INVARIANCE OF PRIMITIVE IDEALS BY
Φ-DERIVATIONS ON BANACH ALGEBRAS

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Abstract. We show that in certain cases a Φ-derivation on a Banach algebra with a nilpotent separating ideal leaves each primitive ideal invariant. We also obtain some sufficient conditions for the separating ideal of a Φ-derivation to be nilpotent.

1. INTRODUCTION

In this paper we study Φ-derivations on Banach algebras. Following [3] by a Φ-derivation on an algebra A, we mean a linear mapping Δ: A → A which satisfies

Δ(xy) = Δ(x)Φ(y) + xΔ(y) \quad (x, y ∈ A),

where Φ is an automorphism on A.

If τ denotes the identity map on A, then τ-derivations would be the ordinary derivations on A. Also for every automorphism Φ on A, τ-Φ is a Φ-derivation, and for each fixed c ∈ A the mapping Δ(x) = cΦ(x) − cx (x ∈ A), is a Φ-derivation which is called an inner Φ-derivation. Moreover, if D is an ordinary derivation on A and if b is an invertible element in A, then the map x ↦ D(x)b is a Φ-derivation on A where Φ is the inner automorphism x ↦ b^{−1}xb.

These objects have been considered extensively in algebraic point of view, see for example [1, 2] and [4]. They also have been used in [2] to study Jordan automorphisms on Banach algebras. Brešar and Villena in [3] obtained some algebraic technical results about Φ-derivations and by applying them they proved some results concerning Φ-derivations of Banach algebras. The following theorem is the final result of [3]. Here Rad(A) denotes the Jacobson radical of A.

Theorem A. Consider the following assertions.

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(i) For every inner automorphism $\Phi$ and every $\Phi$-derivation $\Delta$ of a unital Banach algebra $A$, $\Delta$ leaves each primitive ideal of $A$ invariant.

(ii) For every inner automorphism $\Phi$ and every $\Phi$-derivation $\Delta$ of a unital Banach algebra $A$, $\Delta(a)$ is quasinilpotent whenever $a \in \text{Rad}(A)$ is such that $\Delta^2(a) = 0$.

(iii) For every inner automorphism $\Phi$ and every $\Phi$-derivation $\Delta$ of a unital Banach algebra $A$, $\Delta(a) \neq 1$ for every $a \in \text{Rad}(A)$.

(iv) Every derivation on a Banach algebra $A$ leaves each primitive ideal of $A$ invariant.

(v) Every derivation on a unital Banach algebra $A$ takes invertible values only on such elements $a \in A$ for which the two sided ideal of $A$ generated by $a$ equals $A$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Leftrightarrow (v)$.

Assertion $(iv)$ is the well known noncommutative Singer-Wermer conjecture.

In section 2 we show that if $\Delta$ is a $\Phi$-derivation of a unital Banach algebra with $\Phi$ a continuous automorphism, such that both $\Phi$ and $[\Delta, \Phi] := \Delta\Phi - \Phi\Delta$ leave each nilpotent and each primitive ideal invariant (e.g. $\Phi$ is inner) and if $S(\Delta)$, the separating space of $\Delta$, is nilpotent then $\Delta$ leaves each primitive ideal invariant. This is a generalization of [3, Corollary 3.4]. Also we may add a new assertion to Theorem A as follows.

$(i')$ For every inner automorphism $\Phi$ and every $\Phi$-derivation $\Delta$ of a unital Banach algebra $A$, $\Delta$ has a nilpotent separating ideal.

Then $(i') \Rightarrow (i) \Rightarrow (iii) \Rightarrow (iv) \Leftrightarrow (v)$.

This naturally leads us to the following question.

Q1. Is it true that each $\Phi$-derivation on a Banach algebra has a nilpotent separating space?

It is indeed an open problem for ordinary derivations and it is shown in [8] that for ordinary derivations it is equivalent to the noncommutative Singer-Wermer conjecture. In Section 3 we obtain some sufficient conditions for $\Delta$ to have a nilpotent separating ideal and hence leave each primitive ideal invariant.

