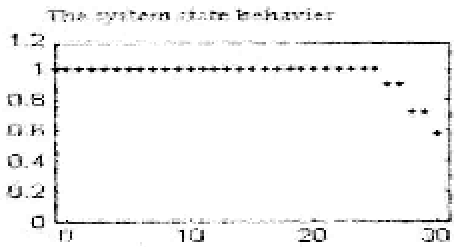


Figure1: The control action and the system responses using the optimal control.

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