Note that if $\Phi$ is an automorphism and if $\Delta$ is a $\Phi$-derivation on a nonunital algebra $A$, then we may extend it to the unitalization of $A$ by defining $\Delta(1) = 0$. Throughout this paper $A$ is a unital Banach algebra, $\Phi$ is a continuous automorphism on $A$ and $\Delta$ is a $\Phi$-derivation of $A$. For a Banach algebra $A$, the sets $R$ and $B$ denote the Jacobson radical and the Baer radical of $A$, respectively. It is clear that
B and R are invariant under each automorphism \( \Phi \) on \( A \). The separating space, \( S(\Delta) \) of \( \Delta \) is defined to be the set 

\[
S(\Delta) := \{ a \in A : \exists \{a_n\} \subseteq A, a_n \to 0, \Delta(a_n) \to a \};
\]

which is a closed subspace of \( A \) and by the closed graph theorem \( S(\Delta) = \{0\} \) if and only if \( \Delta \) is continuous. For a moment consider \( A \) as a Banach \( A \)-bimodule, denoted by \( A^o \), with module operations, \( A \times A^o \to A^o, (a, x) \mapsto a.x = a\Phi(x), (x, a) \mapsto x.a = xa \), for all \( a, x \in A \). Obviously \( \Delta \) is an intertwining map from \( A \) into \( A^o \). Thus by [6, Theorem 5.2.15], \( S(\Delta) \) is a separating submodule and hence a separating ideal of \( A \), by surjectivity of \( \Phi \).

2. \( \Delta \)-INVARIANT IDEALS

Cusack in [5] proved that each derivation on a Banach algebra leaves the Baer radical invariant. Here we prove a similar result for \( \Phi \)-derivations, where \( \Phi \) is a continuous automorphism and \( \Phi, [\Delta, \Phi] \) leave each nilpotent ideal invariant. Clearly these conditions hold if \( \Phi \) is inner.

**Theorem 2.1.** Let \( \Delta \) be a \( \Phi \)-derivation on \( A \), such that \( \Phi \) and \( [\Delta, \Phi] \) leave each nilpotent ideal invariant. Then \( \Delta(B) \subseteq B \).

**Proof.** Let \( I \) be a nilpotent ideal with \( I^k = \{0\} \). Take \( a \in I \) and \( b_1, b_2, ..., b_k \in A \), then by assumption, \( (b_1a)(\Phi^{-1}(b_2a))...(\Phi^{-(k-1)}(b_ka)) = 0 \). Hence by [3, Theorem 2.3]

\[
0 = \Delta^k((b_1a)(\Phi^{-1}(b_2a))...(\Phi^{-(k-1)}(b_ka))) + I = k!\Delta(b_1a)\Delta(b_2a)...\Delta(b_ka) + I.
\]

But \( \Delta(b_1a) + I = b_1\Delta(a) + \Delta(b_1)\Phi(a) + I = b_i\Delta(a) + I \) for \( i = 1, ..., k \). Thus \( (A\Delta(a))^k \subseteq I \subseteq B \). Therefore \( \Delta(a) \in B \) and hence \( \Delta(I) \subseteq B \). Since \( B \) is the algebraic sum of all nilpotent ideals we have the result. \( \blacksquare \)

In [3, Theorem 3.2] it is proved that if \( \Phi \) is a continuous automorphism and \( \Delta \) is a continuous \( \Phi \)-derivation on a Banach algebra \( A \) and if \( J \) is an ideal of \( A \), such that both \( \Phi, [\Delta, \Phi] \) leave \( J \) invariant, then \( \Delta(J)/J \) is a quasinilpotent ideal of \( A/J \). So, if \( J \) is a primitive ideal, then \( \Delta(J)/J \subseteq \text{Rad}(A)/J = \{0\} \) and hence \( \Delta(J) \subseteq J \). We use this fact in the proof of the next theorem.

**Theorem 2.2.** Suppose that \( \Phi, [\Delta, \Phi] \) leave each nilpotent and each primitive ideal invariant. If \( S(\Delta) \) is nilpotent then \( \Delta(P) \subseteq P \) for each primitive ideal \( P \) of \( A \).
Proof. $S(\Delta)$ is a nilpotent ideal, hence $S(\Delta) \subseteq B$. Let $\pi$ be the canonical quotient map from $A$ onto $A/\overline{B}$ then $\pi \circ \Delta$ is continuous. Therefore $\pi(\Delta(\overline{B})) = \{0\}$ and it follows that $\Delta(\overline{B}) \subseteq \overline{B}$. On the other hand, $\Phi$ leaves $B$ invariant and $\Phi$ is continuous, thus $\Phi$ leaves $\overline{B}$ invariant and so it drops to a continuous automorphism $\Phi_0 : A/\overline{B} \rightarrow A/\overline{B}$. Consider $\Delta_0 : A/\overline{B} \rightarrow A/\overline{B}$, $a + \overline{B} \mapsto \Delta(a) + \overline{B}$, which is a continuous $\Phi_0$-derivation on $A/\overline{B}$ and by the argument just before this theorem $\Delta_0(P/\overline{B}) \subseteq P/\overline{B}$ for each primitive ideal $P$ of $A$. Since $\overline{B} \subseteq P$ for every primitive ideal $P$, we have $\Delta(P) \subseteq P$.

Corollary 2.1. If $\Phi$ is an inner automorphism on a Banach algebra $A$ and if $\Delta$ is a $\Phi$-derivation with a nilpotent separating ideal, then $\Delta$ leaves each primitive ideal invariant.

Proof. Clearly for an inner automorphism $\Phi$, $[\Delta, \Phi]$ leaves each ideal invariant. Now the result follows from Theorem 2.2.

3. Nilpotency of the Separating Ideal

Considering (Q1) we obtain some sufficient conditions for the separating ideal of a $\Phi$-derivation on a Banach algebra to be nilpotent or quasinilpotent. Theorem 3.1 (ii) is a generalization of [5, Lemma 4.2] and Corollary 3.2 is [3, Corollary 4.3] which is proved in a different way. Theorem 3.3 and the other results of this section are generalizations of the results in [7]. Throughout this section by (A1) we mean the following assumption:

(A1). The automorphism $\Phi$ is inner or $\Phi$ is continuous (as before), and $[\Delta, \Phi] = 0$.

Under this assumption $S(\Delta)$ is invariant under $[\Delta, \Phi]$ and each $\Phi^j$ $(j \in \mathbb{Z})$.

Theorem 3.1. Let $A$ be a Banach algebra, and let $\Delta$ be a $\Phi$-derivation on $A$. Set $J := S(\Delta) \cap R$. Then the following assertions hold.

(i) Let $Q(A)$ be the set of all quasinilpotent elements of $A$. If $\Delta(J) \subseteq Q(A)$, then $S(\Delta) \subseteq R$.
(ii) Assuming (A1) holds. If $J$ is a nil ideal, then $S(\Delta)$ is a nilpotent ideal of $A$.

Proof.

(i) Let $\Delta(J) \subseteq Q(A)$, but $S(\Delta) \nsubseteq R$. Since $S(\Delta)$ is a separating ideal, $S(\Delta)/J$ is finite dimensional by [6, Lemma 5.2.25]. Therefore $S(\Delta)$ has a strong Wederburn decomposition, that is there exists a finite dimensional subalgebra $U$ of $S(\Delta)$ such that $S(\Delta) = U \oplus J$ and $S(\Delta)$ contains a nonzero idempotent, say $e$ by [6, Theorem 2.8.6]. Let $\{a_n\}$ be a sequence in $A$, with $a_n \rightarrow 0$ and $\Delta(a_n) \rightarrow e$. Then $\{ea_n\} \subseteq S(\Delta)$ and there exist $\{u_n\} \subseteq U$ and $\{r_n\} \subseteq J$, such that $u_n \rightarrow 0$, $r_n \rightarrow 0$, and $ea_n = u_n + r_n$. We
have $\Delta(ea_n) \to e$. Since $U$ is finite dimensional $\Delta(u_n) \to 0$. Therefore $\Delta(r_n) \to e$, and so $e \in \Delta(J) \subseteq \overline{Q(A)}$. Thus by [6, Corollary 2.4.8], the spectrum of $e$ is a connected set containing the origin. It follows that the spectrum of $e$ is nothing but the set $\{0\}$ and this contradicts the fact that $e$ is non-zero. Thus $S(\Delta) \subseteq R$.

(ii) If $J$ is nilpotent, then $J \subseteq B$. Suppose on the contrary that $S(\Delta)$ is not nilpotent, then $S(\Delta) \neq J$. Using the same notation as in the proof of (i), it follows that $e \in \Delta(J) \subseteq \Delta(B)$. Hence by (A1) and Theorem 2.1 $e \in \overline{B} \subseteq R$ which is a contradiction. 

**Corollary 3.2.** Each $\Phi$-derivation $\Delta$ on a semisimple Banach algebra is continuous.

**Proof.** As before let $R$ denote the Jacobson radical. We have $S(\Delta) \cap R \subseteq R = \{0\}$ and by Theorem 3.1(i), $S(\Delta) \subseteq R$. Thus $S(\Delta) = \{0\}$, and $\Delta$ is continuous.

**Theorem 3.3.** Let $\Delta$ be a $\Phi$-derivation on a Banach algebra $A$ such that $[\Delta, \Phi]$ and $\Phi$ are continuous. Let $I$ be a closed ideal of $A$ with $\Phi^{-1}(I) \subseteq I$. Then $S(\Delta) \cap I$ is nilpotent if and only if $\Delta^2 \bigg|_{\prod_{n=1}^{\infty}(S(\Delta) \cap I)^n}$ is continuous.

**Proof.** We have $\Phi^{-1}(S(\Delta \cap I)) \subseteq S(\Delta) \cap I$. Suppose that $\Delta^2$ is continuous on $\bigcap_{n=1}^{\infty}(S(\Delta) \cap I)^n$. Consider $a \in S(\Delta) \cap I$, thus $\Phi^{-1}(a^n) = (\Phi^{-1}(a))^n \in (S(\Delta) \cap I)^n$. Since $S(\Delta)$ is a separating ideal, there exists $N \in \mathbb{N}$ such that $S(\Delta)\Phi^{-1}(a^n) = S(\Delta)\Phi^{-1}(a^N)$ ($n \geq N$). Hence by Mittag-Leffler theorem and the fact that $S(\Delta)\Phi^{-1}(a^n) \subseteq (S(\Delta) \cap I)^n$, we have

$$S(\Delta)\Phi^{-1}(a^N) = \bigcap_{n=1}^{\infty} S(\Delta)\Phi^{-1}(a^n) = \bigcap_{n=1}^{\infty} S(\Delta) \cap \Phi^{-1}(a^n) \subseteq \bigcap_{n=1}^{\infty} (S(\Delta) \cap I)^n.$$

Now, let $\{x_n\} \subseteq A$, $x_n \to 0$ and $\Delta(x_n) \to a^{N+1}$. Take $y_n = x_n\Phi^{-1}(a^{N+1})$, then $y_n \in S(\Delta)\Phi^{-1}(a^N) \subseteq \bigcap_{n=1}^{\infty}(S(\Delta) \cap I)^n$, $y_n \to 0$, and $\Delta(y_n) = \Delta(x_n)a^{N+1} + x_n\Delta(\Phi^{-1}(a^{N+1})) \to a^{2(N+1)}$. Also by the hypothesis, $\Delta^2(y_n) \to 0$ and $\Delta^2(y_n\Phi^{-1}(y_n)) \to 0$. On the other hand, by the continuity of $[\Delta, \Phi]$

$$\Delta^2((y_n)\Phi^{-1}(y_n)) = (y_n)\Delta^2(\Phi^{-1}(y_n)) + \Delta(y_n)^2 + \Delta(y_n)\Phi(\Delta(\Phi^{-1}(y_n))) + \Delta^2(y_n)\Phi(y_n)$$

$\to 2a^{4(N+1)}$ as $n$ tends to $\infty$. Therefore $a^{4(N+1)} = 0$, that is $S(\Delta) \cap I$ is a nil and hence a nilpotent ideal by closedness. The converse is trivial.

Note that the assumptions of Theorem 3.3 hold whenever $\Phi$ is inner.
Corollary 3.4. Let $\Delta$ be a $\Phi$-derivation on a Banach algebra $A$ and let $\Phi$ satisfy (A1), then $S(\Delta)$ is a nilpotent ideal if and only if $\Delta^2 \big| \bigcap_{n=1}^\infty (S(\Delta) \cap R)^n$ is continuous.

Proof. Since $\Phi^{-1}(R) \subseteq R$, then $S(\Delta) \cap R$ is nilpotent, by Theorem 3.3. Now Theorem 3.1 implies that $S(\Delta)$ is nilpotent. The converse is trivial. \hfill \blacksquare

Corollary 3.5. Let $\Delta$ be a $\Phi$-derivation on a Banach algebra $A$ and let $\Phi$ satisfy (A1). If $\dim (\bigcap_{n=1}^\infty (S(\Delta) \cap R)^n) < \infty$, then $S(\Delta)$ is nilpotent, and hence $\Delta$ leaves each primitive ideal of $A$ invariant.

Proof. This is immediate by Corollary 3.4 and Theorem 2.2. \hfill \blacksquare

Remark 3.6. Using the above results, the same notations and slightly different arguments as in [7], we observe that theorems 2.5, 2.6, 2.7 in [7] are also valid in the case of $\Phi$-derivations whenever $\Phi$ satisfies assumption (A1). In particular, [7, Theorem 2.7] together with Corollary 2.1 above, show that "if $\Phi$ is inner and the set $M(\Delta) = \{x \in S(\Delta) \cap R : \Delta(x) \in R\}$ is a nil set, then $\Delta$ leaves each primitive ideal invariant".

REFERENCES

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